Boros Integral involving the multivariable Gimel-function I

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ABSTRACT
In the present paper we evaluate Boros integral with three parameters involving the multivariable Gimel-function. We shall see several corollaries and remarks.

Keywords: Multivariable Gimel-function, multivariable I-function, multivariable H-function, multivarialbe Aleph-function, Aleph-function of two variables, aleph-function of one variable multivariable I-function, I-function of two variables, I-function of one variable, multivariable H-function, Mellin-Barnes integrals contour, Boros integral.

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1. Introduction and preliminaries.

The object of this document is to evaluate the Boros integral involving the multivariable gimel-function ;

Throughout this paper, let $\mathbb{C},\mathbb{R}$ and $\mathbb{N}$ be set of complex numbers, real numbers and positive integers respectively. Also $\mathbb{N}_0=\mathbb{N}\cup\{0\}$. We define a generalized transcendental function of several complex variables.

$$\gimel(z_1,\cdots,z_r)=\gimel^{0,n_2;0,n_3;\cdots;0,n_r:m^{(1)},n^{(1)};m^{(2)},n^{(2)};\cdots;m^{(r)},n^{(r)}}_{p_{i_2},q_{i_2},\tau_{i_2};R_2;p_{i_3},q_{i_3},\tau_{i_3};R_3;\cdots;p_{i_r},q_{i_r},\tau_{i_r}:R_r:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}}\left(\begin{array}{c|}z_1\\ \cdot\\ \cdot\\ \cdot\\ z_r\end{array}\right.$$

$$\begin{array}{c}[(\mathrm{a}_{2j};\alpha_{2j}^{(1)},\alpha_{2j}^{(2)};A_{2j})]_{1,n_2},[\tau_{i_2}(a_{2ji_2};\alpha_{2ji_2}^{(1)},\alpha_{2ji_2}^{(2)};A_{2ji_2})]_{n_2+1,p_{i_2}};[(a_{3j};\alpha_{3j}^{(1)},\alpha_{3j}^{(2)},\alpha_{3j}^{(3)};A_{3j})]_{1,n_3},\\ \cdot\\ [\tau_{i_2}(b_{2ji_2};\beta_{2ji_2}^{(1)},\beta_{2ji_2}^{(2)};B_{2ji_2})]_{1,q_{i_2}};\end{array}$$

$$\begin{array}{ll}[\tau_{i_3}(a_{3ji_3};\alpha_{3ji_3}^{(1)},\alpha_{3ji_3}^{(2)},\alpha_{3ji_3}^{(3)};A_{3ji_3})]_{n_3+1,p_{i_3}};\cdots; & [(\mathrm{a}_{rj};\alpha_{rj}^{(1)},\cdots,\alpha_{rj}^{(r)};A_{rj})]_{1,n_r},\\ \cdot & \cdot\\ [\tau_{i_3}(b_{3ji_3};\beta_{3ji_3}^{(1)},\beta_{3ji_3}^{(2)},\beta_{3ji_3}^{(3)};B_{3ji_3})]_{1,q_{i_3}};\cdots; & \cdots\end{array}$$

$$\begin{array}{ll}[\tau_{i_r}(a_{rji_r};\alpha_{rji_r}^{(1)},\cdots,\alpha_{rji_r}^{(r)};A_{rji_r})]_{n_r+1,p_r}: & [(\mathrm{c}_j^{(1)},\gamma_j^{(1)};C_j^{(1)})]_{1,n^{(1)}},[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)};C_{ji^{(1)}}^{(1)})]_{n^{(1)}+1,p_i^{(1)}}\\ \cdot & \cdot\\ [\tau_{i_r}(b_{rji_r};\beta_{rji_r}^{(1)},\cdots,\beta_{rji_r}^{(r)};B_{rji_r})]_{1,q_r}: & [(d_j^{(1)}),\delta_j^{(1)};D_j^{(1)})]_{1,m^{(1)}},[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)};D_{ji^{(1)}}^{(1)})]_{m^{(1)}+1,q_i^{(1)}}\end{array}$$

$$\left.\begin{array}{l};\cdots;[(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})]_{1,n^{(r)}},[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)};C_{ji^{(r)}}^{(r)})]_{n^{(r)}+1,p_i^{(r)}}\\ \cdot\\ ;\cdots;[(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})]_{1,m^{(r)}},[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)};D_{ji^{(r)}}^{(r)})]_{m^{(r)}+1,q_i^{(r)}}\end{array}\right)$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\xi(s_1,\cdots,s_r)\prod_{k=1}^{r}\phi_k(s_k)z_k^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.1}$$

with $\omega = \sqrt{-1}$

$$\xi(s_1,\cdots,s_r) = \frac{\prod_{j=1}^{n_2}\Gamma^{A_{2j}}(1-a_{2j}+\sum_{k=1}^{2}\alpha_{2j}^{(k)}s_k)}{\sum_{i_2=1}^{R_2}[\tau_{i_2}\prod_{j=n_2+1}^{p_{i_2}}\Gamma^{A_{2ji_2}}(a_{2ji_2}-\sum_{k=1}^{2}\alpha_{2ji_2}^{(k)}s_k)\prod_{j=1}^{q_{i_2}}\Gamma^{B_{2ji_2}}(1-b_{2ji_2}+\sum_{k=1}^{2}\beta_{2ji_2}^{(k)}s_k)]}$$

$$\frac{\prod_{j=1}^{n_3}\Gamma^{A_{3j}}(1-a_{3j}+\sum_{k=1}^{3}\alpha_{3j}^{(k)}s_k)}{\sum_{i_3=1}^{R_3}[\tau_{i_3}\prod_{j=n_3+1}^{p_{i_3}}\Gamma^{A_{3ji_3}}(a_{3ji_3}-\sum_{k=1}^{3}\alpha_{3ji_3}^{(k)}s_k)\prod_{j=1}^{q_{i_3}}\Gamma^{B_{3ji_3}}(1-b_{3ji_3}+\sum_{k=1}^{3}\beta_{3ji_3}^{(k)}s_k)]}$$

