Generalized Expression for the nth Integration of Transcendental and Non-Transcendental Functions

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Abstract: *The study aims to derive the generalized expression in the form of sequences and series for the nth indefinite integration of the following types of integrands: First, non-transcendental function with any rational number as an exponent; Second, the product of the non-transcendental and logarithmic function both having any rational number as an exponent; Third, the product of exponential, sine or cosine function with a non-transcendental function having any integral power. On analyzing the nth indefinite integration of type third integrand, it was observed that the solutions for the integral power of x can also be expressed in the form of Pascal's triangle which revealed a divine beautiful symmetry by each row of Pascal's triangle. In the proposed solution this symmetry generates a key for achieving the generalized expression for any higher integral exponent of x, that will help to study the relationship between the number theory and repeated integration of periodic functions, in which the coefficients of generalized expression can be expressed in terms of binomial coefficients and it can also be derived by rows of Pascal's triangle. Interestingly, all these results are also written in an aesthetic form, which demonstrates the mathematical beauty in the generalized expressions which are discussed in this paper. In the application of fractional calculus, these derived expressions can be used to obtain the compact form of a particular integral of nth order differential equation with constant coefficients in just a single step. Here, the right-hand side of nth order differential equation is the integrand of the following forms of integrals:*

1. $\frac{1}{D^n} = \underbrace{\iiint \cdots\cdots}_{n-times} \int (ax \pm b)^{\pm p/q} (dx)^n$
2. $\frac{1}{D^n} = \underbrace{\iiint \cdots\cdots}_{n-times} \int (ax^{\pm p/q} \pm b)(dx)^n$
3. $\frac{1}{D^n} = \underbrace{\iiint \cdots\cdots}_{n-times} \int (ax \pm b)^{\pm p/q} \log_e (ax \pm b)^{\pm r/s} (dx)^n$
4. $\frac{1}{D^n} = \underbrace{\iiint \cdots\cdots}_{n-times} \int (ax^{\pm p/q} \pm b) \log_e (cx^{\pm r/s})(dx)^n$
5. $\frac{1}{D^n} = \underbrace{\iiint \cdots\cdots}_{n-times} \int (ax^m \pm b) e^{(cx \pm d)} (dx)^n$
6. $\frac{1}{D^n} = \underbrace{\iiint \cdots\cdots}_{n-times} \int (ax^m \pm b) \sin(cx \pm d)(dx)^n$
7. $\frac{1}{D^n} = \underbrace{\iiint \cdots\cdots}_{n-times} \int (ax^m \pm b) \cos(cx \pm d)(dx)^n$

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I. Introduction

In the area of mathematical analysis, there are several methods for evaluating the integration of elementary functions through reduction formulae. In this paper, I have introduced some methods in which the product of functions with higher exponent are integrated repeatedly n times and the results were arranged after every integration in such a manner to generate an exact sequence that produces the multi-dimensional recurrence relation.

This relation will be helpful to deduce the generalized expression for the n^{th} indefinite integration into the finite series.
In this study, the expressions of the following integrals (equation 1-7) were derived:

1. $\frac{1}{D^n} = \underbrace{\iiint \cdots\cdots\cdots \int}_{n-times} (ax \pm b)^{\pm p/q} (dx)^n$ (1)

2. $\frac{1}{D^n} = \underbrace{\iiint \cdots\cdots\cdots \int}_{n-times} (ax^{\pm p/q} \pm b)(dx)^n$ (2)

3. $\frac{1}{D^n} = \underbrace{\iiint \cdots\cdots\cdots \int}_{n-times} (ax \pm b)^{\pm p/q} \log_e (ax \pm b)^{\pm r/s} (dx)^n$ (3)

4. $\frac{1}{D^n} = \underbrace{\iiint \cdots\cdots\cdots \int}_{n-times} (ax^{\pm p/q} \pm b) \log_e (cx^{\pm r/s})(dx)^n$ (4)

5. $\frac{1}{D^n} = \underbrace{\iiint \cdots\cdots\cdots \int}_{n-times} (ax^m \pm b) e^{(cx \pm d)} (dx)^n$ (5)

6. $\frac{1}{D^n} = \underbrace{\iiint \cdots\cdots\cdots \int}_{n-times} (ax^m \pm b) \sin(cx \pm d)(dx)^n$ (6)

7. $\frac{1}{D^n} = \underbrace{\iiint \cdots\cdots\cdots \int}_{n-times} (ax^m \pm b) \cos(cx \pm d)(dx)^n$ (7)

Here, $\frac{1}{D^n}$ is the n^{th} fractional integral with respect to x, where D^n is the differintegral which combined differentiation and integration operator which is denoted by $D = \frac{d}{dx}$, $D^2 = \frac{d^2}{dx^2}$, $D^3 = \frac{d^3}{dx^3}, \cdots D^n = \frac{d^n}{dx^n}$

There are 3 cases for n :-

1. If $n > 0$ Fractional Derivative
2. If $n < 0$ Fractional Integral
3. If $n = 0$ then n^{th} differintegral of a given function is the function itself

In above seven forms of integrals, $\frac{1}{D^n}$ is the n^{th} integral of the given integrands.

Since, we also know the basic formal properties of differentiation and integration for operator D :
Linearity-

- $D^n(f+g) = D^n(f) + D^n(g)$ (8)
- $D^n(af) = aD^n(f)$ (9)

Zero Rule-

- $D^0(f) = f$ (10)

In non-homogenous differential equation of n^{th} order having the right hand side is $f(x)$, then it can be written as-

- $D^n y = f(x)$ (11)

Hence, the particular integral of the differential equation in equation (11) is

- $P.I. = \frac{1}{D^n} \cdot f(x) = \underbrace{\iiint \cdots\cdots\cdots \int}_{n-times} f(x)(dx)^n$ (12)

So, I have to evaluate the n^{th} fractional integral of $f(x)$ to obtain the particular solution of the differential equation that is mentioned in equation (12).
Similarly, if the first four integral forms in equations (1), (2), (3) & (4) are integrated repeatedly, the solution of these integrals will also generate a partial sum of the harmonic series. Here, the number of terms in this series depends upon the number of integrations are performed. On performing a large number of integrations, it will generate the harmonic series with the large number of terms. So in this case, we can also use the asymptotic

approximation formula for the partial sum of the harmonic series, which are given by Euler (1741) and Ramanujan (1998), and this result has been widely studied in [1-6] that are mentioned below.

II. List of main symbolic notations, abbreviations, constants and important results are used in this article:

- $(a,\ b,\ c,\ d,\ p,\ q,\ r,\ s,\ m)\in\Re$
- $n\in\mathbb{N}$
- $\binom{m}{r}=\dfrac{m!}{r!(m-r)!}$ Binomial coefficient. (13)
- $\sum_{i=1}^{n} i=1+2+3+\cdots+n$ Capital Sigma notation for the summation of first n Natural numbers. (14)
- $\dfrac{1}{D^n}\cdot f(x)=\iiint\underbrace{\cdots\cdots}_{n-times}\int f(x)(dx)^n$ n^{th} Fractional integral with respect to x (15)
- $\dfrac{1}{D^p}\cdot f(x)=\iiint\underbrace{\cdots\cdots}_{p-times}\int f(x)(dx)^p$ p^{th} Fractional integral with respect to x (16)
- $\sum_{i=0}^{n-1}\dfrac{C_{i+1}x^i}{i!}$ Constant of n^{th} integration, where C is the any arbitrary constant. (17)
- $\prod_{i=1}^{n} i=1\cdot 2\cdot 3\cdots n$ Capital Pi notation for the product of first n Natural numbers. (18)
- $H_n=\sum_{k=1}^{n}\dfrac{1}{k}$ n^{th} Partial sum of the Harmonic series. (19)
- $H_p=\sum_{k=1}^{p}\dfrac{1}{k}$ p^{th} Partial sum of the Harmonic series. (20)

Here, value of H_n is calculated by following results-

- $H_n=\sum_{k=1}^{n}\dfrac{1}{k}\sim\Upsilon+\log_e(n)=\dfrac{1}{2n}-\dfrac{1}{12n^2}+\dfrac{1}{120n^4}-\cdots$ [1] (21)
- $H_n=\sum_{k=1}^{n}\dfrac{1}{k}=\Upsilon+\log_e(n)-\sum_{k\geq 1}\dfrac{B_{2k}}{2k}\cdot\dfrac{1}{n^{2k}}$ [4] (22)

Here, Υ is the Euler-Mascheroni Constant and B_{2k} are the Bernoulli numbers. Since B_{2k} grows like $\dfrac{2(2k)!}{(2k)^{2k}}$, the asymptotic expansion to given k.

- $H_n\sim\Upsilon+\dfrac{1}{2}\log_e(2m)+\dfrac{1}{12m}-\dfrac{1}{120m^2}+\dfrac{1}{630m^3}-\dfrac{1}{1680m^4}+\dfrac{1}{2310m^5}-\dfrac{191}{360360m^6}+\cdots$ [2] (23)

Where, $m=\dfrac{n(n+1)}{2}$, where m is a positive integer.

- $H_n=\sum_{k=1}^{n}\dfrac{1}{k}\sim\dfrac{1}{2}\log_e(2m)+\Upsilon+\dfrac{1}{2m}\left(\sum_{j=0}^{\infty}\dfrac{a_j(r)}{m^j}\right)^{1/r}$ [3] (24)

It's a general form of an asymptotic expansion of Ramanujan's n^{th} Harmonic number formula with a recursive recurrence relation where $m=\dfrac{n(n+1)}{2}$ is the n^{th} triangular number.

