

## FIXED COEFFICIENTS FOR A NEW SUBCLASSES OF UNIFORMLY $q$ -SPIRALLIKE FUNCTIONS

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**Abstract.** In this note, we give the some properties of the new subclass of  $\lambda$ - $q$ -spirallike function, with negative coefficients and with fixed second coefficients.

**Keywords:** Analytic functions, Starlike functions, Uniformly convex functions,  $q$ -spirallike functions.

AMS Subject Classification: 30C45

### 1. Introduction

Let  $\mathcal{A}$  denote the class of functions of form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (1)$$

defined on the unit disk  $E = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$  normalized by  $f(0) = 0$ ,  $f'(0) = 1$ . Let  $\mathcal{S}$  denote the subclass of function in  $\mathcal{A}$  which are univalent in  $E$ . Spacek [10] introduced the concept of spirallikeness which is a natural generalization of starlikeness. Spirallike functions can be characterized by the following analytic condition: A function  $f$  in  $\mathcal{A}$  is  $\lambda$ -spirallike if and only if,

$$\Re \left\{ e^{i\lambda} \frac{zf'(z)}{f(z)} \right\} > 0, \quad z \in E, \quad (2)$$

where  $-\frac{\pi}{2} < \lambda < \frac{\pi}{2}$ .

Let  $T$  denote the subclass of  $\mathcal{A}$  consisting of functions whose nonzero coefficients from the second on, are negative. That is, a function  $f$  is in  $T$  if it can be expressed as

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n. \quad (3)$$

Selvaraj and Geetha [9] studied the classes  $UCSP(\lambda, \beta)$  and  $SP_p(\lambda, \beta)$  generalizing the classes  $UCSP(\lambda)$  and  $SP_p(\lambda)$  introduced and studied by Rönnng. Using the  $q$ -derivative concept [4] we further extend these classes and study certain characterizing properties.

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In [4], Jackson introduced and studied the concept of the q-derivative operator  $\partial_q$  as follows :

$$\partial_q f(z) = \frac{f(z) - f(qz)}{z(1-q)}, \quad (z \neq 0, 0 < q < 1, \partial_q f(0) = f'(0)). \quad (4)$$

Equivalently (4) may be written as

$$\partial_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}, \quad z \neq 0. \quad (5)$$

Where  $[n]_q = \frac{1-q^n}{1-q}$ , note that as  $q \rightarrow 1^-$ ,  $[n]_q \rightarrow n$ .

**Definition 1.1.** Let  $UCSPT(\lambda, \beta, q)$  be the class of function  $f$  in  $T$ ,  $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$ ,  $a_n \geq 0$  which satisfy the condition

$$\Re \left\{ e^{i\lambda} \left( \frac{\partial_q(z\partial_q f(z))}{\partial_q f(z)} \right) \right\} \geq \left| \frac{\partial_q(z\partial_q f(z))}{\partial_q f(z)} - 1 \right| + \beta, \quad 0 \leq \beta \leq 1, \quad 0 < q < 1, \quad |\lambda| < \frac{\pi}{2}.$$

**Definition 1.2.** Let  $SP_p T(\lambda, \beta, q)$  be the class of functions  $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$  which satisfy the condition

$$\Re \left( e^{i\lambda} \frac{z\partial_q f(z)}{f(z)} \right) \geq \left| \frac{z\partial_q f(z)}{f(z)} - 1 \right| + \beta, \quad (|\lambda| < \frac{\pi}{2}, \quad 0 \leq \beta < 1, \quad 0 < q < 1).$$

**Lemma 1.1.** [6]  $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$  in  $UCSPT(\lambda, \beta, q)$  if and only if,

$$\sum_{n=2}^{\infty} (2[n]_q - \cos \lambda - \beta)[n]_q a_n \leq \cos \lambda - \beta. \quad (6)$$