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$$\frac{\prod_{j=1}^{n_r}\Gamma^{A_{rj}}(1-a_{rj}+\sum_{k=1}^{r}\alpha_{rj}^{(k)}s_k)}{\sum_{i_r=1}^{R_r}[\tau_{i_r}\prod_{j=n_r+1}^{p_{i_r}}\Gamma^{A_{rji_r}}(a_{rji_r}-\sum_{k=1}^{r}\alpha_{rji_r}^{(k)}s_k)\prod_{j=1}^{q_{i_r}}\Gamma^{B_{rji_r}}(1-b_{rji_r}+\sum_{k=1}^{r}\beta_{rji_r}^{(k)}s_k)]} \tag{1.2}$$

and

$$\phi_k(s_k) = \frac{\prod_{j=1}^{m^{(k)}}\Gamma^{D_j^{(k)}}(d_j^{(k)}-\delta_j^{(k)}s_k)\prod_{j=1}^{n^{(k)}}\Gamma^{C_j^{(k)}}(1-c_j^{(k)}+\gamma_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau_{i^{(k)}}\prod_{j=m^{(k)}+1}^{q_{i^{(k)}}}\Gamma^{D_{ji^{(k)}}^{(k)}}(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k)\prod_{j=n^{(k)}+1}^{p_{i^{(k)}}}\Gamma^{C_{ji^{(k)}}^{(k)}}(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k)]} \tag{1.3}$$

1) $[(c_j^{(1)};\gamma_j^{(1)}]_{1,n_1}$ stands for $(c_1^{(1)};\gamma_1^{(1)}),\cdots,(c_{n_1}^{(1)};\gamma_{n_1}^{(1)})$.

2) $n_2,\cdots,n_r,m^{(1)},n^{(1)},\cdots,m^{(r)},n^{(r)},p_{i_2},q_{i_2},R_2,\tau_{i_2},\cdots,p_{i_r},q_{i_r},R_r,\tau_{i_r},p_{i^{(r)}},q_{i^{(r)}},\tau_{i^{(r)}},R^{(r)} \in \mathbb{N}$ and verify :

$0 \leqslant m_2, 0 \leqslant n_2 \leqslant p_{i_2},\cdots,0 \leqslant m_r, 0 \leqslant n_r \leqslant p_{i_r}, 0 \leqslant m^{(1)} \leqslant q_{i^{(1)}},\cdots,0 \leqslant m^{(r)} \leqslant q_{i^{(r)}}$.

3) $\tau_{i_2}(i_2 = 1,\cdots,R_2) \in \mathbb{R}^+; \tau_{i_r} \in \mathbb{R}^+(i_r = 1,\cdots,R_r); \tau_{i^{(k)}} \in \mathbb{R}^+(i = 1,\cdots,R^{(k)}),(k = 1,\cdots,r)$.

4) $\gamma_j^{(k)}, C_j^{(k)} \in \mathbb{R}^+;(j = 1,\cdots,n_k);(k = 1,\cdots,r);\delta_j^{(k)}, D_j^{(k)} \in \mathbb{R}^+;(j = 1,\cdots,m_k);(k = 1,\cdots,r)$.

$\mathrm{C}_{ji^{(k)}}^{(k)} \in \mathbb{R}^+,(j = n^{(k)}+1,\cdots,p^{(k)});(k = 1,\cdots,r);$

$\mathrm{D}_{ji^{(k)}}^{(k)} \in \mathbb{R}^+,(j = m^{(k)}+1,\cdots,q^{(k)});(k = 1,\cdots,r)$.

$\alpha_{kj}^{(l)}, A_{kj} \in \mathbb{R}^+;(j = 1,\cdots,n_k);(k = 2,\cdots,r);(l = 1,\cdots,k)$.

$\alpha_{kji_k}^{(l)}, A_{kji_k} \in \mathbb{R}^+;(j = n_k+1,\cdots,p_{i_k});(k = 2,\cdots,r);(l = 1,\cdots,k)$.

$\beta_{kji_k}^{(l)}, B_{kji_k} \in \mathbb{R}^+;(j = m_k+1,\cdots,q_{i_k});(k = 2,\cdots,r);(l = 1,\cdots,k)$.

$\delta_{ji^{(k)}}^{(k)} \in \mathbb{R}^+;(i = 1,\cdots,R^{(k)});(j = m^{(k)}+1,\cdots,q_{i^{(k)}});(k = 1,\cdots,r)$.

$\gamma_{ji^{(k)}}^{(k)} \in \mathbb{R}^+; (i = 1, \cdots, R^{(k)}); (j = n^{(k)} + 1, \cdots, p_{i^{(k)}}); (k = 1, \cdots, r).$

5) $c_j^{(k)} \in \mathbb{C}; (j = 1, \cdots, n^{(k)}); (k = 1, \cdots, r); d_j^{(k)} \in \mathbb{C}; (j = 1, \cdots, m^{(k)}); (k = 1, \cdots, r).$

$a_{kji_k} \in \mathbb{C}; (j = n_k + 1, \cdots, p_{i_k}); (k = 2, \cdots, r).$

$b_{kji_k} \in \mathbb{C}; (j = m_k + 1, \cdots, q_{i_k}); (k = 2, \cdots, r).$

$d_{ji^{(k)}}^{(k)} \in \mathbb{C}; (i = 1, \cdots, R^{(k)}); (j = m^{(k)} + 1, \cdots, q_{i^{(k)}}); (k = 1, \cdots, r).$

$c_{ji^{(k)}}^{(k)} \in \mathbb{C}; (i = 1, \cdots, R^{(k)}); (j = n^{(k)} + 1, \cdots, p_{i^{(k)}}); (k = 1, \cdots, r).$

The contour L_k is in the $s_k(k = 1, \cdots, r)$- plane and run from $\sigma - i\infty$ to $\sigma + i\infty$ where σ if is a real number with loop, if necessary to ensure that the poles of $\Gamma^{A_{2j}}\left(1 - a_{2j} + \sum_{k=1}^{2} \alpha_{2j}^{(k)} s_k\right) (j = 1, \cdots, n_2), \Gamma^{A_{3j}}\left(1 - a_{3j} + \sum_{k=1}^{3} \alpha_{3j}^{(k)} s_k\right)$ $(j = 1, \cdots, n_3), \cdots, \Gamma^{A_{rj}}\left(1 - a_{rj} + \sum_{i=1}^{r} \alpha_{rj}^{(i)}\right) (j = 1, \cdots, n_r), \Gamma^{C_j^{(k)}}\left(1 - c_j^{(k)} + \gamma_j^{(k)} s_k\right) (j = 1, \cdots, n^{(k)})(k = 1, \cdots, r)$to the left of the contour L_k and the poles of $\Gamma^{D_j^{(k)}}\left(d_j^{(k)} - \delta_j^{(k)} s_k\right) (j = 1, \cdots, m^{(k)})(k = 1, \cdots, r)$ lie to the right of the contour L_k.