For $r=1$, we can determine the coefficients $a_j(r)$

- $H_n=\Upsilon+\dfrac{1}{2}\log_e(2P)-\dfrac{1}{720P^2}+\dfrac{1}{2835P^3}-\dfrac{5}{24192P^4}+\dfrac{r_n}{P^5}$ [4] (25)

Here, $P = m + \frac{1}{6}$, $m = n(n+1)$ where n is a positive integer and n approaches to the infinity.

$0 < r_n < \frac{37}{187110}$ and the coefficient, M_r of $\frac{1}{P_r}$ is given by $M_r = -1\frac{1}{2r6^r} + \sum_{k=0}^{r-1}\binom{r-1}{k}\frac{R_{r-k}}{6^k}$ [4] (26)

Where, R_r is the coefficient of $\frac{1}{m^r}$ in Ramanujan's expansion as mentioned above in equation (23)

The value of Υ i.e. Euler-Mascheroni Constant is calculated by following results-

- $\Upsilon = \lim_{n\to\infty}\left\{\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log_e(n)\right\}$ [1] (27)
- $\Upsilon = \lim_{n\to\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}-\frac{1}{n+1}-\frac{1}{n+2}-\cdots-\frac{1}{n^2}\right)$ [5] (28)

Here-

$\Upsilon = 0.577215664901532860606512090082402431042159335939923598805767234\cdots$ [6] (29)

III. Methodology

To solve the above integrals in equations (1) to (7), I have categorize them under three types i.e. Type-I, Type-II, and Type-III, on the basis of the product of transcendental and non-transcendental functions.

Integrals of Type-I:

$$\frac{1}{D^n} = \underbrace{\iiint\cdots\cdots\int}_{n-times}(Non-Transcendental\ function)^{\pm p/q}(dx)^n$$

In Type-I: there are two forms of integrals mentioned from equations (1) & (2)

- Integral of form 1. $\frac{1}{D^n} = \underbrace{\iiint\cdots\cdots\int}_{n-times}(ax\pm b)^{\pm p/q}(dx)^n$
- Integral of form 2. $\frac{1}{D^n} = \underbrace{\iiint\cdots\cdots\int}_{n-times}(ax^{\pm p/q}\pm b)(dx)^n$

Integrals of Type-II:

$$\frac{1}{D^n} = \underbrace{\iiint\cdots\cdots\int}_{n-times}(Non-Transcendental\ function)^{\pm p/q}\times(Transcendental\ function)^{\pm r/s}(dx)^n$$

In Type-II: there are two forms of integrals mentioned from equations (3) & (4)

- Integral of form 3. $\frac{1}{D^n} = \underbrace{\iiint\cdots\cdots\int}_{n-times}(ax\pm b)^{\pm p/q}\log_e(ax\pm b)^{\pm r/s}(dx)^n$
- Integral of form 4. $\frac{1}{D^n} = \underbrace{\iiint\cdots\cdots\int}_{n-times}(ax^{\pm p/q}\pm b)\log_e(cx^{\pm r/s})(dx)^n$

Integrals of Type-III:

$$\frac{1}{D^n} = \underbrace{\iiint\cdots\cdots\int}_{n-times}(Non-Transcendental\ function)^m\times(Transcendental\ function)(dx)^n$$

In Type-III: there are three forms of integrals mentioned from equations (5), (6) & (7)

- Integral of form 5. $\frac{1}{D^n} = \underbrace{\iiint\cdots\cdots\int}_{n-times}(ax^m\pm b)e^{(cx\pm d)}(dx)^n$
- Integral of form 6. $\frac{1}{D^n} = \underbrace{\iiint\cdots\cdots\int}_{n-times}(ax^m\pm b)\sin(cx\pm d)(dx)^n$
- Integral of form 7. $\frac{1}{D^n} = \underbrace{\iiint\cdots\cdots\int}_{n-times}(ax^m\pm b)\cos(cx\pm d)(dx)^n$

In the above mentioned integrals, all the three types that contain seven forms of integral are solved below in the result section. The solution of these seven forms of integral are also arranged in a special way in order to obtain the generalized expression along with its all the possible cases that are derived.
On successive integration of integrals that are under the category of Type-I & Type-II, the solution of these integrals will also generate a harmonic series. So in this case, we can also use the asymptotic approximation formula for the partial sum of the harmonic series which I have mentioned in [1-6].
I have also expressed each generalized expression in an aesthetic form to demonstrate its mathematical beauty and also to obtain the compact formula. For the aesthetic form for the seven forms of integrals of Type-I, Type-II & Type-III, I have used some symbolic notations that are mentioned in equations (13) to (20).

The objective of the study is to derive the general solution of n^{th} indefinite integrals of Type-I, Type-II and Type-III that are defined above. Significantly, it aims to-

1. Explore more methods for solving the integrals that have the solutions in the form of the series.
2. Reveal the divine beautiful symmetry that produces multi-dimensional recurrence relation which shows the mathematical beauty of these derived formulae in this paper.

IV. Results

Proof for the Integrals of Type-I:
Integral of form 1

1. $D^0=(ax\pm b)^{\pm p/q}$

Solution=

$$\frac{1}{D^1}=\int (ax\pm b)^{\pm p/q}dx=\frac{q^1(ax\pm b)^{(1q\pm p)/q}}{a^1(1q\pm p)}+\sum_{i=0}^{0}\frac{C_{i+1}x^i}{i!} \tag{30}$$

$$\frac{1}{D^2}=\iint (ax\pm b)^{\pm p/q}(dx)^2=\frac{q^2(ax\pm b)^{(2q\pm p)/q}}{a^2(1q\pm p)(2q\pm p)}+\sum_{i=0}^{1}\frac{C_{i+1}x^i}{i!} \tag{31}$$

$$\frac{1}{D^3}=\iiint (ax\pm b)^{\pm p/q}(dx)^3=\frac{q^3(ax\pm b)^{(3q\pm p)/q}}{a^3(1q\pm p)(2q\pm p)(3q\pm p)}+\sum_{i=0}^{2}\frac{C_{i+1}x^i}{i!} \tag{32}$$

$\vdots$

$$\frac{1}{D^n}=\iiint\underbrace{\cdots\cdots}_{n-times}\int (ax\pm b)^{\pm p/q}(dx)^n=\frac{q^n(ax\pm b)^{(nq\pm p)/q}}{a^n(1q\pm p)(2q\pm p)(3q\pm p)\cdots(nq\pm p)}+\sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!} \tag{33}$$

Aesthetic form:

$$\frac{1}{D^n}=\iiint\underbrace{\cdots\cdots}_{n-times}\int (ax\pm b)^{\pm p/q}(dx)^n=\prod_{i=1}^{n}\frac{q^n(ax\pm b)^{(nq\pm p)/q}}{a^n(iq\pm p)}+\sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!} \tag{34}$$

Here, we have 4 forms-

1. If in equation (33), $p=m$ & $q=1$

$$\frac{1}{D^n}=\iiint\underbrace{\cdots\cdots}_{n-times}\int (ax\pm b)^{m}(dx)^n=\frac{(ax\pm b)^{(n+m)}}{a^n(m+1)(m+2)(m+3)\cdots(m+n)}+\sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!} \tag{35}$$

Aesthetic form:

$$\frac{1}{D^n}=\iiint\underbrace{\cdots\cdots}_{n-times}\int (ax\pm b)^{m}(dx)^n=\prod_{i=1}^{n}\frac{(ax\pm b)^{(n+m)}}{a^n(m+i)}+\sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!} \tag{36}$$

2. If in equation (33), $p=1$ & $q=m$ (m^{th} root)

$$\frac{1}{D^n}=\iiint\underbrace{\cdots\cdots}_{n-times}\int (ax\pm b)^{1/m}(dx)^n=\frac{m^n(ax\pm b)^{(nm+1)/m}}{a^n(1m+1)(2m+1)(3m+1)\cdots(nm+1)}+\sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!} \tag{37}$$

Aesthetic form:

$$\frac{1}{D^n}=\iiint\underbrace{\cdots\cdots}_{n-times}\int (ax\pm b)^{1/m}(dx)^n=\prod_{i=1}^{n}\frac{m^n(ax\pm b)^{(nm+1)/m}}{a^n(im+1)}+\sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!} \tag{38}$$

3. If in equation (33), $p=-1$ & $q=m$ (inverse of m^{th} root)

$$\frac{1}{D^n}=\underbrace{\iiint\cdots\cdots\int}_{n-times}(ax\pm b)^{-1/m}(dx)^n=\frac{m^n(ax\pm b)^{(nm-1)/m}}{a^n(1m-1)(2m-1)(3m-1)\cdots(nm-1)}+\sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!} \quad (39)$$

Aesthetic form:

$$\frac{1}{D^n}=\underbrace{\iiint\cdots\cdots\int}_{n-times}(ax\pm b)^{-1/m}(dx)^n=\prod_{i=1}^{n}\frac{m^n(ax\pm b)^{(nm-1)/m}}{a^n(im-1)}+\sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!} \quad (40)$$

4. If in equation (33), $p/q=-m \; here \; m\in\mathbb{N}$. **In this situation we have 3 cases-**

Case 1.1- When $n<m$

$$\frac{1}{D^n}=\underbrace{\iiint\cdots\cdots\int}_{n-times}(ax\pm b)^{-m}(dx)^n=\frac{(-1)^n(ax\pm b)^{(n-m)}}{a^n(m-1)(m-2)(m-3)\cdots(m-n)}+\sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!} \quad (41)$$

