Exponential Transform and its Properties

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Abstract—***In this paper we have introduced the new concept of finite hyperbolic transforms. Transform of some standard functions are obtained and some properties are proved.***

Keywords— ***Generalized Transform, Finite transform, Finite hyperbolic transform, Transform of some standard functions.***

I. Introduction

If the disturbance is $f(t)=e^{at2}$, for $a>0,$ the usual Laplace transform cannot be used to find the solution of an initial value problem because Laplace transform of f does not exist. It is often true that the solution at times later than t would not affect the state at time t. This leads to define Finite Laplace transform.

The finite Laplace transform of a continuous or an almost piecewise continuous function f in (0, T) is denoted by $L_T(f(t))=F(p,T),$

$$L_T(f(t))=F(p,T)=\int_0^T f(t)e^{-pt}dt \tag{1.1}$$

Where p is a real or complex number and T be a finite number which may be positive or negative.

Note : Above definition is defined for any bounded interval $(-T_1,T_2)$.

Finite Laplace transform motivate us to define Finite Sine Hyperbolic transform and RAM Finite Cosine Hyperbolic transform in $0\le t\le T$ in order to extend the power and usefulness of usual Laplace transform in $0\le t<\infty$. section 2 devotes some preliminaries containing some definitions and properties of finite sine hyperbolic transform In section 3.1 shifting properties of Finite Sine Hyperbolic Transform are obtained and In section 3.2 examples are given.

II. Exponential Transform

Definition *:* Let $f(t)$ be function defined for all positive values of t, then

$$\overline{f}(s)=\int_o^\infty a^{-st}f(t)dt, \quad a>1$$

Provided the integral exists is called exponential Transform of $f(t)$. It is denoted as

$$A[f(t)]=\overline{f}(s)=\int_o^\infty a^{-st}f(t)dt, \quad a>1$$

here A is called exponential Transformation operator The parameter s is real or complex number.

In general the parameter s is taken to be a real positive number.

Theorem [Existence of Exponential Transform]

If $f(t)$ is a function of class A. Then Exponential Transform of $f(t)$ exists

or

suppose $f(t)$ is piece-wise continuous in every finite interval and is of exponential order k as $t \to \infty$ Then $f(s)$ exists for all $s \log a > k$ that is exponential transform exists.

Proof : Let $f(t)$ be piece wise continuous function in every finite interval and of exponential order k as $t \to \infty$ To show that $\overline{f}(s)$ exists $\forall s \log a > k$

Let $t_0 > 0$ then

$$A[f(t)] = \overline{f}(s) = \int_o^{\infty} a^{-st} f(t)dt$$

$$= \overline{f}(s) = \int_o^{t_0} a^{-st} f(t)dt + \int_{t_0}^{\infty} a^{-st} f(t)dt$$

continuity of $f(t)$ in the finite interval (o, t_0) implies that $\int_o^{t_0} a^{-st} f(t)$ exists

It remains to show that $\int_{t_0}^{\infty} a^{-st} f(t)dt$ exists $\forall s \log a > k$

$f(t)$ is of exponential order k as $t \to \infty$ implies $\lim_{t \to \infty} e^{-kt} f(t)$ is finite

i.e. given a number t_0 there exists a real number $M > 0$

such that $\left|e^{-kt} f(t)\right| < M \quad \forall t \geq t_0$

$$\left|f(t)\right| < Me^{kt} \quad \forall t \geq t_0$$

Now

$$\left|\int_{t_0}^{\infty} a^{-st} f(t)dt\right| \leq \int_{t_0}^{\infty} \left|a^{-st} f(t)\right|dt$$

$$= \int_{t_0}^{\infty} a^{-st} \left|f(t)\right|dt$$

$$\leq \int_{t_0}^{\infty} a^{-st} Me^{kt} dt$$

$$= \int_{t_0}^{\infty} e^{-st \log a} Me^{kt} dt$$

$$= M \int_{t_0}^{\infty} e^{-(st \log a - k)t} dt$$

$$= \frac{Me^{-(st \log a - k)t_0}}{s \log a - k} \text{ if } s \log a > k$$

Finally

$$\left|\int_{t_0}^{\infty} a^{-st} f(t)dt\right| \le \frac{Me^{-(s\log a-k)t_0}}{s\log a-k} \text{ if } s\log a > k$$

$= \dfrac{Me^{-(st\log a-k)t_0}}{s\log a-k}$ can be made as small as we please choosing t_0 sufficiently largehence

$\int_{t_0}^{\infty} a^{-st} f(t)dt$ exists $\forall s\log a > k$

Remark **:** The conditions given in the Theorem are sufficient for existence of A[f(t)] but are not the necessary conditions.