Using (3), the function  $f(z) \in UCSPT(\lambda, \beta, q)$  will satisfy

$$a_n \leq \frac{\cos \lambda - \beta}{[n]_q(2[n]_q \cos \lambda - \beta)}, \quad (7)$$

$$a_2 \leq \frac{\cos \lambda - \beta}{[2]_q(2[2]_q \cos \lambda - \beta)}. \quad (8)$$

**Lemma 1.2.** [6]  $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$  in  $SP_p T(\lambda, \beta, q)$  if and only if,

$$\sum_{n=2}^{\infty} (2[n]_q - \cos \lambda - \beta)a_n \leq \cos \lambda - \beta. \quad (9)$$

**Definition 1.3.** Let  $UCSPT_c(\lambda, \beta, q)$  be the class of the function in  $UCSPT(\lambda, \beta, q)$  the form

$$f(z) = z - \frac{c(\cos \lambda - \beta)z^2}{[2]_q(2[2]_q \cos \lambda - \beta)} - \sum_{n=3}^{\infty} a_n z^n, \quad (a_n \geq 0), \quad (10)$$

where  $0 \leq c \leq 1$ . When  $c = 1$  we get  $UCSPT_1(\lambda, \beta, q) = UCSPT(\lambda, \beta, q)$ .

**Definition 1.4.** Let  $SP_pT_c(\lambda, \beta, q)$  be the class of the function in  $SP_pT(\lambda, \beta, q)$  of the form

$$f(z) = z - \frac{c(\cos \lambda - \beta)z^2}{(2[2]_q \cos \lambda - \beta)} - \sum_{n=3}^{\infty} a_n z^n, \quad (a_n \geq 0), \quad (11)$$

where  $0 \leq c \leq 1$ . When  $c = 1$  we get  $SP_pT_1(\lambda, \beta, q) = SP_pT(\lambda, \beta, q)$ .

## 2. Main results

**Theorem 2.1.** If  $f(z) \in UCSPT(\lambda, \beta, q)$  then

$$r - \frac{\cos \lambda - \beta}{[2]_q(2[2]_q - \cos \lambda - \beta)} r^2 \leq |f(z)| \leq r + \frac{\cos \lambda - \beta}{[2]_q(2[2]_q - \cos \lambda - \beta)} r^2$$

and

$$1 - \frac{\cos \lambda - \beta}{2[2]_q - \cos \lambda - \beta} r \leq |\partial_q f(z)| \leq 1 + \frac{\cos \lambda - \beta}{2[2]_q - \cos \lambda - \beta} r$$

and the extreme points are

$$f_1(z) = z, \quad f_n(z) = z - \frac{\cos \lambda - \beta}{[n]_q(2[n]_q - \cos \lambda - \beta)} z^n, \quad n = 2, 3, \dots$$

The result is sharp for  $f(z) = z - \frac{\cos \lambda - \beta}{[2]_q(2[2]_q - \cos \lambda - \beta)} z^2$ ,  $z = \pm r$ .

*Proof.*  $f(z) \in UCSPT(\lambda, \beta, q)$ . Hence by Lemma 1.1,

$$\sum_{n=2}^{\infty} (2[n]_q - \cos \lambda - \beta) [n]_q a_n \leq \cos \lambda - \beta.$$

therefore

$$\sum_{n=2}^{\infty} a_n \leq \frac{\cos \lambda - \beta}{[n]_q(2[n]_q - \cos \lambda - \beta)}.$$

From  $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$  with  $|z| = r$ , ( $r < 1$ ) we have

$$\begin{aligned} |f(z)| &\leq r + \sum_{n=2}^{\infty} a_n r^n \\ &\leq r + \sum_{n=2}^{\infty} a_n r^2 \\ &\leq r + \frac{\cos \lambda - \beta}{[2]_q(2[2]_q - \cos \lambda - \beta)} r^2. \end{aligned}$$

□

As  $q \rightarrow 1^-$  in above Theorem we get the result proved by Selvaraj [9].

