On Πgβ-Normal And Πgβ- Regular Spaces

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Abstract - We use the notion of πgβ-closed sets and use it to obtain some new characterizations of πgβ-normal spaces introduced by kumar,chand and rajhbar.We also introduced πgβ-regular spaces and obtained its basic properties.

1. Inroduction

Kumar,Chand and Rajbhar[7] used the notion of πgβ-closed sets to obtain a new characterization of πgβ-normal spaces and obtained some preservation theorems for πgβ-normal spaces. In this paper, we study πgβ-normal spaces further and obtained some new results. Also,
we introduce πgβ-regular spaces and obtain some of its basic properties.

2. Preliminaries

Throughout the present paper, X and Y denote topological spaces. Let A be a subset of X. We denote the interior and closure of A by Int(A) and Cl(A) respectively.
A subset A of a topological space X is said to be β-open [1] or semi-preopen [3] (if A ⊆Cl(Int(Cl(A))). The complement of β-open set is β-closed. The intersection of all β-closed sets containing A is called β-closure [2]) of A and is denoted by βCl(A). Further A is said to be regular open if A=Int(Cl(A)) and it is said to be regular closed if A=Cl(Int(A)).It is said to be π-open[12] if it is finite union of regular open sets.

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Also, A is said to be generalized semi-preclosed[4](briefly.gsp-closed) or gβ-closed (resp.πgβ-closed[11]) if βCl(A) ⊆U, whenever A⊆U and U is open(resp π-open) in X. The complement of πgβ-closed set is πgβ-open

Definition 2.1 [10**].** A space X is said to be gβ-normal if for any two disjoint gβ-closed sets A and B of X, there exists two disjoint β-open sets U and V such that A⊆U and B⊆V.

Definition 2.2 [5**].** A space X is said to be strongly β-normal if for any two disjoint β-closed sets A and B of X, there exists two disjoint β-open sets U and V such that A⊆U and B⊆V.

Definition 2.3 [8**].** A space X is said to be β-normal if for any two disjoint closed sets A and B of X, there exists two disjoint β-open sets U and V such that A⊆U and B⊆V.

Definition 2.4 [9]. A subset A of space X is said to be β-clopen or semi-pre regular if its β-open as well as β-closed.

Definition 2.5. A subset A of space X is said to be locally discrete if
every open set is closed.

Definition 2.6. A space X is said to be weakly πgβ-normal if disjoint
πgβ-closed set can be separated by disjoint closed sets.

Definition 2.7. A function f:X→Y is said to be
(1). contra-β-continuous[6] if $f^{-1}(F)$ is β-open in X for each closed set F of Y.
(2). always πgβ-closed if the image of each πgβ-closed set in X is πgβ-closed set in Y.

3. πgβ-Normal Topological Spaces

In this section, we study more of πgβ-normal spaces and study some of its properties.

Definition 3.1. A space X is said to be πgβ-normal[7] if for any two disjoint πgβ-closed sets A and B of X, there exists two disjoint β-open sets U and V such that $A\subseteq U$ and $B\subseteq V$.

From above definition, its clear that ;

πgβ-normal→gβ-normal→strongly β-normal→ β-normal

None of the implications is reversible as can be seen from the following examples:

Example 3.1. Let X= {a,b,c,d} and let the topology on X be T={φ, X, {b},{d},{b,d},{a,b,c}}.Then (X,T) is normal, hence β-normal but not strongly β-normal since {c,d} and {a} are the pair of disjoint β-closed sets and there is no pair of disjoint β-open sets containing {c,d} and {a}.
Example 3.2. Let X= {a,b,c,d} and let the topology on X be T={φ, X, ,{b,d},{a,b,d},{b,c,d}}.Then (X,T) is β-normal, hence gβ-normal but not πgβ-normal since {b,d} and {a,c} are the pair of disjoint πgβ-closed sets and there is no pair of disjoint β-open sets containing them.

Example 3.3. Let Y and Z be disjoint finite sets and let $X=Y\cup Z$. Then T= {φ, X, Y, Z} is a topology on X. Clearly (X,T) is locally discrete and hence every β-closed set is clopen. Hence (X,T) is strongly β-normal. If A is non empty subset of Y and $B\subseteq Y\text{-}A$, then A is gβ-closed, but A and B cannot be separated by disjoint β-open sets.
Hence (X,T) is not gβ-normal.

Definition 3.2. A space X is said to be πgβ-regular if for every πgβ-closed subset F of X and a point x not in F, there exist disjoint β-open subsets U and V such that $x\in U$ and $F\subseteq V$.

A πgβ-normal space need not be πgβ-regular, as the following example shows:

Example 3.4. Let X= {a,b,c,d} and let the topology on X be T={φ, X,{a,b}}.Then (X,T) is πgβ-normal but not πgβ-regular since {a,b,c} is πgβ-closed but for d not in {a,b,c},there does not exist any pair of disjoint β-open sets containing them.