Exponential Transform of some functions

I) $f(t)=1$

By exponential Transform

$$A[1] = \int_o^{\infty} a^{-st}.1dt = \int_o^{\infty} a^{-st}dt$$

$$= \int_o^{\infty} e^{-st\log a}dt$$

$$= \left[\frac{e^{-st\log a}}{s\log a}\right]_0^{\infty} = \frac{1}{s\log a}\left[\frac{1}{e^{st\log a}}\right]_0^{\infty}$$

$$= \frac{1}{-s\log a}[0-1] = \frac{1}{s\log a}$$

$$A[1] = \frac{1}{s\log a}, a > 1$$

II) $f(t) = t^n$

By Exponential Transform

$$A[f(t)] = A[t^n] = \int_0^{\infty} a^{-st}t^n dt$$

$$A[t^n] = \int_0^{\infty} e^{-st\log a}t^n dt$$

Put $St\log a = x$

$$t = \frac{x}{s\log a}$$

$$\therefore dt = \frac{dx}{s\log a}$$

$$\therefore A[t^n] = \int_0^{\infty} e^{-x}\left(\frac{x}{s\log a}\right)^n \frac{dx}{s\log a}$$

$$= \int_0^{\infty} \frac{e^{-x}x^n}{(s\log a)^{n+1}}dx = \frac{1}{(s\log a)^{n+1}}\int_0^{\infty} e^{-x}x^n dx$$

$$= \frac{n!}{(s\log a)^{n+1}} \qquad \left\{ \because \frac{\Gamma(n+1)}{\Gamma(n+1)} = \int_0^\infty e^{-x}x^n dx \text{ and } = n! \right.$$

$$\therefore A[t^n] = \frac{n!}{[s\log a]^{n+1}}, a > 1$$

III) $f(t) = e^{kt}$

$$\therefore A[f(t)] = A[e^{kt}] = \int_0^\infty a^{-st}e^{kt}dt$$

$$= \int_0^\infty e^{-st\log a}e^{kt}dt$$

$$= \int_0^\infty e^{(-s\log a+k)t}dt$$

$$= \int_0^\infty e^{-(s\log a-k)t}dt$$

$$= \left[\frac{e^{-(s\log a-k)t}}{-(s\log a-k)}\right]_0^\infty$$

$$= \frac{1}{(s\log a-k)}\left[\frac{1}{e^{(s\log a-k)t}}\right]_0^\infty$$

$$= \frac{1}{-(s\log a-k)}[0-1]$$

$$\therefore A[e^{kt}] = \frac{1}{s\log a-k}, a>1$$

IV) $f(t) = \cosh kt$

$$\therefore A[f(t)] = A[\cosh kt] = \int_0^\infty a^{-st}\cosh ktdt$$

$$\therefore A[\cosh kt] = A[\frac{e^{kt}+e^{-kt}}{2}] \qquad \left(\because \cosh kt = \frac{e^{kt}+e^{-kt}}{2}\right)$$

$$= \frac{1}{2}A[e^{kt}] + \frac{1}{2}A[e^{-kt}]$$

$$= \frac{1}{2}\left[\frac{1}{s\log a-k} + \frac{1}{s\log a+k}\right]$$

$$= \frac{1}{2}\left[\frac{s\log ak + s\log a - k}{(s\log a)^2 - k^2}\right]$$

$$= \frac{1}{2}\left[\frac{2s\log a}{(s\log a)^2 - k^2}\right]$$

$$\therefore A[\cosh kt] = \frac{s\log a}{(s\log a)^2 - k^2}, a > 1, (s\log a)^2 > k^2$$

V) $f(t) = \sinh kt$

$$\therefore A[\sinh kt] = \int_0^\infty a^{-st} \sinh ktdt$$

$$\therefore A[\sinh kt] = A[\frac{e^{kt} - e^{-kt}}{2}] \qquad \left(\because \sinh kt = \frac{e^{kt} - e^{-kt}}{2}\right)$$

$$= \frac{1}{2}A[e^{kt}] - \frac{1}{2}A[e^{-kt}]$$

$$= \frac{1}{2}\left[\frac{1}{s\log a - k} - \frac{1}{s\log a + k}\right]$$

$$= \frac{1}{2}\left[\frac{s\log a + k - s\log a + k}{(s\log a)^2 - k^2}\right]$$

$$= \frac{1}{2}\left[\frac{2k}{(s\log a)^2 - k^2}\right]$$

$$\therefore A[\sinh kt] = \frac{k}{(s\log a)^2 - k^2}, a > 1, (s\log a)^2 > k^2$$

