Differential Subordination And Convex Univalent Functions

S.Aarthi[1]
Assistant Professor, Department of Career Development, SRM Institute of Science and Technology, SRM nagar, potheri- 603203,Kancheepuram (District),India.

Abstract *-In this work, we study about the first order differential subordination equation. Then consider the analytic function p and a univalent function q. We are proposing the work to make the function q to satisfy the best dominant's conditions of the differential subordination by making suitable changes for the functions p and q. Finally study the general differential subordination equation wherein we apply some derivatives to get an fascinating results about starlike property.*

KEYWORDS: *Starlikeness, univalent functions, dominant, subordination equations.*

I. INTRODUCTION

Let the class of function f be A and is analytic in the unit disc U. The unit disc is defined by the function $U=\{z:|z|<1\}$ and these functions are normalized by the conditions $f(0)=0$ and $f'(0)=1$. Denoting the subclass of functions f by A' which are analytic by in U and satisfy the conditions $f(0)=0$. Let the analytic function be p and h and q be the univalent function in the unit disc. Then the analytic function should satisfy the following first order differential subordination equation,

$$\left.\begin{aligned}&\Psi(p(z),zp'(z))\prec h(z),z\in U,\\&\Psi(p(0),0)=h(0).\end{aligned}\right\}\qquad(1.1)$$

is defined by the author in [12]

When the univalent function q satisfy the above equation, if $p(0)=q(0)$ and $p\prec q$ for all p then it is known as the dominant of the equation. When a dominant $\tilde{q}$ which satisfies $\tilde{q}\prec q$ for all dominants q of above equation then it is known as the best dominant of the above differential equation.

Consider $\lambda,\ \lambda>0,$ be any real number and α be any complex number with Re $\alpha>0,$ we study the first order differential subordination of the form

$$\phi(\alpha,\lambda;p(z))\prec\phi(\alpha,\lambda;q(z)),z\in U,\qquad(1.2)$$

and here we have to search out the conditions for the function q to become the best dominant of the above differential subordination equation.

II.PRELIMINARIES

Theorem 2.1

Let $L(z,t):U\times[0,\infty)\to C$ be the function which is in the format of $L(z,t)=a_1(t)z+...$ with $a_1(t)\neq 0$ for all $t\geq 0$, and $\lim_{t\to\infty}|a_1(t)|=\infty$, is a subordination chain $\leftrightarrow$ $\mathrm{Re}\left[\dfrac{z\frac{\partial L}{\partial z}}{\frac{\partial L}{\partial t}}\right]>0$ for all $z\in U$ and $t\geq 0$.

Theorem 2.2

Let F and G be analytic function in unit disc U, and $\bar{U}$ respectively. In addition to that let G be an univalent function in $\bar{U}$ having some exception for the points ζ such that $\lim_{z\to\zeta}F(z)=\infty$, with $F(0)=G(0)$. If F and G is not subordinate to eachother in U, then there exist points $z_0\in U$, $\zeta_0\in\partial U$ (boundary of U) and an $m\geq 1$ such that $F(|z|<|z_0|)\subset G(U)$, $F(z_0)=G(\zeta_0)$ and $z_0F'(z_0)=m\zeta_0G'(\zeta_0)$.

III. MAIN RESULTS

Theorem 3.1

Let α be any complex number with Re $\alpha>0$. Suppose the following conditions are satisfied by the convex univalent function $q\in A'$,

(a) Re $q(z)>0$, in U when Re $\alpha\geq|\alpha|^2$;

(b) Re $q(z)>\dfrac{|\alpha|^2-\mathrm{Re}\,\alpha}{2|\alpha|^2}$, in U when Re $\alpha<|\alpha|^2$.

For any real number λ, $\lambda>0$, then the following differential subordination equation is satisfied by the function $p\in A'$

$$\phi(\alpha,\lambda;\mathrm{p}(z))\prec\phi(\alpha,\lambda;q(z)), \tag{3.1.1}$$

in U, then $p(z)\prec q(z)$ in U and q is the best dominant.

is defined by the author in [12].