Aesthetic form:

$$\frac{1}{D^n}=\underbrace{\iiint\cdots\cdots\int}_{n-times}(ax\pm b)^{-m}(dx)^n=\prod_{i=1}^{n}\frac{(-1)^n(ax\pm b)^{(n-m)}}{a^n(m-i)}+\sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!} \quad (42)$$

Case 1.2- When $n = m$

$$\frac{1}{D^n}=\underbrace{\iiint\cdots\cdots\int}_{n-times}(ax\pm b)^{-m}(dx)^n=\frac{(-1)^{m-1}\log_e(ax\pm b)}{a^n(m-1)(m-2)(m-3)\cdots\{m-(m-1)\}}+\sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!} \quad (43)$$

Aesthetic form:

$$\frac{1}{D^n}=\underbrace{\iiint\cdots\cdots\int}_{n-times}(ax\pm b)^{-m}(dx)^n=\prod_{i=1}^{(m-1)}\frac{(-1)^{m-1}\log_e(ax\pm b)}{a^n(m-i)}+\sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!} \quad (44)$$

Case 1.3- When $n > m$. Here, let say $p=(n-m)$

$$\frac{1}{D^n}=\underbrace{\iiint\cdots\cdots\int}_{n-times}(ax\pm b)^{-m}(dx)^n=\frac{(-1)^{m-1}(ax\pm b)^p\left[\log_e(ax\pm b)-\left\{\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{p}\right\}\right]}{a^n(m-1)(m-2)(m-3)\cdots\{m-(m-1)\}\cdot(1\cdot2\cdot3\cdots p)}+ \quad (45)$$

$$\sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!}$$

Aesthetic form:

$$\frac{1}{D^n}=\underbrace{\iiint\cdots\cdots\int}_{n-times}(ax\pm b)^{-m}(dx)^n=\prod_{i=1}^{(m-1)}\frac{(-1)^{m-1}(ax\pm b)^p\left[\log_e(ax\pm b)-H_p\right]}{a^n(m-i)\cdot p!}+\sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!} \quad (46)$$

Integral of form 2

2. $D^0=(ax^{\pm p/q}\pm b)$

Solution=

$$\frac{1}{D^1}=\int(ax^{\pm p/q}\pm b)dx=\frac{q^1ax^{(1q\pm p)/q}}{(1q\pm p)}\pm\frac{bx}{1}+\sum_{i=0}^{0}\frac{C_{i+1}x^i}{i!} \quad (47)$$

$$\frac{1}{D^2}=\iint(ax^{\pm p/q}\pm b)(dx)^2=\frac{q^2ax^{(2q\pm p)/q}}{(1q\pm p)(2q\pm p)}\pm\frac{bx^2}{1\cdot2}+\sum_{i=0}^{1}\frac{C_{i+1}x^i}{i!} \quad (48)$$

$$\frac{1}{D^3}=\iiint(ax^{\pm p/q}\pm b)(dx)^3=\frac{q^3ax^{(3q\pm p)/q}}{(1q\pm p)(2q\pm p)(3q\pm p)}\pm\frac{bx^3}{1\cdot2\cdot3}+\sum_{i=0}^{2}\frac{C_{i+1}x^i}{i!} \quad (49)$$

$$\vdots$$

$$\frac{1}{D^n}=\underbrace{\iiint\cdots\cdots\int}_{n-times}(ax^{\pm p/q}\pm b)(dx)^n=\frac{q^nax^{(nq\pm p)/q}}{(1q\pm p)(2q\pm p)(3q\pm p)\cdots(nq\pm p)}\pm\frac{bx^n}{n!}+\sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!} \quad (50)$$

Aesthetic form:

$$\frac{1}{D^n}=\iiint\underbrace{\cdots\cdots}_{n-times}\int(ax^{\pm p/q}\pm b)(dx)^n=\prod_{i=1}^{n}\frac{q^n ax^{(nq\pm p)/q}}{(iq\pm p)}\pm\frac{bx^n}{n!}+\sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!} \tag{51}$$

Here, we have 4 forms-

1. If in equation (50), $p=m\ \&\ q=1$

$$\frac{1}{D^n}=\iiint\underbrace{\cdots\cdots}_{n-times}\int(ax^{m}\pm b)(dx)^n=\frac{ax^{(n+m)}}{(m+1)(m+2)(m+3)\cdots(m+n)}\pm\frac{bx^n}{n!}+\sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!} \tag{52}$$

Aesthetic form:

$$\frac{1}{D^n}=\iiint\underbrace{\cdots\cdots}_{n-times}\int(ax^{m}\pm b)(dx)^n=\prod_{i=1}^{n}\frac{ax^{(n+m)}}{(m+i)}\pm\frac{bx^n}{n!}+\sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!} \tag{53}$$

2. If in equation (50), $p=1\ \&\ q=m$ (mth root)

$$\frac{1}{D^n}=\iiint\underbrace{\cdots\cdots}_{n-times}\int(ax^{1/m}\pm b)(dx)^n=\frac{m^n ax^{(nm+1)/m}}{(1m+1)(2m+1)(3m+1)\cdots(nm+1)}\pm\frac{bx^n}{n!}+\sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!} \tag{54}$$

Aesthetic form:

$$\frac{1}{D^n}=\iiint\underbrace{\cdots\cdots}_{n-times}\int(ax^{1/m}\pm b)(dx)^n=\prod_{i=1}^{n}\frac{m^n ax^{(nm+1)/m}}{(im+1)}\pm\frac{bx^n}{n!}+\sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!} \tag{55}$$

3. If in equation (50), $p=-1\ \&\ q=m$ (inverse of mth root)

$$\frac{1}{D^n}=\iiint\underbrace{\cdots\cdots}_{n-times}\int(ax^{-1/m}\pm b)(dx)^n=\frac{m^n ax^{(nm-1)/m}}{(1m-1)(2m-1)(3m-1)\cdots(nm-1)}\pm\frac{bx^n}{n!}+\sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!} \tag{56}$$

Aesthetic form:

$$\frac{1}{D^n}=\iiint\underbrace{\cdots\cdots}_{n-times}\int(ax^{-1/m}\pm b)(dx)^n=\prod_{i=1}^{n}\frac{m^n ax^{(nm-1)/m}}{(im-1)}\pm\frac{bx^n}{n!}+\sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!} \tag{57}$$

4. If in equation (50), $p/q=-m\ here\ m\in\mathbb{N}$. **In this situation we have 3 cases-**

Case 2.1- When $n<m$

$$\frac{1}{D^n}=\iiint\underbrace{\cdots\cdots}_{n-times}\int(ax^{-m}\pm b)(dx)^n=\frac{(-1)^n ax^{(n-m)}}{(m-1)(m-2)(m-3)\cdots(m-n)}\pm\frac{bx^n}{n!}+\sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!} \tag{58}$$

Aesthetic form:

$$\frac{1}{D^n}=\iiint\underbrace{\cdots\cdots}_{n-times}\int(ax^{-m}\pm b)(dx)^n=\prod_{i=1}^{n}\frac{(-1)^n ax^{(n-m)}}{(m-i)}\pm\frac{bx^n}{n!}+\sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!} \tag{59}$$

Case 2.2- When $n = m$

$$\frac{1}{D^n}=\iiint\underbrace{\cdots\cdots}_{n-times}\int(ax^{-m}\pm b)(dx)^n=\frac{(-1)^{m-1}a\log_e(x)}{(m-1)(m-2)(m-3)\cdots\{m-(m-1)\}}\pm\frac{bx^n}{n!}+\sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!} \tag{60}$$

Aesthetic form:

$$\frac{1}{D^n}=\iiint\underbrace{\cdots\cdots}_{n-times}\int(ax^{-m}\pm b)(dx)^n=\prod_{i=1}^{(m-1)}\frac{(-1)^{m-1}a\log_e(x)}{(m-i)}\pm\frac{bx^n}{n!}+\sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!} \tag{61}$$

Case 2.3- When $n > m$, Here let say $p=(n-m)$

$$\frac{1}{D^n}=\iiint\underbrace{\cdots\cdots}_{n-times}\int(ax^{-m}\pm b)(dx)^n=\frac{(-1)^{m-1}ax^p\left[\log_e x-\left\{\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{p}\right\}\right]}{(m-1)(m-2)(m-3)\cdots\{m-(m-1)\}\cdot(1\cdot2\cdot3\cdots p)}\pm \tag{62}$$
$$\frac{bx^n}{n!}+\sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!}$$

Aesthetic form:

$$\frac{1}{D^n}=\underbrace{\iiint\cdots\cdots\int}_{n-times}(ax^{-m}\pm b)(dx)^n=\prod_{i=1}^{(m-1)}\frac{(-1)^{m-1}ax^p\left[\log_e x-H_p\right]}{(m-i)\cdot p!}\pm\frac{bx^n}{n!}+\sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!} \tag{63}$$