The multivariable Gimel-function defined by (1.1) is analytic of $z_i(i = 1; \cdots, r)$ if

$$\tau_{i_2} \sum_{j=1}^{p_{i_2}} A_{2ji_2}^{(k)} \alpha_{2ji_2}^{(k)} + \cdots + \tau_{i_r} \sum_{j=1}^{p_{i_r}} A_{rji_r}^{(k)} \alpha_{rji_r}^{(k)} + \tau_{i^{(k)}} \sum_{j=1}^{p_{i^{(k)}}} C_{ji^{(k)}}^{(k)} \gamma_{ji^{(k)}}^{(k)} -$$

$$\tau_{i_2} \sum_{j=1}^{q_{i_2}} B_{2ji_2}^{(k)} \beta_{2ji_2}^{(k)} - \cdots - \tau_{i_r} \sum_{j=1}^{q_{i_r}} B_{rji_r}^{(k)} \beta_{rji_r}^{(k)} - \tau_{i^{(k)}} \sum_{j=1}^{q_{i^{(k)}}} D_{ji^{(k)}}^{(k)} \delta_{ji^{(k)}}^{(k)} < 0 \ (k = 1, \cdots, r) \tag{1.4}$$

The integral defined by (1.1) converges absolutely if

$|arg(z_k)| < \frac{1}{2} A_i^{(k)} \pi$ where

$$A_i^{(k)} = \sum_{j=1}^{m^{(k)}} D_j^{(k)} \delta_j^{(k)} + \sum_{j=1}^{n^{(k)}} C_j^{(k)} \gamma_j^{(k)} + \sum_{j=1}^{n_2} A_{2j} \alpha_{2j}^{(k)} - \tau_{i_2} \sum_{j=n_2+1}^{p_{i_2}} A_{2ji_2} \alpha_{2ji_2}^{(k)} + \sum_{j=1}^{n_3} A_{3j} \alpha_{3j}^{(k)} - \tau_{i_3} \sum_{j=n_3+1}^{p_{i_3}} A_{3ji_3} \alpha_{3ji_3}^{(k)}$$

$$+ \cdots + \sum_{j=1}^{n_r} A_{rj} \alpha_{rj}^{(k)} - \tau_{i_r} \sum_{j=n_r+1}^{p_{i_r}} A_{rji_r} \alpha_{rji_r}^{(k)} - \tau_{i^{(k)}} \left(\sum_{j=m^{(k)}+1}^{q_i^{(k)}} D_{ji^{(k)}}^{(k)} \delta_{ji^{(k)}}^{(k)} + \sum_{j=n^{(k)}+1}^{p_i^{(k)}} C_{ji^{(k)}}^{(k)} \gamma_{ji^{(k)}}^{(k)} \right)$$

$$-\tau_{i_2} \sum_{j=1}^{q_{i_2}} B_{2ji_2} \beta_{2ji_2}^{(k)} - \cdots - \tau_{i_r} \sum_{j=1}^{q_{i_r}} B_{rji_r} \beta_{rji_r}^{(k)} \qquad (k = 1, \cdots, r) \tag{1.5}$$

Following the lines of Braaksma ([3] p. 278), we may establish the the asymptotic expansion in the following convenient form :

$\daleth(z_1, \cdots, z_r) = 0(|z_1|^{\alpha_1}, \cdots, |z_r|^{\alpha_r}), max(|z_1|, \cdots, |z_r|) \rightarrow 0$

$\daleth(z_1, \cdots, z_r) = 0(|z_1|^{\beta_1}, \cdots, |z_r|^{\beta_r}), min(|z_1|, \cdots, |z_r|) \rightarrow \infty$ where $i = 1, \cdots, r :$

$\alpha_i = \min_{1 \leqslant j \leqslant m^{(i)}} Re \left[D_j^{(i)} \left(\frac{d_j^{(i)}}{\delta_j^{(i)}} \right) \right]$ and $\beta_i = \max_{1 \leqslant j \leqslant n^{(i)}} Re \left[C_j^{(i)} \left(\frac{c_j^{(i)} - 1}{\gamma_j^{(i)}} \right) \right]$

In your investigation, we shall use the following notations, see Ayant [2].

$$\mathbb{A} =[(\mathrm{a}_{2j};\alpha_{2j}^{(1)},\alpha_{2j}^{(2)};A_{2j})]_{1,n_2},[\tau_{i_2}(a_{2ji_2};\alpha_{2ji_2}^{(1)},\alpha_{2ji_2}^{(2)};A_{2ji_2})]_{n_2+1,p_{i_2}},[(a_{3j};\alpha_{3j}^{(1)},\alpha_{3j}^{(2)},\alpha_{3j}^{(3)};A_{3j})]_{1,n_3},$$

$$[\tau_{i_3}(a_{3ji_3};\alpha_{3ji_3}^{(1)},\alpha_{3ji_3}^{(2)},\alpha_{3ji_3}^{(3)};A_{3ji_3})]_{n_3+1,p_{i_3}};\cdots;[(\mathrm{a}_{(r-1)j};\alpha_{(r-1)j}^{(1)},\cdots,\alpha_{(r-1)j}^{(r-1)};A_{(r-1)j})]_{1,n_{r-1}},$$

$$[\tau_{i_{r-1}}(a_{(r-1)ji_{r-1}};\alpha_{(r-1)ji_{r-1}}^{(1)},\cdots,\alpha_{(r-1)ji_{r-1}}^{(r-1)};A_{(r-1)ji_{r-1}})]_{n_{r-1}+1,p_{i_{r-1}}} \tag{1.6}$$

$$\mathbf{A} = [(\mathrm{a}_{rj};\alpha_{rj}^{(1)},\cdots,\alpha_{rj}^{(r)};A_{rj})]_{1,n_r},[\tau_{i_r}(a_{rji_r};\alpha_{rji_r}^{(1)},\cdots,\alpha_{rji_r}^{(r)};A_{rji_r})]_{\mathfrak{n}+1,p_{i_r}} \tag{1.7}$$

$$A = [(\mathrm{c}_j^{(1)},\gamma_j^{(1)};C_j^{(1)})]_{1,n^{(1)}},[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)};C_{ji^{(1)}}^{(1)})]_{n^{(1)}+1,p_i^{(1)}};\cdots;$$

$$[(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})]_{1,n^{(r)}},[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)};C_{ji^{(r)}}^{(r)})]_{n^{(r)}+1,p_i^{(r)}} \tag{1.8}$$

$$\mathbb{B} = [\tau_{i_2}(b_{2ji_2};\beta_{2ji_2}^{(1)},\beta_{2ji_2}^{(2)};B_{2ji_2})]_{1,q_{i_2}},[\tau_{i_3}(b_{3ji_3};\beta_{3ji_3}^{(1)},\beta_{3ji_3}^{(2)},\beta_{3ji_3}^{(3)};B_{3ji_3})]_{1,q_{i_3}};\cdots;$$

