Applications of the Fourier-Wiener and Integral Transforms

Dr. Halema Zakaria Yahia Hussein

Department of Mathematics, Shaqra University, College of Science and Humanities, Kingdom of Saudi Arabia, Khartoum, Sudan

Abstract: *We shall study the existence and uniqueness and regularity of solutions for Cauchy problems associated with the following two equations:* $v_t = -R^k v$, $v_{tt} = -R^k v$; *and the elliptic type equation* $-R^k v = f (k > 1)$.*We applications of the Fourier-Wienertransform defined on a class of polynomial cylinder functions on Hilbert space and Wiener space.*

Keywords: *Wiener space,Wiener measure, Hilbert space, exponential type analytic functions ,equicontinuous semigroup of class*(C_0).

I. INTRODUCTION

We study the Fourier-Wiener transform was defined on a class of mean exponentialtype analytic functionals on the classical Wiener space $\mathcal{C}$and applications of the Fourier-Wienertransform are concerned we find that it is desirable to enlarge the domainof definition of the Fourier-Wiener transform as much as possible and define on theclass of exponential type analytic functions $\mathcal{E}_a$ on $[B]$-the complexification of B.

Let ε_a be the class of exponential type analytic functions defined on the complexification $[B]$of B. For each pair of nonzero complex numbers $\alpha, \beta$and $f \in \varepsilon_a$, we define

$$F_{\alpha,\beta} f(y) = \int_B f(\alpha x + \beta y)\, p_1(dx)(y \in [B]).$$

We show that the inverse $F_{\alpha,\beta}^{-1}$ exists and there exist two nonzero complex numbersα', β' such that $F_{\alpha,\beta}^{-1} = F_{\alpha',\beta'}$ where α', β' is a solution of $\beta\beta' = 1$ *and* $(\beta\alpha')^2 + \alpha^2 = 0$.

II. PRELIMINARIES

We shall apply the Fourier-Wiener transform [99] and integration by parts [100] to examine Cauchy problems associatedwith the following types of differential equations:

(i) $(,v_t(x,t) = -R^k v(x,t)$

(ii) $v_{tt}(x,t) = -R^k v(x,t)$, and also the elliptic type differential equation

(iii) $-R^k v(x) = \mathrm{g}(x)$where$k \geq 1$.

Theorem (1.2):For$f \in \mathcal{E}_a$, we have

$$\mathcal{F}_c^{-1}\mathcal{F}_c f(z) = f(z), \ \ z \in [B]. \tag{1.2}$$

Proof: (i) We claim that (1) holds for $f(z) = Tz^m, m = 1,2, \dots$, with T a symmetric operator in $[W_m]$.

In fact,

$$\mathcal{F}_c^{-1}\mathcal{F}_c(T(\cdot)^m)(z) = \int_B \int_B T(x + iy + z)^m p_c(dx) p_c(dy)$$

$$= Tz^m + \int_B \int_B \sum_{j=0}^{m-1} Tz^j (x + iy)^{m-j} p_c(dx) p_c(dy)$$

Let $\int_B \int_B \sum_{j=0}^{m-1} Tz^j(x+iy)^{m-j} p_c(dx)p_c(dy)$=0, by Lemma (4.1.12)[93]

$$\mathcal{F}_c^{-1}\mathcal{F}_c(T(\cdot)^m)(z) == Tz^m + 0$$

$$= Tz^m$$

(ii): By Proposition (4.1.8)[93], we have

$$f(z) = \sum_{m=0}^{\infty} (1/m\,!)D^m f(0)z^m.$$

Consequently,

$$\mathcal{F}_c f(z) = \sum_{m=0}^{\infty} (1/m\,!)\mathcal{F}_c(D^m f(0)(\cdot)^m)(z),$$

by Lebesgue's dominated convergence theorem.