Proof for the Integrals of Type-II:
Integral of form 3

3. $D^0=(ax\pm b)^{\pm p/q}\log_e(ax\pm b)^{\pm r/s}$

Solution=

$$\frac{1}{D^1}=\int(ax\pm b)^{\pm p/q}\log_e(ax\pm b)^{\pm r/s}dx=\pm\frac{r}{s}\left[\frac{q^1(ax\pm b)^{(1q\pm p)/q}}{a^1(1q\pm p)}\left\{\log_e(ax\pm b)-\left(\frac{q}{(1q\pm p)}\right)\right\}\right] +\sum_{i=0}^{0}\frac{C_{i+1}x^i}{i!} \tag{64}$$

$$\frac{1}{D^2}=\iint(ax\pm b)^{\pm p/q}\log_e(ax\pm b)^{\pm r/s}(dx)^2=\pm\frac{r}{s}\left[\frac{q^2(ax\pm b)^{(2q\pm p)/q}}{a^2(1q\pm p)(2q\pm p)}\left\{\log_e(ax\pm b)-\left(\frac{q}{(1q\pm p)}+\frac{q}{(2q\pm p)}\right)\right\}\right]+\sum_{i=0}^{1}\frac{C_{i+1}x^i}{i!} \tag{65}$$

$$\frac{1}{D^3}=\iiint(ax\pm b)^{\pm p/q}\log_e(ax\pm b)^{\pm r/s}(dx)^3=\pm\frac{r}{s}\left[\frac{q^3(ax\pm b)^{(3q\pm p)/q}}{a^3(1q\pm p)(2q\pm p)(3q\pm p)}\left\{\log_e(ax\pm b)-\left(\frac{q}{(1q\pm p)}+\frac{q}{(2q\pm p)}+\frac{q}{(3q\pm p)}\right)\right\}\right]+\sum_{i=0}^{2}\frac{C_{i+1}x^i}{i!} \tag{66}$$

$\vdots$

$$\frac{1}{D^n}=\underbrace{\iiint\cdots\cdots\int}_{n-times}(ax\pm b)^{\pm p/q}\log_e(ax\pm b)^{\pm r/s}(dx)^n=\pm\frac{r}{s}\left[\frac{q^n(ax\pm b)^{(nq\pm p)/q}}{a^n(1q\pm p)(2q\pm p)(3q\pm p)\cdots(nq\pm p)}\cdot\left\{\log_e(ax\pm b)-\left(\frac{q}{(1q\pm p)}+\frac{q}{(2q\pm p)}+\frac{q}{(3q\pm p)}+\cdots+\frac{q}{(nq\pm p)}\right)\right\}\right]+\sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!} \tag{67}$$

Aesthetic form:

$$\frac{1}{D^n}=\underbrace{\iiint\cdots\cdots\int}_{n-times}(ax\pm b)^{\pm p/q}\log_e(ax\pm b)^{\pm r/s}(dx)^n=\pm\frac{r}{s}\left[\prod_{i=1}^{n}\frac{q^n(ax\pm b)^{(nq\pm p)/q}}{a^n(iq\pm p)}\cdot\left\{\log_e(ax\pm b)-\sum_{i=1}^{n}\frac{q}{(iq\pm p)}\right\}\right]+\sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!} \tag{68}$$

Here, we have 4 forms-

1. If in equation (67), $p=m\ \&\ q=1$

$$\frac{1}{D^n}=\underbrace{\iiint\cdots\cdots\int}_{n-times}(ax\pm b)^m\log_e(ax\pm b)^{\pm r/s}(dx)^n=\pm\frac{r}{s}\left[\frac{(ax\pm b)^{(n+m)}}{a^n(m+1)(m+2)(m+3)\cdots(m+n)}\cdot\left\{\log_e(ax\pm b)-\left(\frac{1}{(m+1)}+\frac{1}{(m+2)}+\frac{1}{(m+3)}+\cdots+\frac{1}{(m+n)}\right)\right\}\right]+\sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!} \tag{69}$$

Aesthetic form:

$$\frac{1}{D^n}=\underbrace{\iiint\cdots\cdots\int}_{n-times}(ax\pm b)^m\log_e(ax\pm b)^{\pm r/s}(dx)^n=\pm\frac{r}{s}\left[\prod_{i=1}^{n}\frac{(ax\pm b)^{(n+m)}}{a^n(m+i)}\left\{\log_e(ax\pm b)-\right.\right.$$

$$\left.\left.\sum_{i=1}^{n}\frac{1}{(m+i)}\right\}\right]+\sum_{i=0}^{n-1}\frac{C_{i+1}x^{i}}{i!} \qquad (70)$$

2. If in equation (67), $p=1$ & $q=m$ (m[th] root)

$$\frac{1}{D^{n}}=\underbrace{\iiint\cdots\cdots\int}_{n-times}(ax\pm b)^{1/m}\log_{e}(ax\pm b)^{\pm r/s}(dx)^{n}=\pm\frac{r}{s}\left[\frac{m^{n}(ax\pm b)^{(nm+1)/m}}{a^{n}(1m+1)(2m+1)(3m+1)\cdots(nm+1)}\cdot\left\{\log_{e}(ax\pm b)-\left(\frac{m}{(1m+1)}+\frac{m}{(2m+1)}+\frac{m}{(3m+1)}+\cdots+\frac{m}{(nm+1)}\right)\right\}\right]+\sum_{i=0}^{n-1}\frac{C_{i+1}x^{i}}{i!} \qquad (71)$$

Aesthetic form:

$$\frac{1}{D^{n}}=\underbrace{\iiint\cdots\cdots\int}_{n-times}(ax\pm b)^{1/m}\log_{e}(ax\pm b)^{\pm r/s}(dx)^{n}=\pm\frac{r}{s}\left[\prod_{i=1}^{n}\frac{m^{n}(ax\pm b)^{(nm+1)/m}}{a^{n}(im+1)}\cdot\left\{\log_{e}(ax\pm b)-\sum_{i=1}^{n}\frac{m}{(im+1)}\right\}\right]+\sum_{i=0}^{n-1}\frac{C_{i+1}x^{i}}{i!} \qquad (72)$$

3. If in equation (67), $p=-1$ & $q=m$ (inverse of m[th] root)

$$\frac{1}{D^{n}}=\underbrace{\iiint\cdots\cdots\int}_{n-times}(ax\pm b)^{-1/m}\log_{e}(ax\pm b)^{\pm r/s}(dx)^{n}=\pm\frac{r}{s}\left[\frac{m^{n}(ax\pm b)^{(nm-1)/m}}{a^{n}(1m-1)(2m-1)(3m-1)\cdots(nm-1)}\cdot\left\{\log_{e}(ax\pm b)-\left(\frac{m}{(1m-1)}+\frac{m}{(2m-1)}+\frac{m}{(3m-1)}+\cdots+\frac{m}{(nm-1)}\right)\right\}\right]+\sum_{i=0}^{n-1}\frac{C_{i+1}x^{i}}{i!} \qquad (73)$$

Aesthetic form:

$$\frac{1}{D^{n}}=\underbrace{\iiint\cdots\cdots\int}_{n-times}(ax\pm b)^{-1/m}\log_{e}(ax\pm b)^{\pm r/s}(dx)^{n}=\pm\frac{r}{s}\left[\prod_{i=1}^{n}\frac{m^{n}(ax\pm b)^{(nm-1)/m}}{a^{n}(im-1)}\cdot\left\{\log_{e}(ax\pm b)-\sum_{i=1}^{n}\frac{m}{(im-1)}\right\}\right]+\sum_{i=0}^{n-1}\frac{C_{i+1}x^{i}}{i!} \qquad (74)$$

4. If in equation (67), $p/q=-m\ here\ m\in\mathbb{N}$. **In this situation we have 3 cases-**

Case 3.1- When $n<m$

$$\frac{1}{D^{n}}=\underbrace{\iiint\cdots\cdots\int}_{n-times}(ax\pm b)^{-m}\log_{e}(ax\pm b)^{\pm r/s}(dx)^{n}=\pm\frac{r}{s}\left[\frac{(-1)^{n}(ax\pm b)^{(n-m)}}{a^{n}(m-1)(m-2)(m-3)\cdots(m-n)}\cdot\left\{\log_{e}(ax\pm b)+\left(\frac{1}{(m-1)}+\frac{1}{(m-2)}+\frac{1}{(m-3)}+\cdots+\frac{1}{(m-n)}\right)\right\}\right]+\sum_{i=0}^{n-1}\frac{C_{i+1}x^{i}}{i!} \qquad (75)$$

Aesthetic form:

$$\frac{1}{D^{n}}=\underbrace{\iiint\cdots\cdots\int}_{n-times}(ax\pm b)^{-m}\log_{e}(ax\pm b)^{\pm r/s}(dx)^{n}=\pm\frac{r}{s}\left[\prod_{i=1}^{n}\frac{(-1)^{n}(ax\pm b)^{(n-m)}}{a^{n}(m-i)}\cdot\left\{\log_{e}(ax\pm b)+\sum_{i=1}^{n}\frac{1}{(m-i)}\right\}\right]+\sum_{i=0}^{n-1}\frac{C_{i+1}x^{i}}{i!} \qquad (76)$$

Case 3.2- When $n\ =\ m$

$$\frac{1}{D^{n}}=\underbrace{\iiint\cdots\cdots\int}_{n-times}(ax\pm b)^{-m}\log_{e}(ax\pm b)^{\pm r/s}(dx)^{n}=\pm\frac{r}{s}\left[\frac{(-1)^{m-1}}{a^{n}(m-1)(m-2)(m-3)\cdots\{m-(m-1)\}}\cdot\right.$$

$$\left\{\frac{\log_e{}^2(ax\pm b)}{2}+\left(\frac{\log_e(ax\pm b)}{(m-1)}+\frac{\log_e(ax\pm b)}{(m-2)}+\frac{\log_e(ax\pm b)}{(m-3)}+\cdots+\frac{\log_e(ax\pm b)}{\{m-(m-1)\}}\right)\right\}\Bigg]+\sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!} \tag{77}$$