**Corollary 2.1.** If  $f(z) \in UCSPT(\lambda, \beta)$  then

$$r - \frac{\cos \lambda - \beta}{2(4 - \cos \lambda - \beta)} r^2 \leq |f(z)| \leq r + \frac{\cos \lambda - \beta}{2(4 - \cos \lambda - \beta)} r^2$$

and

$$1 - \frac{\cos \lambda - \beta}{4 - \cos \lambda - \beta} r \leq |f'(z)| \leq 1 + \frac{\cos \lambda - \beta}{4 - \cos \lambda - \beta} r$$

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and the extreme points are

$$f_1(z) = z, \quad f_n(z) = z - \frac{\cos \lambda - \beta}{n(2n - \cos \lambda - \beta)} z^n, \quad n = 2, 3, \dots$$

The result is sharp for  $f(z) = z - \frac{\cos \lambda - \beta}{2(4 - \cos \lambda - \beta)} z^2, \quad z = \pm r$ .

**Theorem 2.2.** If  $f(z) \in SP_p T(\lambda, \beta, q)$  then

$$r - \frac{\cos \lambda - \beta}{(2[2]_q - \cos \lambda - \beta)} r^2 \leq |f(z)| \leq r + \frac{\cos \lambda - \beta}{(2[2]_q - \cos \lambda - \beta)} r^2.$$

The result is sharp for  $f(z) = z - \frac{\cos \lambda - \beta}{[2]_q (2[2]_q - \cos \lambda - \beta)} z^2, \quad z = \pm r$ .

*Proof.* From Lemma 1.2

$$\sum_{n=2}^{\infty} (2[n]_q - \cos \lambda - \beta) a_n \leq \cos \lambda - \beta.$$

therefore

$$\sum_{n=2}^{\infty} a_n \leq \frac{\cos \lambda - \beta}{2[n]_q - \cos \lambda - \beta}.$$

From  $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$  with  $|z| = r, \quad (r < 1)$  we have

$$\begin{aligned} |f(z)| &\leq r + \sum_{n=2}^{\infty} a_n r^n \\ &\leq r + \sum_{n=2}^{\infty} a_n r^2 \\ &\leq r + \frac{\cos \lambda - \beta}{2[2]_q - \cos \lambda - \beta} r^2. \end{aligned}$$

$$1 - \frac{\cos \lambda - \beta}{2[2]_q - \cos \lambda - \beta} r \leq |\partial_q f(z)| \leq 1 + \frac{\cos \lambda - \beta}{2[2]_q - \cos \lambda - \beta} r$$

and the extreme points are

$$f_1(z) = z, \quad f_n(z) = z - \frac{\cos \lambda - \beta}{(2[n]_q - \cos \lambda - \beta)} z^n, \quad n = 2, 3, \dots$$

The result is sharp for  $f(z) = z - \frac{\cos \alpha - \beta}{[2]_q (2[2]_q - \cos \alpha - \beta)} z^2, \quad z = \pm r$ . □

As  $q \rightarrow 1^-$  in above Theorem we get the result proved by Selvaraj [9].

**Corollary 2.2.** If  $f(z) \in SP_p T(\lambda, \beta)$  then

$$r - \frac{\cos \lambda - \beta}{(4 - \cos \lambda - \beta)} r^2 \leq |f(z)| \leq r + \frac{\cos \lambda - \beta}{(4 - \cos \lambda - \beta)} r^2.$$

The result is sharp for  $f(z) = z - \frac{\cos \lambda - \beta}{2(4 - \cos \lambda - \beta)} z^2, \quad z = \pm r$ .

**Theorem 2.3.** The function  $f(z)$  defined by (10) belongs to  $UCSPT_c(\lambda, \beta, q)$  if and only if,

$$\sum_{n=2}^{\infty} (2[n]_q - \cos \lambda - \beta)[n]_q a_n \leq (1 - c)(\cos \lambda - \beta). \quad (12)$$

The result is sharp.