Finally, by Step (i) and the dominated convergence theorem again,

$$\mathcal{F}_c^{-1}\mathcal{F}_c f(z) = \sum_{m=0}^{\infty} (1/m\,!)\mathcal{F}_c^{-1}\mathcal{F}_c(D^m f(0)(\cdot)^m)(z)$$

$$= \sum_{m=0}^{\infty} (1/m\,!)D^n f(0)z^m = f(z).$$

Theorem (2.2): There is a unique solution $u(y,t)$ for the Cauchy problem $u_t(y,t) = -\widetilde{\mathfrak{N}}^k u(y,t);\quad u(y,0) = \mathcal{F}_1 f(y)(y \in B, t > 0)$ such that (i) $u(\cdot,t) \in \tilde{\mathcal{E}}_a$ for each $t \geq 0$, (ii) $u(y,t)$ is (strongly) differentiable in t (i.e. $(\partial/\partial t)u(y,t)$ exists uniformly on bounded sets). Moreover, the solution is given by $u(y,t) = T_t(\mathcal{F}_1 f)(y)$.

Proof: Let $g(y,s)$ be any solution for $u_t(y,t) = -\tilde{R}^k u(y,t);\quad u(y,0) = \mathcal{F}_1 f(y)(y \in B, t > 0)$ such that (i) and (ii) hold.

Fix $t > 0$. Set $F(y,s) = T_{t-s}g(y,s)$for $t \geq s \geq 0$, $y \in [B]$. Since $g(\cdot,s) \in \tilde{\mathcal{E}}_a$, the domain of $-\tilde{R}^k$, it follows by [112, Chapter 9, Theorem 2] that $F(y,s)$ is strongly differentiable and

$$(\partial/\partial s)F(y,s) = \tilde{R}^k T_{t-s}g(y,s) + T_{t-s}(\partial/\partial s)g(y,s)$$

(since T_t, is an equicontinuous semigroup and g satisfies (ii))

$$= T_{t-s}\tilde{R}^k g(y,s) - T_{t-s}\tilde{R}^k g(y,s)$$

$$= 0,\quad y - \text{ uniformly on bounded sets.}$$

This implies that $F(y,s) = C$ (a constant) on $[B]$for $0 \leq s \leq t$. In particular, $F(y,t) = F(y,0) = T_t g(y,0) = T_t(\mathcal{F}_1 f)(y)$. On the other hand, $F(y,t) = T_0 g(y,t) = g(y,t)$. We conclude that $g(y,t) = T_t(\mathcal{F}_1 f)(y)$.

Proposition (2.3):$\{T_t\}$ is an equicontinuous semigroup of class (C_0) [112, Chapter 9].

Proof: Obviously,

$$T_{t+s} = T_t T_s \text{and} T_0 = I. \tag{2.2}$$

By the mean value theorem, we have (for $t_0 \geq 0$)

$$\left\|T_t f - T_{t_0} f\right\|_N \leq \sum_{n=0}^{\infty} (1/m!)\left|e^{-m^k t} - e^{-m^k t_0}\right| \|D^m f(0)x^m\|_M$$

$$\leq |t-t_o| \|f\|_{2M} \left(\sum_{n=1}^{\infty} (n^k/2^m) \right),$$

by$(1/m!)|D^m f(0) x^m| \leq 2^{-m} \|f\|_{2M}$ (3.2)

Thus

$$\lim_{t \to t_0} T_t f = T_{t_0} f \quad \text{for} t_0 \geq 0. \qquad (4.2)$$

Finally, suppose q is any continuous seminorm on $\tilde{\mathcal{E}}_a$. Then there exist N and a constant C_1 such that $q(f) \leq C_1 \|f\|_N$.

Consequently, $q(T_t f) \leq C_1 \|T_t f\|_N \leq 2C_1 \|f\|_{2N}$,, i.e. $\{T_t\}$ is equicontinuous.

III. CLAIMS

Theorem (1.3):

(i)In order that equation $-R^k u(x) = f(x)$ has a solution in $\mathcal{E}_a$, it is necessary andsufficient that $f \in \mathcal{E}_a$ and $\mathcal{F}_1 f(0) = 0$.

(ii)The solution of $-R^k u(x) = f(x)$ is unique in the sensethat if $W_1(x), W_2(x) \in \mathcal{E}_a$ and each solves $-R^k u(x) = f(x)$then $W_1 - W_2$ is a constant.