Aesthetic form:

$$\frac{1}{D^n}=\iiint\underbrace{\cdots\cdots}_{n-times}\int(ax\pm b)^{-m}\log_e(ax\pm b)^{\pm r/s}(dx)^n=\pm\frac{r}{s}\left[\prod_{i=1}^{(m-1)}\frac{(-1)^{m-1}}{a^n(m-i)}\left\{\frac{\log_e{}^2(ax\pm b)}{2}+\sum_{i=1}^{(m-1)}\frac{\log_e(ax\pm b)}{(m-i)}\right\}\right]+\sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!} \tag{78}$$

Case 3.3- When $n > m$. Here, let say $p=(n-m)$

$$\frac{1}{D^n}=\iiint\underbrace{\cdots\cdots}_{n-times}\int(ax\pm b)^{-m}\log_e(ax\pm b)^{\pm r/s}(dx)^n=\pm\frac{r}{s}\left[\frac{(-1)^{m-1}}{a^m(m-1)(m-2)(m-3)\cdots\{m-(m-1)\}}\left\{\frac{1}{D^p}\cdot\frac{\log_e{}^2(ax\pm b)}{2}+\frac{(ax\pm b)^p}{a^p p!}\left\{\log_e(ax\pm b)-\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{p}\right)\right\}\cdot\left(\frac{1}{(m-1)}+\frac{1}{(m-2)}+\frac{1}{(m-3)}+\cdots+\frac{1}{\{m-(m-1)\}}\right)\right\}\right]+\sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!} \tag{79}$$

Aesthetic form:

$$\frac{1}{D^n}=\iiint\underbrace{\cdots\cdots}_{n-times}\int(ax\pm b)^{-m}\log_e(ax\pm b)^{\pm r/s}(dx)^n=\pm\frac{r}{s}\left[\prod_{i=1}^{(m-1)}\frac{(-1)^{m-1}}{a^m(m-i)}\left\{\frac{1}{D^p}\cdot\frac{\log_e{}^2(ax\pm b)}{2}+\sum_{i=1}^{(m-1)}\frac{(ax\pm b)^p\left\{\log_e(ax\pm b)-H_p\right\}}{a^p(m-i)\cdot p!}\right\}\right]+\sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!} \tag{80}$$

Integral of form 4

4. $D^0=(ax^{\pm p/q}\pm b)\log_e(cx^{\pm r/s})$

Solution=

$$\frac{1}{D^1}=\int(ax^{\pm p/q}\pm b)\log_e(cx^{\pm r/s})dx=\left[\frac{q^1ax^{(1q\pm p)/q}}{(1q\pm p)}\left\{\log_e cx^{\pm r/s}\mp\frac{r}{s}\cdot\left(\frac{q}{(1q\pm p)}\right)\right\}\right]\pm\left[\frac{bx^1}{1}\cdot\left\{\log_e cx^{\pm r/s}\mp\frac{r}{s}\cdot\left(\frac{1}{1}\right)\right\}\right]+\sum_{i=0}^{0}\frac{C_{i+1}x^i}{i!} \tag{81}$$

$$\frac{1}{D^2}=\iint(ax^{\pm p/q}\pm b)\log_e(cx^{\pm r/s})(dx)^2=\left[\frac{q^2ax^{(2q\pm p)/q}}{(1q\pm p)(2q\pm p)}\left\{\log_e cx^{\pm r/s}\mp\frac{r}{s}\cdot\left(\frac{q}{(1q\pm p)}+\frac{q}{(2q\pm p)}\right)\right\}\right]\pm\left[\frac{bx^2}{1\cdot 2}\left\{\log_e cx^{\pm r/s}\mp\frac{r}{s}\cdot\left(\frac{1}{1}+\frac{1}{2}\right)\right\}\right]+\sum_{i=0}^{1}\frac{C_{i+1}x^i}{i!} \tag{82}$$

$$\frac{1}{D^3}=\iiint(ax^{\pm p/q}\pm b)\log_e(cx^{\pm r/s})(dx)^3=\left[\frac{q^3ax^{(3q\pm p)/q}}{(1q\pm p)(2q\pm p)(3q\pm p)}\left\{\log_e cx^{\pm r/s}\mp\frac{r}{s}\cdot\right.\right.$$

$$\left(\frac{q}{(1q\pm p)}+\frac{q}{(2q\pm p)}+\frac{q}{(3q\pm p)}\right)\Bigg\}\Bigg]\pm\left[\frac{bx^{3}}{1\cdot 2\cdot 3}\left\{\log_{e}cx^{\pm r/s}\mp\frac{r}{s}\cdot\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}\right)\right\}\right]+\sum_{i=0}^{2}\frac{C_{i+1}x^{i}}{i!} \tag{83}$$

$\vdots$

$$\frac{1}{D^{n}}=\iiint\underbrace{\cdots\cdots}_{n-times}\int(ax^{\pm p/q}\pm b)\log_{e}(cx^{\pm r/s})(dx)^{n}=\Bigg[\frac{q^{n}ax^{(nq\pm p)/q}}{(1q\pm p)(2q\pm p)(3q\pm p)\cdots(nq\pm p)}\cdot$$
$$\left\{\log_{e}cx^{\pm r/s}\mp\frac{r}{s}\cdot\left(\frac{q}{(1q\pm p)}+\frac{q}{(2q\pm p)}+\frac{q}{(3q\pm p)}+\cdots+\frac{q}{(nq\pm p)}\right)\right\}\Bigg]\pm\Bigg[\frac{bx^{n}}{n!}\Bigg\{\log_{e}cx^{\pm r/s}\mp\frac{r}{s}\cdot \tag{84}$$
$$\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)\Bigg\}\Bigg]+\sum_{i=0}^{n-1}\frac{C_{i+1}x^{i}}{i!}$$

Aesthetic form:

$$\frac{1}{D^{n}}=\iiint\underbrace{\cdots\cdots}_{n-times}\int(ax^{\pm p/q}\pm b)\log_{e}(cx^{\pm r/s})(dx)^{n}=\Bigg[\prod_{i=1}^{n}\frac{q^{n}ax^{(nq\pm p)/q}}{(iq\pm p)}\Bigg\{\log_{e}cx^{\pm r/s}\mp\frac{r}{s}\cdot$$
$$\sum_{i=1}^{n}\frac{q}{(iq\pm p)}\Bigg\}\Bigg]\pm\left[\frac{bx^{n}}{n!}\left\{\log_{e}cx^{\pm r/s}\mp\frac{r}{s}\cdot(H_{n})\right\}\right]+\sum_{i=0}^{n-1}\frac{C_{i+1}x^{i}}{i!} \tag{85}$$

Here, we have 4 forms-

1. If in equation (84), $p=m\ \&\ q=1$

$$\frac{1}{D^{n}}=\iiint\underbrace{\cdots\cdots}_{n-times}\int(ax^{m}\pm b)\log_{e}(cx^{\pm r/s})(dx)^{n}=\Bigg[\frac{ax^{(n+m)}}{(m+1)(m+2)(m+3)\cdots(m+n)}\cdot$$
$$\left\{\log_{e}cx^{\pm r/s}\mp\frac{r}{s}\cdot\left(\frac{1}{(m+1)}+\frac{1}{(m+2)}+\frac{1}{(m+3)}+\cdots+\frac{1}{(m+n)}\right)\right\}\Bigg]\pm \tag{86}$$
$$\left[\frac{bx^{n}}{n!}\left\{\log_{e}cx^{\pm r/s}\mp\frac{r}{s}\cdot\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)\right\}\right]+\sum_{i=0}^{n-1}\frac{C_{i+1}x^{i}}{i!}$$

Aesthetic form:

$$\frac{1}{D^{n}}=\iiint\underbrace{\cdots\cdots}_{n-times}\int(ax^{m}\pm b)\log_{e}(cx^{\pm r/s})(dx)^{n}=\left[\prod_{i=1}^{n}\frac{ax^{(n+m)}}{(m+i)}\left\{\log_{e}cx^{\pm r/s}\mp\frac{r}{s}\cdot\sum_{i=1}^{n}\frac{1}{(m+i)}\right\}\right]\pm$$
$$\left[\frac{bx^{n}}{n!}\left\{\log_{e}cx^{\pm r/s}\mp\frac{r}{s}\cdot(H_{n})\right\}\right]+\sum_{i=0}^{n-1}\frac{C_{i+1}x^{i}}{i!} \tag{87}$$

2. If in equation (84), $p=1\ \&\ q=m$ (m[th] root)

$$\frac{1}{D^{n}}=\iiint\underbrace{\cdots\cdots}_{n-times}\int(ax^{1/m}\pm b)\log_{e}(cx^{\pm r/s})(dx)^{n}=\Bigg[\frac{m^{n}ax^{(nm+1)/m}}{(1m+1)(2m+1)(3m+1)\cdots(nm+1)}\cdot$$
$$\left\{\log_{e}cx^{\pm r/s}\mp\frac{r}{s}\cdot\left(\frac{m}{(1m+1)}+\frac{m}{(2m+1)}+\frac{m}{(3m+1)}+\cdots+\frac{m}{(nm+1)}\right)\right\}\Bigg]\pm\Bigg[\frac{bx^{n}}{n!}\Bigg\{\log_{e}cx^{\pm r/s}\mp\frac{r}{s}\cdot \tag{88}$$
$$\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)\Bigg\}\Bigg]+\sum_{i=0}^{n-1}\frac{C_{i+1}x^{i}}{i!}$$