*Proof.* Taking

$$a_2 = \frac{c(\cos \lambda - \beta)z^2}{[2]_q(2[2]_q - \cos \lambda - \beta)}, \quad 0 \leq c \leq 1, \quad (13)$$

in (6) we get the required result . Also the result is sharp for the function

$$f(z) = z - \frac{c(\cos \lambda - \beta)}{[2]_q(2[2]_q - \cos \lambda - \beta)} - \frac{(1 - c)(\cos \lambda - \beta)z^n}{[n]_q(2[n]_q - \cos \lambda - \beta)}, \quad (n \geq 3). \quad (14)$$

□

As  $q \rightarrow 1^-$  in above Theorem we get the result proved by Selvaraj [9].

**Corollary 2.3.** The function  $f(z)$  defined by (10) belongs to  $UCSPT_c(\lambda, \beta)$  if and only if,

$$\sum_{n=2}^{\infty} (2n - \cos \lambda - \beta)n a_n \leq (1 - c)(\cos \lambda - \beta). \quad (15)$$

**Corollary 2.4.** If  $f(z)$  defined by (10) is in the class  $UCSPT_c(\lambda, \beta, q)$  then,

$$a_n = \frac{(1 - c)(\cos \lambda - \beta)}{[n]_q(2[n]_q - \cos \lambda - \beta)}, \quad (0 \leq c \leq 1, n \geq 3). \quad (16)$$

**Theorem 2.4.** The function  $f(z)$  defined by (11) belongs to  $SP_p T_c(\lambda, \beta, q)$  if and only if,

$$\sum_{n=2}^{\infty} (2[n]_q - \cos \lambda - \beta)a_n \leq (1 - c)(\cos \lambda - \beta). \quad (17)$$

The result is sharp.

*Proof.* Taking

$$a_2 = \frac{c(\cos \lambda - \beta)z^2}{(2[2]_q - \cos \lambda - \beta)}, \quad 0 \leq c \leq 1, \quad (18)$$

in (9) we get the required result . Also the result is sharp for the function

$$f(z) = z - \frac{c(\cos \lambda - \beta)}{(2[2]_q - \cos \lambda - \beta)} - \frac{(1 - c)(\cos \lambda - \beta)z^n}{(2[n]_q - \cos \lambda - \beta)}, \quad (n \geq 3). \quad (19)$$

□

As  $q \rightarrow 1^-$  in above Theorem we get the result proved by Geetha [2].

**Corollary 2.5.** The function  $f(z)$  defined by (11) belongs to  $SP_p T_c(\lambda, \beta)$  if and only if,

$$\sum_{n=2}^{\infty} (2n - \cos \lambda - \beta)a_n \leq (1 - c)(\cos \lambda - \beta). \quad (20)$$

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**Corollary 2.6.** If  $f(z)$  defined by (11) is in the class  $SP_pT_c(\lambda, \beta, q)$  then,

$$a_n = \frac{(([2]_q - 1) - c)(\cos \lambda - \beta)}{(2[n]_q - \cos \lambda - \beta)}, \quad (0 \leq c \leq 1, n \geq 3). \quad (21)$$

### 3. Closure Theorem

**Theorem 3.1.** The class  $UCSPT_c(\alpha, \beta, q)$  is closed under  $q$ -spirallike linear combination.

*Proof.* Let  $f(z)$  defined by (10) be in  $UCSPT_c(\alpha, \beta, q)$ . Now define  $h(z)$  by

$$h(z) = z - \frac{c(\cos \lambda - \beta)z^2}{[2]_q(2[2]_q \cos \lambda - \beta)} - \sum_{n=3}^{\infty} c_n z^n, \quad (c_n \geq 0). \quad (22)$$

If  $f(z)$  and  $h(z)$  in  $UCSPT_c(\lambda, \beta, q)$  then it is sufficient to show that the function  $G(z)$  defined by

$$G(z) = \lambda f(z) + (1 - \lambda)h(z), \quad (0 \leq \lambda \leq 1), \quad (23)$$

is also in  $UCSPT_c(\lambda, \beta, q)$ .