Proof:

(i)a. Necessity. Since $R(\mathcal{E}_a) \subset \mathcal{E}_a$ (Lemma (4.1.17))[93], if $u(x)$ is a solution in$\mathcal{E}_a$then $R^k u(x) \in \mathcal{E}_a$ so does f.

Next, applying the Fourier-Wiener transform to $-R^k u(x) = f(x)$, we get

$$-\tilde{R}^k (\mathcal{F}_1 u)(y) = \mathcal{F}_1 f(y). \qquad (1.3)$$

Therefore$\mathcal{F}_1 f(0) = -\tilde{R}^k (\mathcal{F}_1 u)(0) = 0$.

b. Sufficiency (existence). Suppose $f \in \mathcal{E}_a$and $\mathcal{F}_1 f(0) = 0$. We let $\mathrm{g}(y) = \mathcal{F}_1 f(y)$. Since $\mathrm{g} \in \mathcal{E}_a$, $\mathrm{g}(y) = \sum_{m=1}^{\infty} (1/m!) D^m \mathrm{g}(0) y^m$ (since $\mathrm{g}(0) = 0$). It is easy to see that

$$v(y) = \mathcal{F}_1 u(y) = v(0) - \sum_{m=1}^{\infty} (1/m^k m!) D^m \mathrm{g}(0) y^m$$

solves equation (1.3). Consequently,

$$u(x) = \mathcal{F}_1^{-1} v(x) = v(0) - \mathcal{F}_1^{-1} \left(\sum_{m=1}^{\infty} (1/m^k m!) D^m \mathrm{g}(0) y^m \right)(x) \qquad (2.3)$$

solves $-R^k u(x) = f(x)$.

(ii) Uniqueness. Observe that if $h \in \mathcal{E}_a$, then $\tilde{R}h = 0$ iff $h \equiv a$ const. It followsthat if W_1, W_2 are two solutions for $-R^k u(x) = f(x)$ such that $W_1, W_2 \in \mathcal{E}_a$ and we let$H = W_1 - W_2$, then we must have $\Re H = 0$; hence $\tilde{R}(\mathcal{F}_1 H) \equiv 0$.

According to our observation $\mathcal{F}_1 H \equiv C_1$, a constant. Therefore$H \equiv \mathcal{F}_1^{-1}(C_1) \equiv C_1$.

Theorem (2.3)*:* Let $\alpha, \beta, \alpha', \beta'$ be nonzero complex numbers. Then in order that $\mathrm{F}_{\alpha',\beta'}$ $(\mathrm{F}_{\alpha,\beta}\ f)(z) = f(z)$ for all f in E_a it is necessary and sufficient that

$$\beta\beta' = 1 \ \ and \ \ (\beta\alpha')^2 + \alpha^2 = 0. \qquad (3.3)$$

Proof: By Corollary (4.2.6) [114], we see that,forany$f \in \mathrm{E}_a$,

$$\mathrm{F}_{\alpha',\beta'}\ (\mathrm{F}_{\alpha,\beta}\ f)(\mathrm{z}) = \int_B \int_B f\big(\alpha x + \beta(\alpha' y + \beta' z)\big) p_1(dx) p_1(dy)$$

$$= \int_B f\left(\beta\beta' z + \sqrt{\alpha^2 + (\beta\alpha')^2}\, y\right) p_1(dy). \quad (4.3)$$

$$= \int_B f(z) p_1(dy).$$

Now, it becomes obvious that the theorem follows immediately from (4.3).

IV.CONCLUION

There the Fourier-Wiener transform was defined on a class of mean exponentialtype analytic functionals on the classical Wiener space $\mathcal{C}$.

As for as theapplications of the Fourier-Wienertransform are concerned we find that it is desirable to enlarge the domainof definition of the Fourier-Wiener transform as much as possible(so that it atleast also contains a large class of functions other than cylinder functions).

It was shown that the Fourier-Wiener transform is a useful tool for solving differential equations on infinite dimensional spaces.

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