Aesthetic form:

$$\frac{1}{D^{n}}=\iiint\underbrace{\cdots\cdots}_{n-times}\int(ax^{1/m}\pm b)\log_{e}(cx^{\pm r/s})(dx)^{n}=\left[\prod_{i=1}^{n}\frac{m^{n}ax^{(nm+1)/m}}{(im+1)}\left\{\log_{e}cx^{\pm r/s}\mp\frac{r}{s}\cdot\sum_{i=1}^{n}\frac{m}{(im+1)}\right\}\right]$$

$$\pm\left[\frac{bx^n}{n!}\left\{\log_e cx^{\pm r/s} \mp \frac{r}{s}\cdot\left(H_n\right)\right\}\right]+\sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!} \tag{89}$$

3. If in equation (84), $p=-1$ & $q=m$ (inverse of m[th] root)

$$\frac{1}{D^n}=\iiint\underbrace{\cdots\cdots}_{n-times}\int(ax^{-1/m}\pm b)\log_e(cx^{\pm r/s})(dx)^n=\left[\frac{m^n ax^{(nm-1)/m}}{(1m-1)(2m-1)(3m-1)\cdots(nm-1)}\cdot\right.$$
$$\left.\left\{\log_e cx^{\pm r/s}\mp\frac{r}{s}\cdot\left(\frac{m}{(1m-1)}+\frac{m}{(2m-1)}+\frac{m}{(3m-1)}+\cdots+\frac{m}{(nm-1)}\right)\right\}\right]\pm\left[\frac{bx^n}{n!}\left\{\log_e cx^{\pm r/s}\mp\frac{r}{s}\cdot\right.\right. \tag{90}$$
$$\left.\left.\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)\right\}\right]+\sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!}$$

Aesthetic form:

$$\frac{1}{D^n}=\iiint\underbrace{\cdots\cdots}_{n-times}\int(ax^{-1/m}\pm b)\log_e(cx^{\pm r/s})(dx)^n=\left[\prod_{i=1}^{n}\frac{m^n ax^{(nm-1)/m}}{(im-1)}\left\{\log_e cx^{\pm r/s}\mp\frac{r}{s}\cdot\right.\right.$$
$$\left.\left.\sum_{i=1}^{n}\frac{m}{(im-1)}\right\}\right]\pm\left[\frac{bx^n}{n!}\left\{\log_e cx^{\pm r/s}\mp\frac{r}{s}\cdot\left(H_n\right)\right\}\right]+\sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!} \tag{91}$$

4. If in equation (84), $p/q=-m$ here $m\in\mathbb{N}$. **In this situation we have 3 cases-**

Case 4.1- When $n<m$

$$\frac{1}{D^n}=\iiint\underbrace{\cdots\cdots}_{n-times}\int(ax^{-m}\pm b)\log_e(cx^{\pm r/s})(dx)^n=\left[\frac{(-1)^n ax^{(n-m)}}{(m-1)(m-2)(m-3)\cdots(m-n)}\cdot\right.$$
$$\left.\left\{\log_e cx^{\pm r/s}\pm\frac{r}{s}\cdot\left(\frac{1}{(m-1)}+\frac{1}{(m-2)}+\frac{1}{(m-3)}+\cdots+\frac{1}{\{m-(m-1)\}}\right)\right\}\right]\pm \tag{92}$$
$$\left[\frac{bx^n}{n!}\left\{\log_e cx^{\pm r/s}\mp\frac{r}{s}\cdot\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)\right\}\right]+\sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!}$$

Aesthetic form:

$$\frac{1}{D^n}=\iiint\underbrace{\cdots\cdots}_{n-times}\int(ax^{-m}\pm b)\log_e(cx^{\pm r/s})(dx)^n=\left[\prod_{i=1}^{n}\frac{(-1)^n ax^{(n-m)}}{(m-i)}\left\{\log_e cx^{\pm r/s}\pm\frac{r}{s}\cdot\right.\right.$$
$$\left.\left.\sum_{i=1}^{(m-1)}\frac{1}{(m-i)}\right\}\right]\pm\left[\frac{bx^n}{n!}\left\{\log_e cx^{\pm r/s}\mp\frac{r}{s}\cdot\left(H_n\right)\right\}\right]+\sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!} \tag{93}$$

Case 4.2- When $n = m$

$$\frac{1}{D^n}=\iiint\underbrace{\cdots\cdots}_{n-times}\int(ax^{-m}\pm b)\log_e(cx^{\pm r/s})(dx)^n=\left[\frac{(-1)^{m-1}a}{(m-1)(m-2)(m-3)\cdots\{m-(m-1)\}}\left\{\pm\frac{s}{r}\cdot\right.\right.$$
$$\left.\left.\frac{\log^2{}_e cx^{\pm r/s}}{2}\pm\frac{r}{s}\left(\frac{\log_e x}{(m-1)}+\frac{\log_e x}{(m-2)}+\frac{\log_e x}{(m-3)}+\cdots+\frac{\log_e x}{\{m-(m-1)\}}\right)\right\}\right]\pm\left[\frac{bx^n}{n!}\left\{\log_e cx^{\pm r/s}\mp\frac{r}{s}\cdot\right.\right. \tag{94}$$
$$\left.\left.\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)\right\}\right]+\sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!}$$

Aesthetic form:

$$\frac{1}{D^n}=\iiint\underbrace{\cdots\cdots}_{n-times}\int(ax^{-m}\pm b)\log_e(cx^{\pm r/s})(dx)^n=\left[\prod_{i=1}^{(m-1)}\frac{(-1)^{m-1}a}{(m-i)}\left\{\pm\frac{s}{r}\cdot\frac{\log^2{}_e cx^{\pm r/s}}{2}\pm\frac{r}{s}\cdot\right.\right.$$

$$\sum_{i=1}^{(m-1)}\frac{\log_e x}{(m-i)}\Bigg\}\Bigg]\Bigg]\pm\left[\frac{bx^n}{n!}\left\{\log_e cx^{\pm r/s}\mp\frac{r}{s}\cdot\left(H_n\right)\right\}\right]+\sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!} \tag{95}$$

Case 4.3- When $n > m$, Here let say $p=(n-m)$

$$\frac{1}{D^n}=\iiint\underbrace{\cdots\cdots}_{n-times}\int(ax^{-m}\pm b)\log_e(cx^{\pm r/s})(dx)^n=\Bigg[\frac{(-1)^{m-1}a}{(m-1)(m-2)(m-3)\cdots\{m-(m-1)\}}\Bigg\{\pm\frac{s}{r}\cdot$$

$$\frac{1}{D^p}\cdot\frac{\log^2{}_e cx^{\pm r/s}}{2}\pm\frac{r}{s}\cdot\frac{x^p}{p!}\left(\log_e x-\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{p}\right)\right)\left(\frac{1}{(m-1)}+\frac{1}{(m-2)}+\frac{1}{(m-3)}+\cdots+\right. \tag{96}$$

$$\left.\frac{1}{\{m-(m-1)\}}\right)\Bigg\}\Bigg]\pm\left[\frac{bx^n}{n!}\left\{\log_e cx^{\pm r/s}\mp\frac{r}{s}\cdot\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)\right\}\right]+\sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!}$$

Aesthetic form:

$$\frac{1}{D^n}=\iiint\underbrace{\cdots\cdots}_{n-times}\int(ax^{-m}\pm b)\log_e(cx^{\pm r/s})(dx)^n=\Bigg[\prod_{i=1}^{(m-1)}\frac{(-1)^{m-1}a}{(m-i)}\Bigg\{\pm\frac{s}{r}\cdot\frac{1}{D^p}\cdot\frac{\log^2{}_e cx^{\pm r/s}}{2}\pm\frac{r}{s}\cdot$$

$$\sum_{i=1}^{(m-1)}\frac{x^p\left(\log_e x-H_p\right)}{(m-i)\cdot p!}\Bigg\}\Bigg]\pm\left[\frac{bx^n}{n!}\left\{\log_e cx^{\pm r/s}\mp\frac{r}{s}\cdot\left(H_n\right)\right\}\right]+\sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!} \tag{97}$$

Proof for the Integrals of Type-III:

Integral of form 5- Initially, I have found the expression for the n^{th} integral of these integrands $(ax^0\pm b)e^{(cx\pm d)}$, $(ax^1\pm b)e^{(cx\pm d)}$, $(ax^2\pm b)e^{(cx\pm d)}$, & $(ax^3\pm b)e^{(cx\pm d)}$ that are solved below-

$$D^0=(ax^0\pm b)e^{(cx\pm d)}$$

Solution=

$$\frac{1}{D^1}=\int(ax^0\pm b)e^{(cx\pm d)}dx=\left[\frac{e^{(cx\pm d)}(ax^0\pm b)}{c^{0+1}}\right]+\sum_{i=0}^{0}\frac{C_{i+1}x^i}{i!} \tag{98}$$

$$\frac{1}{D^2}=\iint(ax^0\pm b)e^{(cx\pm d)}(dx)^2=\left[\frac{e^{(cx\pm d)}(ax^0\pm b)}{c^{0+2}}\right]+\sum_{i=0}^{1}\frac{C_{i+1}x^i}{i!} \tag{99}$$

$$\frac{1}{D^3}=\iiint(ax^0\pm b)e^{(cx\pm d)}(dx)^3=\left[\frac{e^{(cx\pm d)}(ax^0\pm b)}{c^{0+3}}\right]+\sum_{i=0}^{2}\frac{C_{i+1}x^i}{i!} \tag{100}$$

$$\vdots$$

$$\frac{1}{D^n}=\iiint\underbrace{\cdots\cdots}_{n-times}\int(ax^0\pm b)e^{(cx\pm d)}(dx)^n=\left[\frac{e^{(cx\pm d)}(ax^0\pm b)}{c^{0+n}}\right]+\sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!} \tag{101}$$