$$G(z) = z - \frac{c(\cos \lambda - \beta)z^2}{[2]_q(2[2]_q \cos \lambda - \beta)} - \sum_{n=3}^{\infty} (\lambda a_n + (1 - \lambda)c_n)z^n. \quad (24)$$

Using Theorem 2.3 we get

$$\sum_{n=3}^{\infty} (2[n]_q - \cos \lambda - \beta)[n]_q(\lambda a_n + (1 - \lambda)c_n) \leq (1 - c)(\cos \lambda - \beta). \quad (25)$$

Hence  $G(z)$  is in  $UCSPT_c(\lambda, \beta, q)$ . Thus  $UCSPT_c(\lambda, \beta, q)$  is closed under  $q$ -spirallike linear combination.  $\square$

As  $q \rightarrow 1^-$  in above Theorem we get the result proved by Geetha [2].

**Corollary 3.1.** The class  $UCSPT_c(\alpha, \beta)$  is closed under convex linear combination.

**Theorem 3.2.** The class  $SP_pT_c(\alpha, \beta, q)$  is closed under  $q$ -spirallike linear combination.

*Proof.* Let  $f(z)$  defined by (11) be in  $SP_pT_c(\alpha, \beta, q)$ . Now define  $h(z)$  by

$$h(z) = z - \frac{c(\cos \lambda - \beta)z^2}{(2[2]_q \cos \lambda - \beta)} - \sum_{n=3}^{\infty} c_n z^n, \quad (c_n \geq 0). \quad (26)$$

If  $f(z)$  and  $h(z)$  in  $SP_pT_c(\lambda, \beta, q)$  then it is sufficient to show that the function  $G(z)$  defined by

$$G(z) = \lambda f(z) + (1 - \lambda)h(z), \quad (0 \leq \lambda \leq 1), \quad (27)$$

is also in  $SP_pT_c(\lambda, \beta, q)$ .

$$G(z) = z - \frac{c(\cos \lambda - \beta)z^2}{(2[2]_q \cos \lambda - \beta)} - \sum_{n=3}^{\infty} (\lambda a_n + (1 - \lambda)c_n)z^n. \quad (28)$$

Using Theorem 2.4 we get

$$\sum_{n=3}^{\infty} (2[n]_q - \cos \lambda - \beta)[n]_q(\lambda a_n + (1 - \lambda)c_n) \leq (1 - c)(\cos \lambda - \beta). \quad (29)$$

Hence  $G(z)$  is in  $SP_p T_c(\lambda, \beta, q)$ . Thus  $SP_p T_c(\lambda, \beta, q)$  is closed under  $q$ -spirallike linear combination.  $\square$

As  $q \rightarrow 1^-$  in above Theorem we get the result proved by Geetha [2].

**Corollary 3.2.** *The class  $SP_p T_c(\alpha, \beta)$  is closed under convex linear combination.*

**Theorem 3.3.** *Let the functions*

$$f_i(z) = z - \frac{c(\cos \lambda - \beta)z^2}{[2]_q(2[2]_q - \cos \lambda - \beta)} - \sum_{n=3}^{\infty} a_{n,i}z^n, (a_{n,i} \geq 0), \quad (30)$$

be in the class  $UCSPT_c(\lambda, \beta, q)$  for every  $i = 1, 2, 3, \dots, m$ . Then the function  $H(z)$  defined by

$$H(z) = \sum_{i=1}^m d_i f_i(z), (d_i \geq 0) \quad (31)$$

is also in the same class  $UCSPT_c(\lambda, \beta, q)$  where

$$\sum_{i=1}^m d_i = 1. \quad (32)$$

*Proof.* Using (30) and (32) in (31) we have

$$H(z) = z - \frac{c(\cos \lambda - \beta)z^2}{[2]_q(2[2]_q - \cos \lambda - \beta)} - \sum_{n=3}^{\infty} \left[ \sum_{i=1}^m d_i a_{n,i} \right] z^n. \quad (33)$$