Note:

$$\phi(\alpha,\lambda;p(z))=(1-\alpha)p(z)+\alpha(p(z))^2+\alpha\lambda zp'(z),$$

put $p(z)=\frac{zf'(z)}{f(z)}$,

By the general differential subordination of the form,

$$\phi\left(\alpha,\lambda;\frac{zf'(z)}{f(z)}\right)=\frac{zf'(z)}{f(z)}\left(1-\alpha+\alpha(1-\lambda)\frac{zf'(z)}{f(z)}+\alpha\lambda\left(1+\frac{zf''(z)}{f'(z)}\right)\right), \qquad (3.1.2)$$

putting $\lambda=1$ in (3.1.2), we get

$$\phi\left(\alpha,1;\frac{zf'(z)}{f(z)}\right)=\frac{zf'(z)}{f(z)}\left(1-\alpha+\alpha(1-1)\frac{zf'(z)}{f(z)}+\alpha(1)\left(1+\frac{zf''(z)}{f'(z)}\right)\right),$$

$$\phi\left(\alpha,1;\frac{zf'(z)}{f(z)}\right)=\frac{zf'(z)}{f(z)}\left(1-\alpha+\alpha(0)\frac{zf'(z)}{f(z)}+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\right),$$

$$\phi\left(\alpha,1;\frac{zf'(z)}{f(z)}\right)=\frac{zf'(z)}{f(z)}\left(1-\alpha+\alpha+\alpha\frac{zf''(z)}{f'(z)}\right),$$

$$\phi\left(\alpha,1;\frac{zf'(z)}{f(z)}\right)=\frac{zf'(z)}{f(z)}+\alpha\left(\frac{zf'(z)}{f(z)}\right)\left(\frac{zf''(z)}{f'(z)}\right),$$

$$\phi\left(\alpha,1;\frac{zf'(z)}{f(z)}\right)=\frac{zf'(z)}{f(z)}+\alpha\left(\frac{z^2f''(z)}{f(z)}\right),$$

By putting $\alpha=1$ in (3.1.2), we get

$$\phi\left(1,\lambda;\frac{zf'(z)}{f(z)}\right)=\frac{zf'(z)}{f(z)}\left(1-1+1(1-\lambda)\frac{zf'(z)}{f(z)}+(1)\lambda\left(1+\frac{zf''(z)}{f'(z)}\right)\right),$$

$$\phi\left(1,\lambda;\frac{zf'(z)}{f(z)}\right)=\frac{zf'(z)}{f(z)}\left(1-1+(1-\lambda)\frac{zf'(z)}{f(z)}+\lambda\left(1+\frac{zf''(z)}{f'(z)}\right)\right),$$

$$\phi\left(1,\lambda;\frac{zf'(z)}{f(z)}\right)=\frac{zf'(z)}{f(z)}\left((1-\lambda)\frac{zf'(z)}{f(z)}+\lambda\left(1+\frac{zf''(z)}{f'(z)}\right)\right).$$

Theorem 3.2

The general subordination theorem's conditions

Let α,λ and q be a complex number with Re $\alpha>0$. If $f\in A,\ \frac{f(z)}{z}\neq 0$ be a function in U satisfies the following,

$$\phi(\alpha,\lambda;\mathrm{p}(z))\prec\phi(\alpha,\lambda;q(z)),\ z\in U.$$

Setting $p(z)=\dfrac{zf'(z)}{f(z)}$ in above first order differential subordination equation, we get

$$\phi\left(\alpha,\lambda;\frac{zf'(z)}{f(z)}\right)\prec\phi(\alpha,\lambda;q(z)),\ z\in U,$$

so

$$\frac{zf'(z)}{f(z)}\prec q(z),\forall \mathrm{z}\in \mathrm{U}.$$

Theorem 3.3

Let α and q be a complex number with Re $\alpha>0$. If $f\in A$, $\dfrac{f(z)}{z}\neq 0$ be a function in U, satisfies the following,