Definition 3.3. A topological space (X,T) is said to be (β, πgβ)-R_0 if
$\beta Cl(\{x\})\subseteq U$ whenever U is πgβ-open and $x\in U$.

Theorem 3.1. Every πgβ-normal (β, πgβ)-R_0 space is πgβ-regular

Proof. Let F be a πgβ-closed subset of X and $x\in X$ such that x is not in F.Then $x\in X\text{-}F$, where X-F is a πgβ-open set in X.Since X is a (β, πgβ)-R_0 space, we have $\beta Cl(\{x\})\subseteq X\text{-}F$.Thus F and βCl({x}) are disjoint πgβ-closed sets in X.By πgβ-normality of X, there exist disjoint β-open subsets U and V such that $F\subseteq U$ and $\beta Cl(\{x\})\subseteq V$.Therefore, , there exist disjoint β-open subsets U and V such that $x\in U$ and $F\subseteq V$. Hence X is a πgβ-regular.

Following is some new characterization of πgβ-normality:

Theorem. **3.2.** The following properties are equivalent for a space X:
(i). X is πgβ-normal.
(ii).For every πgβ-closed set A and every πgβ-open set B containing A, there is a β-clopen set V such that $A\subseteq V\subseteq B$.

Proof.(i)→(ii). Let A be πgβ-closed set and U be πgβ-open set with $A\subseteq U$. Now we have $A\cap(X\text{-}U)=\emptyset$, hence there exists disjoint β-open sets W_1 and W_2 such that $A\subseteq W_1$ and $X\text{-}U\subseteq W_2$. If $V=\beta Cl(W_1)$, then V is a β-clopen set satisfying $A\subseteq V\subseteq U$.

(ii) → (i). Obviuos.

Theorem 3.3. If f:(X,τ)→(Y,σ) is an injective always πgβ-closed function and (Y,σ) is weakly πgβ-normal, then (X,τ) is πgβ-normal.

Proof. Suppose that A_1, A_2 ⊆X are πgβ-closed and disjoint. Since f is always πgβ-closed and injective, f(A_1), f(A_2)⊆Y are πgβ-closed and disjoint. Since (Y,σ) is weakly πgβ-normal, f(A_1) and f(A_2) can be separated by disjoint closed sets B_1, B_2⊆Y. Moreover as f is contra β-continuous, A_1 and A_2 can be separated by disjoint β-open sets $f^{-1}(B_1)$ and $f^{-1}(B_2)$.Thus (X,τ) is πgβ-normal.

4.πgβ-regular spaces

Definition 4.1. A topological space (X, τ) is said to be gβ-regular[10] if for each gβ-closed set A and each point x∈X such that x∉A, there exist disjoint β-open sets U, V⊆X such that A⊆V and x∈U.

Definition 4.2. A topological space (X, τ) is said to be πgβ-regular if for each πgβ-closed set A and each point x∈X such that x∉A, there exist disjoint β-open sets U, V⊆X such that A⊆V and x∈U.

Definition 4.3. A topological space (X, τ) is said to be πgβ-$T_{1/2}$[11] if every πgβ-closed set is gβ-closed.

Theorem 4.1. A space (X,τ) is πgβ-regular if and only if (X,τ) is gβ-regular and πgβ-$T_{1/2}$.
Proof. Suppose that (X,τ) is πgβ-regular. Then clearly (X,τ) is gβ-regular. Now, let A⊆X be πgβ-closed. For each x∉A, there exists a gβ-open set Vx containing x such that Vx∩A= Ø. If V=∪{Vx:x∉A}, then V is a gβ-open set and V=X~A . Hence A is gβ-closed. Converse is obvious.

Remark 4.1. There exist a topological space which is gβ-regular but not πgβ-regular as can be seen in Example 3.1.

Our next result characterizes πgβ-regular spaces.

Theorem 4.2. For a topological space (X, τ), the following are equivalent:
(i). (X,τ) is πgβ-regular.
(ii). Every πgβ-open set U is a union of β-clopen sets.
(iii). Every πgβ-closed set A is intersection of β-clopen sets.
Proof.(i)→(ii).Let U be πgβ-open and x∈U. If A=X~U, then A is πgβ-closed. By assumption, there exist disjoint β-open subsets W_1 and W_2 of X such that x∈W_1 and A⊆W_2. If V=βCl(W_1) , then V is β-clopen ([9], Theorem 3.1) and V∩A⊆V∩ W_2=Ø. It follows that x∈V⊆U. Thus U is a union of β-clopen sets.

(ii)→(iii).obvious.
(iii)→(i). Let A be πgβ-closed and let x∉A. By assumption, there exists a β-clopen set V such that A⊆V and x∉V. If U=X~V, then U is a β-open set containing x and U∩V= Ø.Thus (X, T) is πgβ-regular.

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