Each  $f_i(z) \in UCSPT_c(\lambda, \beta, q)$  for  $i = 1, 2, 3, \dots, m$ . Theorem 2.3 gives

$$\sum_{n=3}^{\infty} (2[n]_q - \cos \lambda - \beta)[n]_q a_{n,i} \leq (1 - c)(\cos \lambda - \beta),$$

$$\begin{aligned} \text{for } i = 1, 2, 3, \dots, m. \text{ Hence we get } & \sum_{n=3}^{\infty} [n]_q (2[n]_q - \cos \lambda - \beta) [\sum_{i=1}^m d_i a_{n,i}] \\ &= \sum_{i=1}^m d_i \left[ \sum_{n=3}^{\infty} [n]_q (2[n]_q - \cos \lambda - \beta) a_{n,i} \right] \\ &\leq (1 - c)(\cos \lambda - \beta). \end{aligned}$$

This implies  $H(z) \in UCSPT_c(\alpha, \beta, q)$  by Theorem 2.3  $\square$

As  $q \rightarrow 1^-$  in above Theorem we get the result proved by Geetha [1]

**Corollary 3.3.** *Let the functions*

$$f_i(z) = z - \frac{c(\cos \lambda - \beta)z^2}{2(4 - \cos \lambda - \beta)} - \sum_{n=3}^{\infty} a_{n,i}z^n, (a_{n,i} \geq 0), \quad (34)$$

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be in the class  $UCSPT_c(\lambda, \beta)$  for every  $i = 1, 2, 3, \dots, m$ . Then the function  $H(z)$  defined by

$$H(z) = \sum_{i=1}^m d_i f_i(z), \quad (d_i \geq 0) \quad (35)$$

is also in the same class  $UCSPT_c(\lambda, \beta)$  where

$$\sum_{i=1}^m d_i = 1. \quad (36)$$

**Theorem 3.4.** Let

$$f_2(z) = z - \frac{c(\cos \lambda - \beta)z^2}{[2]_q(2[2]_q - \cos \lambda - \beta)} \quad (37)$$

and

$$f_n(z) = z - \frac{c(\cos \lambda - \beta)z^2}{[2]_q(2[2]_q - \cos \lambda - \beta)} - \frac{(1-c)(\cos \lambda - \beta)z^n}{[n]_q(2[n]_q - \cos \lambda - \beta)}, \quad (38)$$

for  $n = 3, 4, \dots$ . Then  $f(z) \in UCSPT_c(\lambda, \beta, q)$  if and only if, it can be expressed the form

$$f(z) = \sum_{n=2}^{\infty} \lambda_n f_n(z), \quad (39)$$

where  $\lambda_n \geq 0$  and  $\sum_{n=2}^{\infty} \lambda_n = 1$ .

*Proof.* First assume that  $f(z)$  can be expressed in the form (24). Then we have

$$f_n(z) = z - \frac{c(\cos \alpha - \beta)z^2}{[2]_q(2[2]_q - \cos \alpha - \beta)} - \sum_{n=3}^{\infty} \frac{(1-c)(\cos \alpha - \beta)}{[n]_q(2[n]_q - \cos \alpha - \beta)} \lambda_n z^n. \quad (40)$$

But

$$\sum_{n=3}^{\infty} \frac{(1-c)(\cos \alpha - \beta)}{[n]_q(2[n]_q - \cos \alpha - \beta)} \lambda_n ([n]_q(2[n]_q - \cos \alpha - \beta)) = (1-c)(\cos \alpha - \beta)(1 - \lambda_n) \leq (1-c)(\cos \alpha - \beta). \quad (41)$$