$$\frac{zf'(z)}{f(z)}+\alpha\frac{z^2f''(z)}{f(z)}\prec(1-\alpha)q(z)+\alpha(q(z))^2+\alpha zq'(z),z\in U,$$

then

$$\frac{zf'(z)}{f(z)}\prec q(z),\forall \mathrm{z}\in U.$$

PROOF

Let,

$$\phi\left(\alpha,\lambda;\frac{zf'(z)}{f(z)}\right)=\frac{zf'(z)}{f(z)}\left(1-\alpha+\alpha(1-\lambda)\frac{zf'(z)}{f(z)}+\alpha\lambda\left(1+\frac{zf''(z)}{f'(z)}\right)\right),$$

putting $\lambda=1$ in above equation, we get

$$\phi\left(\alpha,1;\frac{zf'(z)}{f(z)}\right)=\frac{zf'(z)}{f(z)}\left(1-\alpha+\alpha(1-1)\frac{zf'(z)}{f(z)}+\alpha(1)\left(1+\frac{zf''(z)}{f'(z)}\right)\right),$$

$$\phi\left(\alpha,1;\frac{zf'(z)}{f(z)}\right)=\frac{zf'(z)}{f(z)}\left(1-\alpha+\alpha(0)\frac{zf'(z)}{f(z)}+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\right),$$

$$\phi\left(\alpha,1;\frac{zf'(z)}{f(z)}\right)=\frac{zf'(z)}{f(z)}\left(1-\alpha+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\right),$$

$$\phi\left(\alpha,1;\frac{zf'(z)}{f(z)}\right)=\frac{zf'(z)}{f(z)}\left(1-\alpha+\alpha+\alpha\frac{zf''(z)}{f'(z)}\right),$$

$$\phi\left(\alpha,1;\frac{zf'(z)}{f(z)}\right)=\frac{zf'(z)}{f(z)}\left(1+\alpha\frac{zf''(z)}{f'(z)}\right),$$

$$\phi\left(\alpha,1;\frac{zf'(z)}{f(z)}\right)=\frac{zf'(z)}{f(z)}+\alpha\left(\frac{zf'(z)}{f(z)}\right)\left(\frac{zf''(z)}{f'(z)}\right),$$

$$\phi\left(\alpha,1;\frac{zf'(z)}{f(z)}\right)=\frac{zf'(z)}{f(z)}+\alpha\left(\frac{z^2f''(z)}{f(z)}\right). \qquad (3.3.1)$$

Let,

$$L(z,t)=(1-\alpha)q(z)+\alpha(q(z))^2+\alpha\lambda tzq'(z),$$

Setting $\lambda=1$ in above function, we get

$$L(z,t)=(1-\alpha)q(z)+\alpha(q(z))^2+\alpha(1)tzq'(z),$$

$$L(z,t)=(1-\alpha)q(z)+\alpha(q(z))^2+\alpha tzq'(z). \qquad (3.3.2)$$

From (3.3.1) and (3.3.2) we get,

$$\frac{zf'(z)}{f(z)}+\alpha\frac{z^2f''(z)}{f(z)}\prec(1-\alpha)q(z)+\alpha(q(z))^2+\alpha zq'(z),z\in U,$$

then

$$\frac{zf'(z)}{f(z)}\prec q(z),\forall z\in U.$$

Definition 3.4

Let the function $f\in A$ is known to be α - *convex*, (Bazilevic Functions and Generalized Convexity) if

$$\operatorname{Re}\left((1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\right)>0,z\in U.$$

Theorem3.5

Subordination theorem:

Let λ be a positive real number. Assume that $q\in A'$ is convex univalent in U and

Re $\left(q(z)\right)>0,\ z\in U$. If a function $f\in A,\ \frac{f(z)}{z}\neq 0$ in U, satisfies the differential subordination

$$\frac{zf'(z)}{f(z)}\left((1-\lambda)\frac{zf'(z)}{f(z)}+\lambda\left(1+\frac{zf''(z)}{f'(z)}\right)\right)\prec q(z)\left(q(z)+\frac{\lambda zq'(z)}{q(z)}\right), z\in U,$$

then

$$\frac{zf'(z)}{f(z)}\prec q(z), \forall z\in U.$$

PROOF

We get the proof by let $\alpha=1$ in Theorem 3.2.