$$[\tau_{i_{r-1}}(b_{(r-1)ji_{r-1}};\beta_{(r-1)ji_{r-1}}^{(1)},\cdots,\beta_{(r-1)ji_{r-1}}^{(r-1)};B_{(r-1)ji_{r-1}})]_{1,q_{i_{r-1}}} \tag{1.9}$$

$$\mathbf{B} = [\tau_{i_r}(b_{rji_r};\beta_{rji_r}^{(1)},\cdots,\beta_{rji_r}^{(r)};B_{rji_r})]_{1,q_{i_r}} \tag{1.10}$$

$$\mathrm{B} = [(\mathrm{d}_j^{(1)},\delta_j^{(1)};D_j^{(1)})]_{1,m^{(1)}},[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)};D_{ji^{(1)}}^{(1)})]_{m^{(1)}+1,q_i^{(1)}};\cdots;$$

$$[(\mathrm{d}_j^{(r)},\delta_j^{(r)};D_j^{(r)})]_{1,m^{(r)}},[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)};D_{ji^{(r)}}^{(r)})]_{m^{(r)}+1,q_i^{(r)}} \tag{1.11}$$

$$U = 0,n_2;0,n_3;\cdots;0,n_{r-1};V = m^{(1)},n^{(1)};m^{(2)},n^{(2)};\cdots;m^{(r)},n^{(r)} \tag{1.12}$$

$$X = p_{i_2},q_{i_2},\tau_{i_2};R_2;\cdots;p_{i_{r-1}},q_{i_{r-1}},\tau_{i_{r-1}}:R_{r-1};Y = p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)} \tag{1.13}$$

2. Boros integral.

We have the following integral

Lemma

$$\int_0^\infty \left(\frac{x^2}{a^2x^4+(2ab+c)x^2+b^2}\right)^{p+1}\mathrm{d}x = \frac{B\left(p+\frac{1}{2},\frac{1}{2}\right)}{2a\left[2a(b+|b|)+c\right]^{p+\frac{1}{2}}} \tag{2.1}$$

$b>0, 2a(b+|b|)+c>0, Re(p)>-\dfrac{1}{2}, B(.,.)$ where is the beta function

Concerning the proof, see Boros et al [3].

3. Main integral.

In the paper , we shall note $X = \dfrac{x^2}{a^2x^4+(2ab+c)x^2+b^2}$ and $Y = \dfrac{1}{[2a(b+|b|)+c]}$

We have the general formula :

Theorem.

$$\int_0^\infty X^{l+1}\gimel(z_1X^{\alpha_1},\cdots,z_rX^{\alpha_r})\mathrm{d}x=\frac{\Gamma\left(\frac{1}{2}\right)Y^{l+\frac{1}{2}}}{2a}\gimel^{U;0,n_r+1:V}_{X;p_{i_r}+1,q_{i_r}+1,\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c} z_1Y^{\alpha_1} & \mathbb{A};\left(\frac{1}{2}-l;\alpha_1,\cdots,\alpha_r;1\right),\mathbf{A}:A \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ z_rY^{\alpha_r} & \mathbb{B};\mathbf{B},(-l;\alpha_1,\cdots,\alpha_r;1):B \end{array}\right) \quad (3.1)$$

Provided that

a) $\alpha_i>0,i=1,\cdots,r,b>0,2a(b+|b|)+c>0$

b) $Re(l)+\sum_{i=1}^r\alpha_i\min_{1\leqslant j\leqslant m^{(i)}}D_j^{(i)}Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>-\frac{1}{2}$

c) $|arg(z_iX^{\alpha_i})|<\frac{1}{2}A_i^{(k)}\pi$ where $A_i^{(k)}$ is defined by (1.5).

Proof

To establish (3.1), replace the modified of multivariable Gimel-function by its Mellin-Barnes integrals contour from (1.1) and get the following form of integral (say G)

$$G=\int_0^\infty X^{l+1}\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{i=1}^r\theta_i(s_i)z_i^{s_i}X^{\alpha_is_i}\,\mathrm{d}s_1\cdots\mathrm{d}s_r\mathrm{d}x \quad (3.2)$$

ow we interchange the order of integrations which is justified due to the absolute convergence of integrals involved in the process, we get

$$G=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{i=1}^r\theta_i(s_i)z_i^{s_i}\int_0^\infty X^{l+1+\sum_{i=1}^r\alpha_is_i}\,\mathrm{d}r\mathrm{d}s_1\cdots\mathrm{d}s_r \quad (3.3)$$

Now using the lemma and definition of beta function, we obtain

$$\int_0^\infty X^{l+1+\sum_{i=1}^r\alpha_is_i}\,\mathrm{d}r=\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(l+\sum_{i=1}^r\alpha_is_i+\frac{1}{2}\right)Y^{l+\sum_{i=1}^r\alpha_is_i+\frac{1}{2}}}{2a\Gamma\left(l+\sum_{i=1}^r\alpha_is_i+1\right)} \quad (3.4)$$

and

$$G=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{i=1}^r\theta_i(s_i)z_i^{s_i}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(l+\sum_{i=1}^r\alpha_is_i+\frac{1}{2}\right)Y^{l+\sum_{i=1}^r\alpha_is_i+\frac{1}{2}}}{2a\Gamma\left(l+\sum_{i=1}^r\alpha_is_i+1\right)}\mathrm{d}s_1\cdots\mathrm{d}s_r=$$

$$\frac{\Gamma\left(\frac{1}{2}\right)Y^{l+\frac{1}{2}}}{2a}\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{i=1}^r\theta_i(s_i)z_i^{s_i}\frac{\Gamma\left(l+\sum_{i=1}^r\alpha_is_i+\frac{1}{2}\right)Y^{\sum_{i=1}^r\alpha_is_i}}{\Gamma\left(l+\sum_{i=1}^r\alpha_is_i+1\right)}\mathrm{d}s_1\cdots\mathrm{d}s_r \quad (3.5)$$

Now, we interpret the resulting by means of Mellin-Barnes multiple integrals contour to get the required result.

4. Special cases.

In this section, we shall see several corollaries and remarks.