VI) $f(t) = \sin kt$

$$\therefore A[f(t)] = A[\sin kt] = \int_0^\infty a^{-st} \sin ktdt$$

$$\therefore A[\sin kt] = A[\frac{e^{ikt} - e^{-ikt}}{2i}] \qquad \left(\because \sin kt = \frac{e^{ikt} - e^{-ikt}}{2i}\right)$$

$$= \frac{1}{2i}\left[A[e^{ikt}] - A[e^{-ikt}]\right]$$

$$= \frac{1}{2i}\left[\frac{1}{s\log a - ik} - \frac{1}{s\log a + ik}\right]$$

$$= \frac{1}{2i}\left[\frac{s\log a + ik - s\log a + ik}{(s\log a)^2 - (ik)^2}\right]$$

$$= \frac{1}{2i}\left[\frac{2ik}{(s\log a)^2 - (ik)^2}\right]$$

$$\therefore A[\sin kt] = \frac{k}{(s\log a)^2 + k^2}$$

VII) $f(t) = \cos kt$

$$\therefore A[f(t)] = A[\cos kt] = \int_0^\infty a^{-st} \cos kt dt$$

$$\therefore A[\cos kt] = A[\frac{e^{ikt} + e^{-ikt}}{2}] \qquad \left[\because \cos kt = \frac{e^{ikt} + e^{-ikt}}{2}\right]$$

$$= \frac{1}{2} A\left[e^{ikt}\right] + \frac{1}{2} A\left[e^{-ikt}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s\log a - ik} + \frac{1}{s\log a + ik}\right]$$

$$= \frac{1}{2}\left[\frac{s\log a + ik + s\log a - ik}{(s\log a)^2 - (ik)^2}\right]$$

$$= \frac{1}{2}\left[\frac{2s\log a}{(s\log a)^2 - k^2}\right]$$

$$\therefore A[\cos kt] = \frac{s\log a}{(s\log a)^2 + k^2}$$

Properties of Exponential Transform

I) Linear Property

$$A[k_1 f_1(t) + k_2 f_2(t)] = k_1 A[f_1(t)] + k_2 A[f_2(t)]$$

Proof : $A[k_1 f_1(t) + k_2 f_2(t)] = \int_0^\infty a^{-st}[k_1 f_1(t) + k_2 f_2(t)]dt$

$$= \int_0^\infty [k_1 a^{-st} f_1(t) + k_2 a^{-st} f_2(t)]dt$$

$$= k_1 \int_0^\infty a^{-st} f_1(t)dt + k_2 \int_0^\infty a^{-st} f_2(t)dt$$

$$= k_1 A[f_1(t)] + k_2 A[f_2(t)]$$

$$A[k_1 f_1(t) + k_2 f_2(t)] = k_1 A[f_1(t)] + k_2 A[f_2(t)]$$

II) Shifting Property

If $A[f(t)] = \overline{f}(s)$, then $A[e^{kt} f(t)] = \overline{f}(s - \frac{k}{\log a})$

Proof : $A[e^{kt} f(t)] = \int_0^\infty a^{-st} e^{kt} f(t)dt$

$$= \int_0^\infty e^{-st\log a} e^{kt} f(t)dt$$

$$= \int_0^\infty e^{(-s\log a + k)t} f(t)dt$$

$$= \int_0^\infty e^{-(s\log a - k)t} f(t)dt$$

$$= \int_0^\infty e^{-\log a(s-\frac{k}{\log a})t} f(t)dt$$

$$= \int_0^\infty a^{-(s-\frac{k}{\log a})t} f(t)dt$$

$$= \bar{f}(s - \frac{k}{\log a})$$

$$A[e^{kt} f(t)] = \bar{f}(s - \frac{k}{\log a})$$

Remark : With The help of First Shifting Theorem, we can have
The Following Important Results