Combination form:

$$\frac{1}{D^n}=\iiint\underbrace{\cdots\cdots}_{n-times}\int(ax^0\pm b)e^{(cx\pm d)}(dx)^n=\frac{e^{(cx\pm d)}}{c^{0+n}}\left[c^0(ax^0\pm b)(-1)^0\binom{0}{0}\right]+\sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!} \tag{102}$$

$$D^0=(ax^1\pm b)e^{(cx\pm d)}$$

Solution=

$$\frac{1}{D^1}=\int(ax^1\pm b)e^{(cx\pm d)}dx=\left[\frac{e^{(cx\pm d)}\{c(ax^1\pm b)-a\}}{c^{1+1}}\right]+\sum_{i=0}^{0}\frac{C_{i+1}x^i}{i!} \tag{103}$$

$$\frac{1}{D^2}=\iint(ax^1\pm b)e^{(cx\pm d)}(dx)^2=\left[\frac{e^{(cx\pm d)}\{c(ax^1\pm b)-2a\}}{c^{1+2}}\right]+\sum_{i=0}^{1}\frac{C_{i+1}x^i}{i!} \tag{104}$$

$$\frac{1}{D^3}=\iiint(ax^1\pm b)e^{(cx\pm d)}(dx)^3=\left[\frac{e^{(cx\pm d)}\{c(ax^1\pm b)-3a\}}{c^{1+3}}\right]+\sum_{i=0}^{2}\frac{C_{i+1}x^i}{i!} \tag{105}$$

$$\vdots$$

$$\frac{1}{D^n}=\iiint\underbrace{\cdots\cdots}_{n-times}\int(ax^1\pm b)e^{(cx\pm d)}(dx)^n=\left[\frac{e^{(cx\pm d)}\{c(ax^1\pm b)-na\}}{c^{1+n}}\right]+\sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!} \tag{106}$$

Combination form:

$$\frac{1}{D^n}=\iiint\underbrace{\cdots\cdots}_{n-times}\int(ax^1\pm b)e^{(cx\pm d)}(dx)^n=\frac{e^{(cx\pm d)}}{c^{1+n}}\left[c^1(ax^1\pm b)(-1)^0\binom{1}{0}+na(cx)^0(-1)^1\binom{1}{1}\right]+\sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!} \tag{107}$$

$$D^0=(ax^2\pm b)e^{(cx\pm d)}$$

Solution=

$$\frac{1}{D^1}=\int(ax^2\pm b)e^{(cx\pm d)}dx=\left[\frac{e^{(cx\pm d)}\{c^2(ax^2\pm b)-2acx+2a\}}{c^{2+1}}\right]+\sum_{i=0}^{0}\frac{C_{i+1}x^i}{i!} \tag{108}$$

$$\frac{1}{D^2}=\iint(ax^2\pm b)e^{(cx\pm d)}(dx)^2=\left[\frac{e^{(cx\pm d)}\{c^2(ax^2\pm b)-4acx+6a\}}{c^{2+2}}\right]+\sum_{i=0}^{1}\frac{C_{i+1}x^i}{i!} \tag{109}$$

$$\frac{1}{D^3}=\iiint(ax^2\pm b)e^{(cx\pm d)}(dx)^3=\left[\frac{e^{(cx\pm d)}\{c^2(ax^2\pm b)-6acx+12a\}}{c^{2+3}}\right]+\sum_{i=0}^{2}\frac{C_{i+1}x^i}{i!} \tag{110}$$

$$\vdots$$

$$\frac{1}{D^n}=\iiint\underbrace{\cdots\cdots}_{n-times}\int(ax^2\pm b)e^{(cx\pm d)}(dx)^n=\left[\frac{e^{(cx\pm d)}\{c^2(ax^2\pm b)-2nacx+n(n+1)a\}}{c^{2+n}}\right]+\sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!} \tag{111}$$

Combination form:

$$\frac{1}{D^n}=\iiint\underbrace{\cdots\cdots}_{n-times}\int(ax^2\pm b)e^{(cx\pm d)}(dx)^n=\frac{e^{(cx\pm d)}}{c^{2+n}}\left[c^2(ax^2\pm b)(-1)^0\binom{2}{0}+na(cx)^1(-1)^1\binom{2}{1}+n(n+1)a(cx)^0(-1)^2\binom{2}{2}\right]+\sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!} \tag{112}$$

$$D^0=(ax^3\pm b)e^{(cx\pm d)}$$

Solution=

$$\frac{1}{D^1}=\int(ax^3\pm b)e^{(cx\pm d)}dx=\left[\frac{e^{(cx\pm d)}\{c^3(ax^3\pm b)-3ac^2x^2+6acx-6a\}}{c^{3+1}}\right]+\sum_{i=0}^{0}\frac{C_{i+1}x^i}{i!} \tag{113}$$

$$\frac{1}{D^2}=\iint(ax^3\pm b)e^{(cx\pm d)}(dx)^2=\left[\frac{e^{(cx\pm d)}\{c^3(ax^3\pm b)-6ac^2x^2+18acx-24a\}}{c^{3+2}}\right]+\sum_{i=0}^{1}\frac{C_{i+1}x^i}{i!} \tag{114}$$

$$\frac{1}{D^3}=\iiint (ax^3\pm b)e^{(cx\pm d)}(dx)^3=\left[\frac{e^{(cx\pm d)}\{c^3(ax^3\pm b)-9ac^2x^2+36acx-60a\}}{c^{3+3}}\right]+\sum_{i=0}^{2}\frac{C_{i+1}x^i}{i!} \tag{115}$$

$\vdots$

$$\frac{1}{D^n}=\iiint \underbrace{\cdots\cdots}_{n-times}\int (ax^3\pm b)e^{(cx\pm d)}(dx)^n=\left[\frac{e^{(cx\pm d)}\{c^3(ax^3\pm b)-3nac^2x^2+3n(n+1)acx-n(n+1)(n+2)a\}}{c^{3+n}}\right]+\sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!} \tag{116}$$

Combination form:

$$\frac{1}{D^n}=\iiint \underbrace{\cdots\cdots}_{n-times}\int (ax^3\pm b)e^{(cx\pm d)}(dx)^n=\frac{e^{(cx\pm d)}}{c^{3+n}}\left[c^3(ax^3\pm b)(-1)^0\binom{3}{0}+na(cx)^2(-1)^1\binom{3}{1}+n(n+1)a(cx)^1(-1)^2\binom{3}{2}+n(n+1)(n+2)a(cx)^0(-1)^3\binom{3}{3}\right]+\sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!} \tag{117}$$

Now to get the expression for the n^{th} integral of $(ax^m\pm b)e^{(cx\pm d)}$, I did some analysis in all the solutions of integrals that are solved above in equations (102), (107), (112) & (117). Let us consider a Pascal's triangle having rows starting from 0^{th} row to m^{th} row with elements in the form of binomial coefficients.

$$\begin{array}{ccccccccccccccccc}
 & & & & 1 & & & & \overset{0^{th}}{Row} & & & & \binom{0}{0} & & & & \\
 & & & 1 & & 1 & & & \overset{1^{st}}{Row} & & & \binom{1}{0} & & \binom{1}{1} & & & \\
 & & 1 & & 2 & & 1 & & \overset{2^{nd}}{Row} & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\
 & 1 & & 3 & & 3 & & 1 & \overset{3^{rd}}{Row} & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\
\iddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
\binom{m}{0} & \binom{m}{1} & \binom{m}{2} & \binom{m}{3} & \cdots & \binom{m}{m} & & & \overset{m^{th}}{Row} & \binom{m}{0} & \binom{m}{1} & \binom{m}{2} & \binom{m}{3} & \cdots & \binom{m}{m} & &
\end{array} \tag{118}$$

Now in equation (118), I have introduced a sequence diagonally:

$1,\ n,\ n(n+1),\ n(n+1)(n+2),\ \cdots,\ n(n+1)(n+2)(n+3)\cdots\{n+(m-1)\}$ and we get-

$$\begin{array}{cccccccccc}
 & & & & \swarrow 1 & & & & & \\
 & & & \binom{0}{0} & & \swarrow n & & & & \\
 & & \binom{1}{0} & & \binom{1}{1} & & \swarrow n(n+1) & & & \\
 & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \swarrow n(n+1)(n+2) & & \\
\binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & & \\
\iddots\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \swarrow n(n+1)(n+2)(n+3)\cdots\{n+(m-1)\} & \\
\binom{m}{0} & \binom{m}{1} & \binom{m}{2} & \binom{m}{3} & \cdots & \binom{m}{m} & & & &
\end{array} \tag{119}$$