If $n_2=\cdots=n_{r-1}=m_2=m_3=\cdots=m_r=p_{i_2}=q_{i_2}=\cdots=p_{i_{r-1}}=q_{i_{r-1}}=0$ and $\tau_{i_2}=\cdots=\tau_{i_r}=\tau_{i^{(1)}}=\cdots=\tau_{i^{(r)}}=R_2=\cdots=R_r=R^{(1)}=\cdots=R^{(r)}=1$, then the multivariable Gimel-function reduces to multivariable I-function defined by Prathima [18], see [19-22,28-30,34,35]. We have

Corollary 1.

$$\int_0^{\infty} X^{l+1}\mathrm{I}\left(z_1X^{\alpha_1},\cdots,z_rX^{\alpha_r}\right)\mathrm{d}x = \frac{\Gamma\left(\frac{1}{2}\right)Y^{l+\frac{1}{2}}}{2a}\,\mathrm{I}_{p+1,q+1:Y}^{0,n+1:X}\left(\begin{array}{c|c} \mathrm{z}_1Y^{\alpha_1} & \left(\frac{1}{2}-l;\alpha_1,\cdots,\alpha_r;1\right),\mathbf{A}:A \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_rY^{\alpha_r} & \mathbf{B},\,(\text{-l};\alpha_1,\cdots,\alpha_r;1):B \end{array}\right) \tag{4.1}$$

under the same conditions that (3.1)

a) $\alpha_i > 0, i = 1,\cdots,r, b > 0, 2a(b+|b|)+c > 0$
b)$\left|arg(z_iX^{\alpha_i})\right| < \frac{1}{2}\Omega_i'\pi$ where

$$\Omega_i' = \sum_{k=1}^{n^{(i)}} A_k^{(i)}\alpha_k^{(i)} - \sum_{k=n^{(i)}+1}^{p^{(i)}} A_k^{(i)}\alpha_k^{(i)} + \sum_{k=1}^{m^{(i)}} B_k^{(i)}\beta_k^{(i)} - \sum_{k=m^{(i)}+1}^{q^{(i)}} B_k^{(i)}\beta_k^{(i)}$$

$$+\sum_{k=1}^{n} A_k^{(i)}\alpha_k^{(i)} - \sum_{k=n+1}^{p} A_k^{(i)}\alpha_k^{(i)} - \sum_{k=1}^{q} B_k^{(i)}\beta_k^{(i)} \tag{4.2}$$

c) $Re(l) + \sum_{i=1}^{r}\alpha_i \min_{1\leqslant j\leqslant m_i} Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right) > -\frac{1}{2}$

If $r = 2$, the multivariable I-function reduces to I-function of two variables defined by Kumari et al. [8], see [9-14,17]. Let

$$A = \{(a_i;\alpha_i,A_i;\mathbf{A}_i)\}_{1,p_1};A' = \{(e_i;E_i;\mathbf{E}_i)\}_{1,p_2},\{(g_i;G_i;\mathbf{G}_i)\}_{1,p_3} \tag{4.3}$$

$$B = \{(b_i;\beta_i,B_i;\mathbf{B}_i)\}_{1,q_1};B' = \{(f_i;F_i;\mathbf{F}_i)\}_{1,q_2},\{(h_i;H_i;\mathbf{H}_i)\}_{1,q_4} \tag{4.4}$$

We have,

Corollary 2.

$$\int_0^{\infty} X^{l+1}\,I_{p_1,q_1;p_2,q_2;p_3,q_3}^{0,n_1;m_2,n_2,m_3,n_3}(z_1X^{\alpha_1},z_2X^{\alpha_2})\mathrm{d}x = \frac{\Gamma\left(\frac{1}{2}\right)Y^{l+\frac{1}{2}}}{2a}$$

$$I_{p_1+1,q_1+1;p_2,q_2;p_3,q_3}^{0,n_1+1;m_2,n_2,m_3,n_3}\left(\begin{array}{c|c} \mathrm{z}_1Y^{\alpha_1} & \left(\frac{1}{2}-l;\alpha_1,\alpha_2;1\right),A:A' \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_2Y^{\alpha_2} & \mathrm{B},\,(\text{-l};\alpha_1,\alpha_2;1):B' \end{array}\right) \tag{4.5}$$

under the same conditions that (4.1) with $r = 2$.

If $r = 1$, the multivariable I-function reduces to I-function of one variable defined by Rathie [23].

Let

$A_1 = (a_j,\alpha_j:A_j)_{1,p};B_1 = (b_j,\beta_j:B_j)_{1,q}$

we obtain the following result

Corollary 3.

$$\int_0^{\infty} X^{l+1}\,I_{p,q}^{m,n}(zX^{\alpha})\mathrm{d}x = \frac{\Gamma\left(\frac{1}{2}\right)Y^{l+\frac{1}{2}}}{2a}\,I_{p+1,q+1}^{m,n+1}\left(\begin{array}{c|c} \mathrm{z}\mathrm{Y}^{\alpha} & \left(\frac{1}{2}-l;\alpha;1\right),A_1: \\ & \cdot \\ & \mathrm{B}_1,(-l;\alpha;1) \end{array}\right) \tag{4.6}$$

under the same conditions that (4.1) with $r=1$.

4. Multivariable Aleph-function.

If $n_2=\cdots=n_{r-1}=p_{i_2}=p_{i_3}=\cdots=p_{i_{r-1}}=0, m_2=\cdots=m_{r-1}=q_{i_2}=q_{i_3}=\cdots=q_{i_{r-1}}=0$, and $A_{2j}=A_{2ji_2}=B_{2ji_2}=\cdots=A_{rj}=A_{rji_r}=B_{rji_r}=1$, $A_{rj}=A_{rji_r}=B_{rji_r}=1$ then the multivariable Gimel function reduces to multivariable Aleph-function.

For convenience, we will use the following notations in this paper (see Ayant [1])

$$V=m_1,n_1;\cdots;m_r,n_r \tag{4.4}$$

$$\mathrm{W}=p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)},\cdots,p_{i^{(r)}},q_{i^{(r)}},\tau_{i^{(r)}};R^{(r)} \tag{4.5}$$

$$A=[(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)})]_{1,\mathfrak{n}},[\tau_i(a_{ji};\alpha_{ji}^{(1)},\cdots,\alpha_{ji}^{(r)})]_{\mathfrak{n}+1,p_i} \tag{4.6}$$

$$C=[(c_j^{(1)};\gamma_j^{(1)})]_{1,n_1},\ [\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)};\gamma_{ji^{(1)}}^{(1)})]_{n_1+1,p_{i^{(1)}}};\cdots;[(c_j^{(r)};\gamma_j^{(r)})]_{1,n_r},[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)};\gamma_{ji^{(r)}}^{(r)})]_{n_r+1,p_{i^{(r)}}} \tag{4.7}$$

$$B=[(b_j;\beta_j^{(1)},\cdots,\beta_j^{(r)})]_{1,m},[\tau_i(b_{ji};\beta_{ji}^{(1)},\cdots,\beta_{ji}^{(r)})]_{m+1,q_i} \tag{4.9}$$

$$D=[(d_j^{(1)};\delta_j^{(1)})]_{1,m_1},\tau_{i^{(1)}}[(d_{ji^{(1)}}^{(1)}];\delta_{ji^{(1)}}^{(1)})]_{m_1+1,q_{i^{(1)}}}\ ;\cdots;[(d_j^{(r)};\delta_j^{(r)})]_{1,m_r},[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)};\delta_{ji^{(r)}}^{(r)})]_{m_r+1,q_{i^{(r)}}} \tag{4.10}$$