By having suitable changes to the functions p and q in $\phi\left(\alpha,\lambda;\mathrm{p}(z)\right)\prec\phi\left(\alpha,\lambda;q(z)\right),\ z\in U.$

Hereby

$$\frac{zf'(z)}{f(z)}\left(1-\alpha+\alpha(1-\lambda)\frac{zf'(z)}{f(z)}+\alpha\lambda\left(1+\frac{zf''(z)}{f'(z)}\right)\right)\prec\frac{zg'(z)}{g(z)}\left(1-\alpha+\alpha(1-\lambda)\frac{zg'(z)}{g(z)}+\alpha\lambda\left(1+\frac{zg''(z)}{g'(z)}\right)\right),$$

From equation (3.1.2), we have

$$\phi\left(\alpha,\lambda;\frac{zf'(z)}{f(z)}\right)=\frac{zf'(z)}{f(z)}\left(1-\alpha+\alpha(1-\lambda)\frac{zf'(z)}{f(z)}+\alpha\lambda\left(1+\frac{zf''(z)}{f'(z)}\right)\right),$$

setting $\alpha=1$ in above equation, we get

$$\phi\left(1,\lambda;\frac{zf'(z)}{f(z)}\right)=\frac{zf'(z)}{f(z)}\left(1-1+1(1-\lambda)\frac{zf'(z)}{f(z)}+(1)\lambda\left(1+\frac{zf''(z)}{f'(z)}\right)\right),$$

$$\phi\left(1,\lambda;\frac{zf'(z)}{f(z)}\right)=\frac{zf'(z)}{f(z)}\left(1-1+(1-\lambda)\frac{zf'(z)}{f(z)}+\lambda\left(1+\frac{zf''(z)}{f'(z)}\right)\right),$$

$$\phi\left(1,\lambda;\frac{zf'(z)}{f(z)}\right)=\frac{zf'(z)}{f(z)}\left((1-\lambda)\frac{zf'(z)}{f(z)}+\lambda\left(1+\frac{zf''(z)}{f'(z)}\right)\right). \quad (3.5.1)$$

by

$$\phi\left(\alpha,\lambda;\mathrm{p}(z)\right)\prec\phi\left(\alpha,\lambda;q(z)\right),\ z\in U,$$

setting $\alpha=1$,

$$\phi(1,\lambda;p(z)) \prec \phi(1,\lambda;q(z)),\ z \in U.$$

Already we have an function,

$$L(z,t) = (1-\alpha)q(z) + \alpha(q(z))^2 + \alpha\lambda tzq'(z),$$

setting $\alpha = 1$ in above equation, we get

$$L(z,t) = (1-1)q(z) + (1)(q(z))^2 + (1)\lambda tzq'(z),$$

$$L(z,t) = (0)q(z) + (1)(q(z))^2 + (1)\lambda tzq'(z),$$

$$L(z,t) = (q(z))^2 + \lambda zq'(z),$$

$$L(z,t) = q(z)q(z) + \lambda zq'(z),$$

$$L(z,t) = q(z)\left(q(z) + \frac{\lambda zq'(z)}{q(z)}\right), z \in U. \quad (3.5.2)$$

From (3.5.1) and (3.5.2), we have

$$\frac{zf'(z)}{f(z)}\left((1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right)\right) \prec q(z)\left(q(z) + \frac{\lambda zq'(z)}{q(z)}\right), z \in U,$$

then

$$\frac{zf'(z)}{f(z)} \prec q(z), \forall z \in U.$$

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