Hence from (12) it follows that  $f(z) \in UCSPT_c(\alpha, \beta, q)$ . Conversely, we assume that  $f(z)$  defined by (10),  $f(z) \in UCSPT_c(\alpha, \beta, q)$ . Then by using (16) we get

$$a_n \leq \frac{(1-c)(\cos \alpha - \beta)}{[n]_q(2[n]_q - \cos \alpha - \beta)}, \quad (n = 3, 4, \dots). \quad (42)$$

Taking

$$\lambda_n = \frac{[n]_q(2[n]_q - \cos \alpha - \beta)a_n}{(1-c)(\cos \alpha - \beta)}, \quad (n = 3, 4, \dots) \quad (43)$$

and

$$\lambda_2 = 1 - \sum_{n=3}^{\infty} \lambda_n, \quad (44)$$

we have (41).  $\square$

As  $q \rightarrow 1^-$  in above Theorem we get the result proved by Geetha [1].

**Corollary 3.4.** Let

$$f_2(z) = z - \frac{c(\cos \lambda - \beta)z^2}{2(4 - \cos \lambda - \beta)} \quad (45)$$

and

$$f_n(z) = z - \frac{c(\cos \lambda - \beta)z^2}{2(4 - \cos \lambda - \beta)} - \frac{((1 - c)(\cos \lambda - \beta)z^n)}{n(2n - \cos \lambda - \beta)}, \quad (46)$$

for  $n = 3, 4, \dots$ . Then  $f(z) \in UCSPT_c(\lambda, \beta)$  if and only if, it can be expressed the form

$$f(z) = \sum_{n=2}^{\infty} \lambda_n f_n(z), \quad (47)$$

where  $\lambda_n \geq 0$  and  $\sum_{n=2}^{\infty} \lambda_n = 1$ .

**Theorem 3.5.** Let the function  $f(z)$  define by (10) in the class  $UCSPT_c(\alpha, \beta, q)$ . Then  $f(z)$  is spirallike of order  $\rho$ , ( $0 \leq \rho < 1$ ) in the disk  $|z| < r_1(\alpha, \beta, c, \rho, q)$  where  $r_1(\alpha, \beta, c, \rho, q)$  is the largest value for which

$$\frac{c(\cos \alpha - \beta)([2]_q - \rho)r}{[2]_q(2[2]_q - \cos \alpha - \beta)} + \frac{(([2]_q - 1) - c)(\cos \alpha - \beta)([n]_q - \rho)r^{n-1}}{[n]_q(2[n]_q - \cos \alpha - \beta)} \leq 1 - \rho, \quad (48)$$

for  $n \leq 3$ . The result is sharp with the extremal function

$$f_n(z) = z - \frac{c(\cos \lambda - \beta)([2]_q - \rho)z^2}{[2]_q(2[2]_q - \cos \lambda - \beta)} - \frac{(([2]_q - 1) - c)(\cos \alpha - \beta)([n]_q - \rho)z^n}{[n]_q(2[n]_q - \cos \lambda - \beta)}, \quad (49)$$

for some  $n$ .

*Proof.* It suffices to show that

$$\left| \frac{z\partial_q f(z)}{f(z)} - 1 \right| \leq 1 - \rho, \quad (0 \leq \rho < 1),$$

for  $|z|r_1(\alpha, \beta, c, \rho, q)$ . Note that

$$\begin{aligned} \left| \frac{z\partial_q f(z)}{f(z)} - 1 \right| &\leq \frac{\frac{c(\cos \alpha - \beta)r}{[2]_q(2[2]_q - \cos \alpha - \beta)} + \sum_{n=3}^{\infty} ([n]_q - 1)a_n r^{n-1}}{1 - \frac{c(\cos \alpha - \beta)r}{[2]_q(2[2]_q - \cos \alpha - \beta)} - \sum_{n=3}^{\infty} a_n r^{n-1}} \\ &\leq 1 - \rho, \end{aligned}$$

for  $|z| < r$  if and only if,

$$\frac{c(\cos \lambda - \beta)([2]_q - \rho)r}{[2]_q(2[2]_q - \cos \lambda - \beta)} + \frac{(([2]_q - 1) - c)(\cos \lambda - \beta)([n]_q - \rho)r^{n-1}}{[n]_q(2[n]_q - \cos \lambda - \beta)} + \sum_{n=3}^{\infty} ([n]_q - \rho)a_n r^{n-1} \leq 1 - \rho.$$