We have the following result

Corollary 4.

$$\int_0^\infty X^{l+1}\aleph\left(z_1X^{\alpha_1},\cdots,z_rX^{\alpha_r}\right)\mathrm{d}x=\frac{\Gamma\left(\frac{1}{2}\right)Y^{l+\frac{1}{2}}}{2a}\aleph_{p_i+1,q_i+1,\tau_i;R:W}^{m,\mathfrak{n}+1:V}\left(\begin{array}{c|c} \mathrm{z}_1Y^{\alpha_1} & \left(\frac{1}{2}-l;\alpha_1,\cdots,\alpha_r\right),A:C\\ \cdot & \cdot\\ \cdot & \cdot\\ \cdot & \cdot\\ \mathrm{z}_rY^{\alpha_r} & \mathrm{B},\ (\text{-l};\alpha_1,\cdots,\alpha_r):D\end{array}\right) \tag{4.11}$$

a) $|arg(z_iX^{\alpha_i})|<\frac{1}{2}A_i'^{(k)}\pi$, where

$$A_i'^{(k)}=\sum_{j=1}^{\mathfrak{n}}\alpha_j^{(k)}-\tau_i\sum_{j=\mathfrak{n}+1}^{p_i}\alpha_{ji}^{(k)}-\tau_i\sum_{j=1}^{q_i}\beta_{ji}^{(k)}+\sum_{j=1}^{n_k}\gamma_j^{(k)}-\tau_{i^{(k)}}\sum_{j=n_k+1}^{p_{i^{(k)}}}\gamma_{ji^{(k)}}^{(k)}$$

$$+\sum_{j=1}^{m_k}\delta_j^{(k)}-\tau_{i^{(k)}}\sum_{j=m_k+1}^{q_{i^{(k)}}}\delta_{ji^{(k)}}^{(k)}>0,\ \text{with}\ k=1,\cdots,r,\ i=1,\cdots,R\ ,\ i^{(k)}=1,\cdots,R^{(k)} \tag{4.12}$$

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable Aleph-function.

b) $\alpha_i>0, i=1,\cdots,r, b>0, 2a(b+|b|)+c>0$

c) $Re(l)+\sum_{i=1}^r\alpha_i\min_{1\leqslant j\leqslant m_i}Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>-\frac{1}{2}$

Let

$$\mathrm{A}_2 = (a_j, \alpha_j, A_j)_{1,n}, [\tau_i(a_{ji}, \alpha_{ji}, A_{ji})]_{n+1,P_i}; \mathrm{B}_2 = [\tau_i(b_{ji}, \beta_{ji}, B_{ji})]_{1,Q_i} \tag{4.13}$$

$$C_2 = (c_j, \gamma_j)_{1n_1}, [\tau_{i'}(c_{ji'}, \gamma_{ji'})]_{n_1+1;P_{i'}}; (e_j, E_j)_{1n_2}, [\tau_{i''}(e_{ji''}, \gamma_{ji''})]_{n_2+1;P_{i''}} \tag{4.14}$$

$$\mathrm{D}_2 = (d_j, \delta_j)_{1m_1}, [\tau_{i'}(d_{ji'}, \delta_{ji'})]_{m_1+1;P_{i''}}; (f_j, F_j)_{1m_2}, [\tau_{i''}(f_{ji''}, F_{ji''})]_{m_2+1;Q_{i''}} \tag{4.15}$$

The multivariable Aleph-function reduces to Aleph-function of two variables defined by Kumar [7] and Sharma [25], we have

Corollary 5.

$$\int_0^{\infty} X^{l+1} \aleph\,(z_1 X^{\alpha_1}, z_2 X^{\alpha_2}) \mathrm{d}x = \frac{\Gamma\left(\frac{1}{2}\right) Y^{l+\frac{1}{2}}}{2a} \aleph^{0,n+1;m_1,n_1;m_2,n_2}_{P_i+1,Q_i+1,\tau_i;r;P_{i'},Q_{i'},\tau_{i'};r';P_{i''},Q_{i''},\tau_{i''};r''}$$

$$\left(\begin{array}{c|c} \mathrm{z}_1 Y^{\alpha_1} & \left(\frac{1}{2} - l; \alpha_1, \alpha_2\right), A_2 : C_2 \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_2 Y^{\alpha_2} & \mathrm{B}_2, (-l; \alpha_1, \alpha_2) : D_2 \end{array}\right) \tag{4.16}$$

a) Existence conditions of the integral (4.16) are

$$A_1 = \tau_i \sum_{j=1}^{P_i} \alpha_{ji} - \tau_i \sum_{j=1}^{Q_i} \beta_{ji} + \tau_{i'} \sum_{j=1}^{P_{i'}} \gamma_{ji} - \tau_{i'} \sum_{j=1}^{Q_{i'}} \delta_{ji'} < 0 \tag{4.17}$$

$$A_2 = \tau_i \sum_{j=1}^{P_i} A_{ji} - \tau_i \sum_{j=1}^{Q_i} B_{ji} + \tau_{i''} \sum_{j=1}^{P_{i''}} E_{ji''} - \tau_{i''} \sum_{j=1}^{Q_{i''}} F_{ji''} < 0 \tag{4.18}$$

b) The integral defined by (4.16) is converges absolutely, if

$$\varphi = \sum_{j=1}^{\mathbf{n}} \alpha_j - \tau_i \sum_{j=n+1}^{P_i} \alpha_{ji} - \tau_i \sum_{j=1}^{Q_i} \beta_{ji} + \sum_{j=1}^{n_1} \gamma_j - \tau_{i'} \sum_{j=n_1+1}^{P_{i'}} \gamma_{ji'} + \sum_{j=1}^{n_2} E_j - \tau_{i''} \sum_{j=n_2+1}^{P_{i''}} \gamma_{ji''} > 0 \tag{4.19}$$

$$\Lambda = \sum_{j=1}^{\mathbf{n}} A_j - \tau_i \sum_{j=n+1}^{P_i} A_{ji} - \tau_i \sum_{j=1}^{Q_i} B_{ji} + \sum_{j=1}^{m_1} \delta_j - \tau_{i'} \sum_{j=m_1+1}^{Q_{i'}} \delta_{ji'} + \sum_{j=1}^{m_2} F_j - \tau_{i''} \sum_{j=m_2+1}^{Q_{i''}} F_{ji''} > 0 \tag{4.20}$$

and

c) $|arg(z_1 X^{\alpha_1})| < \dfrac{1}{2}\pi\varphi$ and $|arg(z_2 X^{\alpha_2})| < \dfrac{1}{2}\pi\Lambda$

