Solving Ordinary Differential Equations by using a new numerical method

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Abstract. *In this manuscript, a new numerical technique has been established to solve some complex initial valueproblems of ordinary differential equations. The complete analysis of the derivation of thisnewtechnique is introduced here. In future study, we will investigate on the main properties of the technique such as consistency, feasibility and convergenceby using few Initial valueproblems. The augmentation of this new numerical system may be illustrated and comparison is also be made with some existingmethods.*

Keywords: *Numerical method, ordinary differential equations, numerical system expansion*.

I. Introduction

Numerical methods for the solution of initial value problems in ordinary differential equations made enormous progress during this century for several reasons. For centuries, differential equations have been an important concept in many branches of science. They arise spontaneously in physics, engineering, chemistry, biology, economics and a lot of fields in between. Many Ordinary Differential Equations (ODEs) have been solved analytically to obtain solutions in a closed form. However, the range of Differential Equations that can be solved by straightforward analytical methods is relatively restricted. In many cases, where a Differential Equation and known boundary conditions are given, an approximate solution is often obtainable by the application of numerical methods.

The first technique is to reduce the differential equation in to one that can be solved exactly and then use the result of the reduced equation to approximate the solution to the original problem. Another technique, which we will prove in this article, uses methods of approximations the solution of original problem. This is the technique that is generally taken as the approximation methods give more perfect results and relative error information.

Many Numerical researchers namely Ibijola [1], Liu, Turner [2], Butcher [3], Fatunla [4] and even Ogunrinde [5] and so on, have established schemes in order to solve some initial value problems of ordinary differential equations. Enright, Fellen and Sedgwick [6] have developed a numerical method which compares the numerical solution of ordinary differential equations. Yuan and Agrawal [7], established a scheme for fractional derivatives, on the other hand Obaymi [8,9] also studied on some approximation techniques which were used to derive stable non-standard finite difference schemes. And Ibijola [10] focus on the convergence, consistency and stability of a method of integration of ordinary differential equations. [11-12] solved ODE with numerical examples. The efficiency of all their efforts made for stability, accuracy, convergence and consistency of the methods. The accuracy property of different methods can be considered an order and convergence as well as truncation error co-efficient. Numerical methods are usually used for solving the mathematical problems that are articulated in the field of science and engineering in which case the determination of the exact solution is so hard or impossible. Only a few numbers of differential equations can be solved analytically. Consequently, to obtain the analytical solution for differential equation there exist different methods. A huge number of differential equations are unable to determine the solution in closed form using familiar analytical methods, in which case we apply numerical technique for solving a differential equation under certain initial restriction or restrictions. There exist different kinds of practical numerical methods for finding the solution of initial value problem of ordinary differentialequations.
In this paper, we have established a new numerical technique with some particular properties to determine the solution of initial value problems of ordinary differential equations based on the local representation of theoretical form.

Let us consider

$$y' = \frac{dy}{dx} = f(x,y), \qquad y(a) = y_0 \tag{01}$$

is interpolating by the function

$$F(x) = C_0 + C_1x^2 + C_2e^{x^2} + E\sin x^2 \tag{02}$$

where C_O, C_1, $C_2$and E are real undetermined coefficients.

II.Derivation of the new method

Let us suppose that y_k is a numerical estimation of the theoretical solution $y(x)$and $f_k = f(x_k, y_k)$.
Now we define the mesh points as follows: $x_k = a + kh; \quad k = 0,1,2,3 \ldots \ldots \ldots \ldots \ldots$
We proceed to derive the new technique is as follows:

$$F'(x) = 2C_1x + 2C_2xe^{x^2} + 2E\sin x \tag{03}$$

$$F''(x) = 2C_1 + 2C_2\{e^{x^2}(2x^2+1) + 2E\cos x\} \tag{04}$$

Similarly

$$F'''(x) = 2C_2\left\{4xe^{x^2} + 2xe^{x^2}(2x^2+1)\right\} - 2E\sin x \tag{05}$$

$$F^{\mathrm{IV}}(x) = 2C_2\left\{4e^{x^2} + 8xe^{x^2} + 4x^2e^{x^2}(2x^2+1) + 16x^2e^{x^2}\right\} - 2E\cos x \tag{06}$$

From equation (2) we have,

$$F(x) = C_0 + C_1x^2 + C_2e^{x^2} + E\sin x^2$$

$$\text{Or, } C_0 = F(x) - C_1x^2 - C_2e^{x^2} - E\sin x^2 \tag{07}$$

And from equation (3) we get,

$$C_1 = \frac{F'(x)}{2x} - C_2e^{x^2} - \frac{E}{x}\sin x \tag{08}$$

From equation (4) we get,

$$C_2 = \frac{F''(x)}{2\{e^{x^2}(2x^2+1)\}} - \frac{C_1}{e^{x^2}(2x^2+1)} - \frac{E\cos x}{e^{x^2}(2x^2+1)} \tag{09}$$