Since  $f(z)$  is in  $UCSPT_c(\lambda, \beta, q)$  from (24) we may take

$$a_n = \frac{(([2]_q - 1) - c)(\cos \lambda - \beta)\lambda_n}{[n]_q(2[n]_q - \cos \lambda - \beta)}, \quad (n \geq 3),$$

where  $\lambda_n \geq 0$ , ( $n \geq 3$ ) and  $\sum_{n=3}^{\infty} \lambda_n \leq 1$ . For each fixed  $r$ , we choose the positive integer  $n_o = n_o(r)$  for which  $\frac{([n]_q - \rho)r^{n-1}}{[n]_q}$  is maximal. Then it follows that

$$\sum_{n=3}^{\infty} ([n]_q - \rho)a_n r^{n-1} \leq \frac{(([2]_q - 1) - c)(\cos \lambda - \beta)([n_o]_q - \rho)r^{n_o-1}}{[n_o]_q(2[n_o]_q - \cos \lambda - \beta)}.$$

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□

Hence  $f(z)$  is  $q$ -spirallike of order  $\rho$  in  $|z| < r_1(\lambda, \beta, c, \rho, q)$  provided that

$$\frac{c(\cos \lambda - \beta)([2]_q - \rho)r}{[2]_q(2[2]_q - \cos \lambda - \beta)} + \frac{(([2]_q - 1) - c)(\cos \lambda - \beta)([n_0]_q - \rho)r^{n_0-1}}{[n_0]_q(2[n_0]_q - \cos \lambda - \beta)} \leq 1 - \rho.$$

We find the value  $r_0(\lambda, \beta, c, \rho, q)$  and the corresponding integer  $n_0(r_0)$  so that

$$\frac{c(\cos \lambda - \beta)([2]_q - \rho)r_0}{[2]_q(2[2]_q - \cos \lambda - \beta)} + \frac{(([2]_q - 1) - c)(\cos \alpha - \beta)([n_0]_q - \rho)r_0^{n_0-1}}{[n_0]_q(2[n_0]_q - \cos \lambda - \beta)} \leq 1 - \rho.$$

Then this value  $r_0$  is the radius of  $q$ -spirallikeness of order  $\rho$  for functions  $f(z)$  belonging to the class  $UCSPT_c(\lambda, \beta, q)$ .

As  $q \rightarrow 1^-$  in above Theorem we get the result proved by Geetha [1].

**Corollary 3.5.** Let the function  $f(z)$  define by (10) in the class  $UCSPT_c(\alpha, \beta)$ . Then  $f(z)$  is starlike of order  $\rho$ , ( $0 \leq \rho < 1$ ) in the disk  $|z| < r_1(\alpha, \beta, c, \rho)$  where  $r_1(\alpha, \beta, c, \rho)$  is the largest value for which

$$\frac{c(\cos \alpha - \beta)(2 - \rho)r}{2(4 - \cos \alpha - \beta)} + \frac{(1 - c)(\cos \alpha - \beta)(n - \rho)r^{n-1}}{n(2n - \cos \alpha - \beta)} \leq 1 - \rho, \quad (50)$$

for  $n \leq 3$ . The result is sharp with the extremal function

$$f_n(z) = z - \frac{c(\cos \lambda - \beta)(2 - \rho)z^2}{2(4 - \cos \lambda - \beta)} - \frac{(1 - c)(\cos \alpha - \beta)(n - \rho)z^n}{n(2n - \cos \lambda - \beta)}, \quad (51)$$

for some  $n$ .

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