Let

$$\mathrm{A}'_2 = (a_j, \alpha_j, A_j)_{1,n}, [(a_{ji}, \alpha_{ji}, A_{ji})]_{n+1,P_i}; \mathrm{B}'_2 = [(b_{ji}, \beta_{ji}, B_{ji})]_{1,Q_i} \tag{4.21}$$

$$C'_2 = (c_j, \gamma_j)_{1n_1}, [(c_{ji'}, \gamma_{ji'})]_{n_1+1;P_{i'}}; (e_j, E_j)_{1n_2}, [(e_{ji''}, \gamma_{ji''})]_{n_2+1;P_{i''}} \tag{4.22}$$

$$\mathrm{D}'_2 = (d_j, \delta_j)_{1m_1}, [(d_{ji'}, \delta_{ji'})]_{m_1+1;_{i''}}; (f_j, F_j)_{1m_2}, [(f_{ji''}, F_{ji''})]_{m_2+1;Q_{i''}} \tag{4.23}$$

The Aleph-function of two variables reduces to I-function of two variables defined by Sharma and Mishra [27], we have

Corollary 6.

$$\int_0^\infty X^{l+1} I\left(z_1 X^{\alpha_1}, z_2 X^{\alpha_2}\right) \mathrm{d}x = \frac{\Gamma\left(\frac{1}{2}\right) Y^{l+\frac{1}{2}}}{2a} I^{0,n+1;m_1,n_1:m_2,n_2}_{P_i+1,Q_i+1r;P_{i'},Q_{i'};r';P_{i''},Q_{i''};r''} \left(\begin{array}{c|c} \mathrm{z}_1 Y^{\alpha_1} & \left(\frac{1}{2}-l;\alpha_1,\alpha_2\right), A'_2 : C'_2 \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_2 Y^{\alpha_2} & \mathrm{B}'_2, (-l;\alpha_1,\alpha_2) : D'_2 \end{array} \right) \quad (4.24)$$

under the same conditions that (4,16) with $\tau_i, \tau_{i'}, \tau_{i''} \to 1$.

The multivariable Aleph-function reduces to Aleph-function of one variable defined by Sudland [34].
Let

$$A = (a_j, A_j)_{1,n} \cdots [\tau_i(a_{ji}, A_{ji})]_{n+1,p_i} \quad (4.25)$$

$$B = (b_j, B_j)_{1,m} \cdots [\tau_i(b_{ji}, B_{ji})]_{m+1,q_i} \quad (4.26)$$

We have

Corollary 7.

$$\int_0^\infty X^{l+1} \aleph\left(z X^{\alpha}\right) \mathrm{d}x = \frac{\Gamma\left(\frac{1}{2}\right) Y^{l+\frac{1}{2}}}{2a} \aleph^{m,n+1}_{P_i+1,Q_i+1,\tau_i;r} \left(\begin{array}{c|c} \mathrm{z}Y^{\alpha} & \left(\frac{1}{2}-l;\alpha\right), A \\ & \cdot \\ & \mathrm{B},\ (\text{-l};\alpha) \end{array} \right) \quad (4.27)$$

under the same conditions that (4,11), with $r = 1$.

The aleph-function of one variable reduces to I-function of one variable defined by Saxena [24]. Let

$$A' = (a_j, A_j)_{1,n} \cdots [(a_{ji}, A_{ji})]_{n+1,p_i} \quad (4.28)$$

$$B' = (b_j, B_j)_{1,m} \cdots [(b_{ji}, B_{ji})]_{m+1,q_i} \quad (4.29)$$

We have

Corollary 8.

$$\int_0^\infty X^{l+1} I\left(z X^{\alpha}\right) \mathrm{d}x = \frac{\Gamma\left(\frac{1}{2}\right) Y^{l+\frac{1}{2}}}{2a} I^{m,n+1}_{P_i+1,Q_i+1;r} \left(\begin{array}{c|c} \mathrm{z}Y^{\alpha} & \left(\frac{1}{2}-l;\alpha\right), A' \\ & \cdot \\ & \mathrm{B'},\ (\text{-l};\alpha) \end{array} \right) \quad (4.30)$$

under the same conditions that (4,11), with $r = 1$ and $\tau_i \to 1$

Remarks
We have the same integrals with the H-function of two variables [6] and H-function of one variable.

If $A_{2j} = A_{2ji_2} = B_{2ji_2} = \cdots = A_{rj} = A_{rji_r} = B_{rji_r} = 1$ and $\tau_{i_2} = \cdots = \tau_{i_r} = \tau_{i^{(1)}} = \cdots = \tau_{i^{(r)}} = R_2 = \cdots = R_r = R_2 = \cdots = R_r = R^{(1)} = \cdots = R^{(r)} = 1$, then the multivariable Gimel-function reduces to multivariable I-function defined by Prasad [22]. We have

We note

$$A = (a_{2j}; \alpha'_{2j}, \alpha''_{2j})_{1,p_2}; \cdots ; (a_{(r-1)j}; \alpha'_{(r-1)j}, \cdots, \alpha^{r-1}_{(r-1)j})_{1,p_{r-1}} \quad (4.31)$$

$$B = (b_{2j}; \beta'_{2j}, \beta''_{2j})_{1,q_2}; \cdots ; (b_{(r-1)j}; \beta'_{(r-1)j}, \cdots, \beta^{r-1}_{(r-1)j})_{1,q_{r-1}} \quad (4.32)$$

$$\mathbf{A} = (\mathrm{a}_{rj}; \alpha'_{rj}, \cdots, \alpha^{(r)}_{rj})_{1,p_r} : \mathfrak{A} = (a'_j, \alpha'_j)_{1,p'}; \cdots ; (a^{(r)}_j, \alpha^{(r)}_j)_{1,p^{(r)}} \quad (4.33)$$

$$\mathbf{B} = (\mathrm{b}_{rj}; \beta'_{rj}, \cdots, \beta^{(r)}_{rj})_{1,q_r} : \mathfrak{B} = (b'_j, \beta'_j)_{1,q'}; \cdots ; (b^{(r)}_j, \beta^{(r)}_j)_{1,q^{(r)}} \quad (4.34)$$

$$X = p_2, q_2; p_3, q_3; \cdots ; p_{r-1}, q_{r-1}; U = m_2, n_2; m_3, n_3; \cdots ; m_{r-1}, n_{r-1} \tag{4.35}$$

$$Y = (p', q'); \cdots ; (p^{(r)}, q^{(r)}); V = (m', n'); \cdots ; (m^{(r)}, n^{(r)}) \tag{4.36}$$