I) $A[e^{kt}t^n] = \dfrac{n!}{(s - \frac{k}{\log a})^{n+1}}$

II) $A[e^{kt}\cosh bt] = \dfrac{s - \frac{k}{\log a}}{\left(s - \frac{k}{\log a}\right)^2 - b^2}$ where $a > 1, \left(s - \frac{k}{\log a}\right)^2 > b^2$

III) $A[e^{kt}\sinh bt] = \dfrac{b}{\left(s - \frac{k}{\log a}\right) - b^2}$ where $a > 1, \left(s - \frac{k}{\log a}\right) > b^2$

IV) $A[e^{kt}\sin bt] = \dfrac{b}{\left(s - \frac{k}{\log a}\right)^2 + b^2}$ where $a > 1, \left(s - \frac{k}{\log a}\right)^2 > b^2$

V) $A[e^{kt}\cos bt] = \dfrac{s - \frac{k}{\log a}}{\left(s - \frac{k}{\log a}\right)^2 + b^2}$

III) Change of Scale Property

If $A[f(t)] = \overline{f}(s)$ then $A[f(kt)] = \frac{1}{k}\overline{f}\left(\frac{s}{k}\right)$

Proof :

$$A[f(t)] = \overline{f}(s) \text{ then}$$

$$A[f(kt)] = \int_0^\infty a^{-st} f(kt)dt$$

$$= \int_0^\infty e^{-st\log a} f(kt)dt$$

Put $kt = x,\ t = \frac{x}{k} \therefore dt = \frac{dx}{k}$

$$= \int_0^\infty e^{\frac{-sx\log a}{k}} f(x)\frac{dx}{k}$$

$$= \frac{1}{k}\int_0^\infty e^{\frac{-sx\log a}{k}} f(x)dx$$

$$= \frac{1}{k}\int_0^\infty e^{\frac{-sx\log a}{k}} f(t)dt$$

Put $\frac{s}{k} = p$

$$= \frac{1}{k}\int_0^\infty e^{-pt\log a} f(t)dt$$

$$= \frac{1}{k}\int_0^\infty a^{-pt} f(t)dt$$

$$= \frac{1}{k} f(p)$$

$$\therefore A[f(kt) = \frac{1}{k}\overline{f}\left(\frac{s}{k}\right)$$

IV) Second Shifting Theorem

If $A[f(t)] = \overline{f}(s)$ and $G(t) = \begin{cases} F(t-k), & t > k \\ 0, & t < k \end{cases}$

Then

$$A[G(t)] = a^{-ks}\overline{f}(s)$$

Proof : $A[f(t)] = \overline{f}(s)$

$$G(t) = \begin{cases} F(t-k), if & t > k \\ 0, \quad if & t < k \end{cases}$$

$$A[G(t)] = \int_0^\infty a^{-st} G(t)dt$$

$$= \int_0^\infty e^{-st\log a} G(t)dt$$

$$= \int_0^k e^{-st\log a} G(t)dt + \int_k^\infty e^{-st\log a} G(t)dt$$

$$= \int_0^k e^{-st\log a} 0dt + \int_k^\infty e^{-st\log a} G(t)dt$$

$$= 0 + \int_k^\infty e^{-st\log a} F(t-k)dt$$

$$A[G(t)] = \int_k^\infty e^{-st\log a} F(t-k)dt \text{ ------------------(1)}$$

Put $t - k = x$

$\therefore dt = dx$

When $t = k, \quad x = 0$

$t = \infty, \quad x = \infty$

$\therefore$ equation (1) becomes

$$A[G(t)] = \int_0^\infty e^{-s(x+k)\log a} f(x)dx$$

$$= e^{-sk\log a} \int_0^\infty e^{-sx\log a} f(x)dx$$

$$= a^{-sk} \int_0^\infty a^{-st} f(t)dt$$

$$A[G(t)] = a^{-sk} \overline{f}(s)$$

$$A[G(t)] = a^{-ks} \overline{f}(s)$$

Remark :Second Shifting Theorem can also be stated as

If $\overline{f}(s)$ is exponential Transform of $f(t)$ and $k > 0$, then $a^{-ks}\overline{f}(s)$ is the exponential transform of $F(t-k)H(t-k)$, where

$$H(t) = \begin{cases} 1 & if \quad t > 0 \\ 0 & if \quad t < 0 \end{cases}$$

III. Discussion and Conclusions

As like Laplace Transform we observe that ; linearity , Shifting , Change of scale , Second shifting Properties also satisfied by using newly defined Exponential Transform

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