Also from equation (5) we get,

$$E = \frac{F''(x) - 2C_2\{4xe^{x^2} + 2xe^{x^2}(2x^2+1)\}}{-2\sin x} \tag{10}$$

Now substituting the value of C_1 from (8) into (9) we get,

$$C_2 = \frac{xF''(x) - F'(x) - 2xC_2e^{x^2} - 2E\sin x - 2xE\cos x}{2xe^{x^2}(2x^2+1)} \tag{11}$$

Putting the value of C_2 from (11) into (10) we get,

$$E = \frac{xe^{x^2}(2x^2+1)F''(x) - \{xF''(x) - F'(x) - 2xC_2e^{x^2} - 2E\sin x - 2xE\cos x\}\{4xe^{x^2} + 2xe^{x^2}((2x^2+1))\}}{\{xe^{x^2}(2x^2+1)(-2\sin x)\}} \tag{12}$$

By substituting the value of E from (12) into (11) we get,

$$C_2 = \frac{\left\{xe^{x^2}(2x^2+1)(-2\sin x)\right\}\left\{xF''(x) - F'(x) - 2xC_2e^{x^2}\right\} - [xe^{x^2}(2x^2+1)F''(x) - (xF''(x) - F'(x) - 2xC_2e^{x^2} - 2E\sin x - 2xE\cos x)\{4xe^{x^2} + 2xe^{x^2}(2x^2+1)\}(2\sin x + 2x\cos x)]}{\left\{xe^{x^2}(2x^2+1)(-2\sin x)\right\}\{2xe^{x^2}((2x^2+1)\}} \tag{13}$$

Putting the value of C_2 from (13) and E from (12) into (8), we get

$$C_1 = \frac{F^{'}(x)}{2x} - C_2 e^{x^2} - \frac{E}{x}\sin x$$

$$= \frac{F^{'}(x)}{2x} - \left\{ \frac{\{xe^{x^2}(2x^2+1)(-2\sin x)\}\{xF^{''}(x)-F^{'}(x)-2xC_2e^{x^2}\}-[xe^{x^2}(2x^2+1)F^{''}(x)-(xF^{''}(x)-F^{'}(x)-2xC_2e^{x^2}-2E\sin x-2xE\cos x)\{4xe^{x^2}+2xe^{x^2}(2x^2+1)\}(2\sin x+2x\cos x)]e^{x^2}}{\{xe^{x^2}(2x^2+1)(-2\sin x)\}\{2xe^{x^2}((2x^2+1)\}} \right\} - \left\{ \frac{xe^{x^2}(2x^2+1)F^{''}(x)-\{xF^{''}(x)-F^{'}(x)-2xC_2e^{x^2}-2E\sin x-2xE\cos x\}\{4xe^{x^2}+2xe^{x^2}((2x^2+1)\}\sin x}{\{xe^{x^2}(2x^2+1)(-2\sin x)\}} \right\} \quad (14)$$

Let us consider,

$$P = \left\{ \frac{\{xe^{x^2}(2x^2+1)(-2\sin x)\}\{xF^{''}(x)-F^{'}(x)-2xC_2e^{x^2}\}-[xe^{x^2}(2x^2+1)F^{''}(x)-(xF^{''}(x)-F^{'}(x)-2xC_2e^{x^2}-2E\sin x-2xE\cos x)\{4xe^{x^2}+2xe^{x^2}(2x^2+1)\}(2\sin x+2x\cos x)]e^{x^2}}{\{xe^{x^2}(2x^2+1)(-2\sin x)\}\{2xe^{x^2}((2x^2+1)\}} \right\}$$

And $Q = \left\{ \frac{xe^{x^2}(2x^2+1)F^{''}(x)-\{xF^{''}(x)-F^{'}(x)-2xC_2e^{x^2}-2E\sin x-2xE\cos x\}\{4xe^{x^2}+2xe^{x^2}((2x^2+1)\}\sin x}{\{xe^{x^2}(2x^2+1)(-2\sin x)\}} \right\}$

Hence the equation (14) becomes,

$$C_1 = \frac{F^{'}(x)}{2x} - P - Q \quad (15)$$

Now applying the following restrictions on the interpolating function (2) in the following way:

Condition1. The interpolating function (2) must be coincide with the theoretical solution at $x = x_k$ and $x = x_{k+1}$ such that,

$$F(x_k) = C_0 + C_1 {x_k}^2 + C_2 e^{{x_k}^2} + E\sin {x_k}^2$$

And $F(x_{k+1}) = C_0 + C_1 x^2{}_{k+1} + C_2 e^{x^2{}_{k+1}} + E\sin x^2{}_{k+1}$

Condition 2. The derivatives $F^{'}(x), F^{''}(x), F^{'''}(x)$ and $F^{k}(x)$ are coincide with $f(x), f^{'}(x), f^{''}(x)$ and $f^{k-1}(x)$ respectively, that is,

$$F^{'}(x) = f_k$$
$$F^{''}(x) = {f_k}^{'}$$
$$F^{'''}(x) = {f_k}^{''}$$
$$F^{\mathrm{IV}}(x) = {f_k}^{'''}$$

From the above conditions (1) and (2), it follows that, if $F(x_{k+1})- F(x_k)= y_{k+1} - y_k$

Then we have,

$$C_0 + C_1 x^2{}_{k+1} + C_2 e^{x^2{}_{k+1}} + E\sin x^2{}_{k+1} - \{C_0 + C_1 {x_k}^2 + C_2 e^{{x_k}^2} + E\sin {x_k}^2 = y_{k+1} - y_k\}$$

So, $y_{k+1} = y_k + C_1(x^2{}_{k+1} - {x_k}^2) + C_2\left(e^{x^2{}_{k+1}} - e^{{x_k}^2}\right) + E(\sin x^2{}_{k+1} - \sin {x_k}^2)$ (16)

Let us assume that

$\mathrm{x_k} = a + kh$ then ${x_k}^2 = (a+kh)^2 = a^2 + 2akh + (kh)^2$ (17)

Also $x_{k+1} = a + (k+1)h$ so $x^2{}_{k+1} = \{a + (k+1)h\}^2$

Hence, $x^2{}_{k+1} = a^2 + 2akh + 2ah + (kh)^2 + 2kh^2 + h^2$ (18)

Now wecalculate, $x^2{}_{k+1} - x_k{}^2 = 2h(a + kh) + h^2$ (19)

Similarly,

$$e^{x^2{}_{k+1}} - e^{x_k{}^2} = e^{a^2+2akh+2ah+(kh)^2+2kh^2+h^2} - e^{a^2+2akh+(kh)^2}$$

$$e^{x^2{}_{k+1}} - e^{x_k{}^2} = e^{a^2+2akh+(kh)^2}[e^{2h(a+kh)+h^2} - 1] \quad (20)$$

And $\sin x^2{}_{k+1} - \sin x_k{}^2 = \sin\{a^2 + 2akh + 2ah + (kh)^2 + 2kh^2 + h^2\} -$

$\sin\{a^2 + 2akh + (kh)^2\}$ (21)

Putting (17) through (21) into equation (16) then we obtain our required numerical approach

$$y_{k+1} =$$
$$y_k + \left[\frac{F'(x_k)}{2x_k} - P - Q\right] \times \{2h(a + kh) + h^2\} +$$
$$\left[\frac{\left\{xe^{x^2}(2x^2+1)(-2\sin x)\right\}\left\{xF''(x)-F'(x)-2xC_2e^{x^2}\right\}-[xe^{x^2}(2x^2+1)F''(x)-(xF''(x)-F'(x)-2xC_2e^{x^2}-2E\sin x-2xE\cos x)\{4xe^{x^2}+2xe^{x^2}(2x^2+1)\}(2\sin x+2x\cos x)]e^{x^2}}{\left\{xe^{x^2}(2x^2+1)(-2\sin x)\right\}\{2xe^{x^2}((2x^2+1)\}e^{x^2}}\right] \times e^{\{a^2+2akh+(kh)^2\}} \times \left\{e^{2ah+2akh+h^2} -\right.$$

*1+xex22x2+1F''x−xF''x−F'x−2xC2ex2−2E*sin*x−2xE*cos*x{4xex2+2xex2(2x2+1}*sin*xxex22x2+1−2*sin*x×[s*
in*{a2+2akh+2ah+(kh)2+2kh2+h2}−*sin*{a2+2akh+(kh)2}]*
(22)

Equation (22) is the New Numerical Approach for the solution of ordinary differential equations with the given initial values.

III. Conclusion

The main purpose is to introduce a new numerical method as a recommendation numerical approximate values may be agreed or tends to some existing method of solution of various initial value problems. Here this article has the ability to establish the new proposal scheme. In future we will dedicate to check the authenticity of this methods with few numerical examples. Then we will validate the basic properties as the accuracy, reliability, of the proposed new numerical method by solving (1) and then analysis the truncation error, relative error on the comparison with some present standardmethods.

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