Corollary 9.

$$\int_0^\infty X^{l+1} \mathrm{I}\,(z_1 X^{\alpha_1}, \cdots, z_r X^{\alpha_r})\mathrm{d}x = \frac{\Gamma\left(\frac{1}{2}\right) Y^{l+\frac{1}{2}}}{2a} \mathrm{I}^{V;0,n_r+1:X}_{U;p_r+1,q_r+1:Y} \left(\begin{array}{c|c} \mathrm{z}_1 Y^{\alpha_1} & \mathrm{A}; \left(\frac{1}{2} - l; \alpha_1, \cdots, \alpha_r\right), \mathbf{A} : \mathfrak{A} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_r Y^{\alpha_r} & \mathrm{B}; \mathbf{B},\ (\text{-l}; \alpha_1, \cdots, \alpha_r) : \mathfrak{B} \end{array} \right) \tag{4.31}$$

Provided that

a) $\alpha_i > 0, i = 1, \cdots, r, b > 0, 2a(b + |b|) + c > 0$

b) $Re(l) + \sum_{i=1}^{r} \alpha_i \min_{1 \leqslant j \leqslant m^{(i)}} Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right) > -\frac{1}{2}$

c) $|arg(z_i X^{\alpha_i})| < \frac{1}{2}\Omega_i'' \pi$ where

$$\Omega_i'' = \sum_{k=1}^{n^{(i)}} \alpha_k^{(i)} - \sum_{k=n^{(i)}+1}^{p^{(i)}} \alpha_k^{(i)} + \sum_{k=1}^{m^{(i)}} \beta_k^{(i)} - \sum_{k=m^{(i)}+1}^{q^{(i)}} \beta_k^{(i)} + \sum_{k=1}^{n_2} \alpha_{2k}^{(i)} - \sum_{k=n_2+1}^{p_2} \alpha_{2k}^{(i)} + \sum_{k=1}^{n_3} \alpha_{3k}^{(i)}$$

$$- \sum_{k=n_3+1}^{p_3} \alpha_{3k}^{(i)} + \cdots + \sum_{k=1}^{n_r} \alpha_{rk}^{(i)} - \sum_{k=n_r+1}^{p_r} \alpha_{rk}^{(i)} - \sum_{k=1}^{q_2} \beta_{2k}^{(i)} - \sum_{k=1}^{q_3} \beta_{3k}^{(i)} - \sum_{k=1}^{q_3} \beta_{3k}^{(i)} \cdots - \sum_{k=1}^{q_r} \beta_{rk}^{(i)} > 0 (i = 1, \cdots, r) \tag{4.32}$$

Now, the multivariable I-function defined by Prasad [15] reduces to multivariable H-function defined by Srivastava and Panda [31,32], we have $U = V = A = B = 0$ and

Corollary 10.

$$\int_0^\infty X^{l+1} H\,(z_1 X^{\alpha_1}, \cdots, z_r X^{\alpha_r})\mathrm{d}x = \frac{\Gamma\left(\frac{1}{2}\right) Y^{l+\frac{1}{2}}}{2a} \mathrm{H}^{0,n+1:X}_{p+1,q+1:Y} \left(\begin{array}{c|c} \mathrm{z}_1 Y^{\alpha_1} & \left(\frac{1}{2} - l; \alpha_1, \cdots, \alpha_r\right), \mathbf{A} : \mathfrak{A} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_r Y^{\alpha_r} & \mathbf{B},\ (\text{-l}; \alpha_1, \cdots, \alpha_r) : \mathfrak{B} \end{array} \right) \tag{4.33}$$

Provided that

a) $\alpha_i > 0, i = 1, \cdots, r, b > 0, 2a(b + |b|) + c > 0$

b) $Re(l) + \sum_{i=1}^{r} \alpha_i \min_{1 \leqslant j \leqslant m_i} Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right) > -\frac{1}{2}$

c) $|arg(z_i X^{\alpha_i})| < \frac{1}{2}\Omega_i'''' \pi$ where

$$\Omega_i'''' = \sum_{k=1}^{n^{(i)}} \alpha_k^{(i)} - \sum_{k=n^{(i)}+1}^{p^{(i)}} \alpha_k^{(i)} + \sum_{k=1}^{m^{(i)}} \beta_k^{(i)} - \sum_{k=m^{(i)}+1}^{q^{(i)}} \beta_k^{(i)} + \sum_{k=1}^{n} \alpha_k^{(i)} - \sum_{k=n+1}^{p} \alpha_k^{(i)} - \sum_{k=1}^{q} \beta_k^{(i)} > 0$$

Remarks.

We obtain the same results about the multivariable A-function defined by Gautam et al. [4], the Aleph-function of two variable defined by Sharma [24], Kumar [6], the multivariable I-function defined by Sharma and Ahmad [26], the modified multivariable H-function defined by Prasad and Singh [16], the I-function of two variables defined by Sharma and Mishra [27], the I-function of two variables defined by [8], see [9-14,17] , the Aleph-function of one variable [33], the I-function of one variable defined by Saxena [24], the I-function defined by Rathie [23], the H-function of two variable defined by Mittal and Goyal [6], the H-function.

5. Conclusion.

The importance of our all the results lies in their manifold generality. First, by specializing the various parameters as well as variables in multivariable Gimel-function, we obtain a large number of results involving remarkably wide variety of useful special functions (or product of such special functions) which are expressible in terms , Aleph-function of several variables, Aleph-function of two variables, Aleph-function, I-function defined by Rathie [23], I-function defined by Saxena [24], H-function, Meijer’s G-function, E-function, and hypergeometric function of one and several variables . Hence the formulae derived in this paper are most general in character and may prove to be useful in several interesting cases appearing in literature of Pure and Applied Mathematics and Mathematical Physics. Secondly, by specialising the parameters of the Boros integral, we obtain a large number integrals involving the special functions.

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