After introducing this sequence in equation (119), I have multiplied the entire diagonal elements by the respective terms of this sequence and we get-

$$\begin{array}{cccccccccc}
 & & & 1\binom{0}{0} & & & & & & \\
 & & 1\binom{1}{0} & & n\binom{1}{1} & & & & & \\
 & 1\binom{2}{0} & & n\binom{2}{1} & & n(n+1)\binom{2}{2} & & & & \\
1\binom{3}{0} & & n\binom{3}{1} & & n(n+1)\binom{3}{2} & & n(n+1)(n+2)\binom{3}{3} & & & \\
\iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \\
1\binom{m}{0} & n\binom{m}{1} & n(n+1)\binom{m}{2} & & n(n+1)(n+2)\binom{m}{3} & & \cdots & & n(n+1)(n+2)(n+3)\cdots\{n+(m-1)\}\binom{m}{m} &
\end{array} \tag{120}$$

Now again in equation (120), I have introduced a sequence diagonally:
$c^m(ax^m \pm b),\ (cx)^{m-1},\ (cx)^{m-2},\ (cx)^{m-3},\ \cdots\ ,\ (cx)^{m-m}$ and we get-

$$\begin{array}{cccccccccc}
 & & & & \swarrow c^m(ax^m \pm b) & & & & & \\
 & & & 1\binom{0}{0} & & \swarrow a(cx)^{m-1} & & & & \\
 & & 1\binom{1}{0} & & n\binom{1}{1} & & \swarrow a(cx)^{m-2} & & & \\
 & 1\binom{2}{0} & & n\binom{2}{1} & & n(n+1)\binom{2}{2} & & \swarrow a(cx)^{m-3} & & \\
1\binom{3}{0} & & n\binom{3}{1} & & n(n+1)\binom{3}{2} & & n(n+1)(n+2)\binom{3}{3} & & & \\
\iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \swarrow a(cx)^{m-m} \\
1\binom{m}{0} & n\binom{m}{1} & n(n+1)\binom{m}{2} & & n(n+1)(n+2)\binom{m}{3} & & \cdots & & n(n+1)(n+2)(n+3)\cdots\{n+(m-1)\}\binom{m}{m} &
\end{array} \tag{121}$$

After introducing this sequence in equation (121), I have multiplied the entire diagonal elements by the respective terms of this sequence and we get-

$$\begin{array}{ccccccccc}
 & & & 1c^m(ax^m \pm b)\binom{0}{0} & & & & & \\
 & & 1c^m(ax^m \pm b)\binom{1}{0} & & na(cx)^{m-1}\binom{1}{1} & & & & \\
 & 1c^m(ax^m \pm b)\binom{2}{0} & & na(cx)^{m-1}\binom{2}{1} & & n(n+1)a(cx)^{m-2}\binom{2}{2} & & & \\
1c^m(ax^m \pm b)\binom{3}{0} & & na(cx)^{m-1}\binom{3}{1} & & n(n+1)a(cx)^{m-2}\binom{3}{2} & & n(n+1)(n+2)a(cx)^{m-3}\binom{3}{3} & & \\
\iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
1c^m(ax^m \pm b)\binom{m}{0} & na(cx)^{m-1}\binom{m}{1} & n(n+1)a(cx)^{m-2}\binom{m}{2} & & n(n+1)(n+2)a(cx)^{m-3}\binom{m}{3} & & \cdots & & n(n+1)(n+2)(n+3)\cdots\{n+(m-1)\}a(cx)^{m-m}\binom{m}{m}
\end{array} \tag{122}$$

Now again in equation (122), I have introduced a sequence: $(-1)^0,\ (-1)^1,\ (-1)^2,\ (-1)^3,\ \cdots\ ,\ (-1)^m$ and we get-

$$\begin{array}{cccccccccc}
 & & & & \swarrow(-1)^0 & & & & & \\
 & & & 1c^m(ax^m \pm b)\binom{0}{0} & & \swarrow(-1)^1 & & & & \\
 & & 1c^m(ax^m \pm b)\binom{1}{0} & & na(cx)^{m-1}\binom{1}{1} & & \swarrow(-1)^2 & & & \\
 & 1c^m(ax^m \pm b)\binom{2}{0} & & na(cx)^{m-1}\binom{2}{1} & & n(n+1)a(cx)^{m-2}\binom{2}{2} & & \swarrow(-1)^3 & & \\
1c^m(ax^m \pm b)\binom{3}{0} & & na(cx)^{m-1}\binom{3}{1} & & n(n+1)a(cx)^{m-2}\binom{3}{2} & & n(n+1)(n+2)a(cx)^{m-3}\binom{3}{3} & & & \\
\iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \swarrow(-1)^m \\
1c^m(ax^m \pm b)\binom{m}{0} & na(cx)^{m-1}\binom{m}{1} & n(n+1)a(cx)^{m-2}\binom{m}{2} & & n(n+1)(n+2)a(cx)^{m-3}\binom{m}{3} & & \cdots & & n(n+1)(n+2)(n+3)\cdots\{n+(m-1)\}a(cx)^{m-m}\binom{m}{m} &
\end{array} \tag{123}$$

After introducing this sequence in equation (123), I have multiplied the entire diagonal elements by the respective terms of this sequence and we get-

$$\begin{array}{ccccccccc}
 & & & 1c^m(ax^m \pm b)(-1)^0\binom{0}{0} & & & & & \\
 & & 1c^m(ax^m \pm b)(-1)^0\binom{1}{0} & & na(cx)^{m-1}(-1)^1\binom{1}{1} & & & & \\
 & 1c^m(ax^m \pm b)(-1)^0\binom{2}{0} & & na(cx)^{m-1}(-1)^1\binom{2}{1} & & n(n+1)a(cx)^{m-2}(-1)^2\binom{2}{2} & & & \\
1c^m(ax^m \pm b)(-1)^0\binom{3}{0} & & na(cx)^{m-1}(-1)^1\binom{3}{1} & & n(n+1)a(cx)^{m-2}(-1)^2\binom{3}{2} & & n(n+1)(n+2)a(cx)^{m-3}(-1)^3\binom{3}{3} & & \\
\iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
1c^m(ax^m \pm b)(-1)^0\binom{m}{0} & na(cx)^{m-1}(-1)^1\binom{m}{1} & n(n+1)a(cx)^{m-2}(-1)^2\binom{m}{2} & & n(n+1)(n+2)a(cx)^{m-3}(-1)^3\binom{m}{3} & & \cdots & & n(n+1)(n+2)(n+3)\cdots\{n+(m-1)\}a(cx)^{m-m}(-1)^m\binom{m}{m}
\end{array} \tag{124}$$

Now in equation (124), we can observe that, on putting $m = 0,\ 1,\ 2,\ 3,\ \cdots\ ,\ m$ in $0^{th}\,Row,\ 1^{st}\,Row,\ 2^{nd}\,Row,\ 3^{rd}\,Row,\ \cdots\ ,\ m^{th}\,Row$ respectively. Then each row of Pascal's triangle is representing the coefficients of n^{th} indefinite integral of the following integrands: $(ax^0 \pm b)e^{(cx\pm d)},\ (ax^1 \pm b)e^{(cx\pm d)},\ (ax^2 \pm b)e^{(cx\pm d)},\ (ax^3 \pm b)e^{(cx\pm d)},\ \cdots\ ,\ (ax^m \pm b)e^{(cx\pm d)}$

$$\begin{array}{c}
0^{th}\,Row \Rightarrow \frac{1}{D^n} = \iiint \underbrace{\cdots\cdots}_{n-times} \int (ax^0 \pm b)e^{(cx\pm d)}(dx)^n = \frac{e^{(cx\pm d)}}{c^{0+n}}\left[c^0(ax^0\pm b)(-1)^0\binom{0}{0}\right] + \sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!} \\
1^{st}\,Row \Rightarrow \frac{1}{D^n} = \iiint \underbrace{\cdots\cdots}_{n-times} \int (ax^1 \pm b)e^{(cx\pm d)}(dx)^n = \frac{e^{(cx\pm d)}}{c^{1+n}}\left[c^1(ax^1\pm b)(-1)^0\binom{1}{0} + na(cx)^0(-1)^1\binom{1}{1}\right] + \sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!} \\
2^{nd}\,Row \Rightarrow \frac{1}{D^n} = \iiint \underbrace{\cdots\cdots}_{n-times} \int (ax^2 \pm b)e^{(cx\pm d)}(dx)^n = \frac{e^{(cx\pm d)}}{c^{2+n}}\left[c^2(ax^2\pm b)(-1)^0\binom{2}{0} + na(cx)^1(-1)^1\binom{2}{1} + n(n+1)a(cx)^0(-1)^2\binom{2}{2}\right] + \sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!} \\
3^{rd}\,Row \Rightarrow \frac{1}{D^n} = \iiint \underbrace{\cdots\cdots}_{n-times} \int (ax^3 \pm b)e^{(cx\pm d)}(dx)^n = \frac{e^{(cx\pm d)}}{c^{3+n}}\left[c^3(ax^3\pm b)(-1)^0\binom{3}{0} + na(cx)^2(-1)^1\binom{3}{1} + n(n+1)a(cx)^1(-1)^2\binom{3}{2} + n(n+1)(n+2)a(cx)^0(-1)^3\binom{3}{3}\right] + \sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!} \\
\vdots \\
m^{th}\,Row \Rightarrow \frac{1}{D^n} = \iiint \underbrace{\cdots\cdots}_{n-times} \int (ax^m \pm b)e^{(cx\pm d)}(dx)^n = \frac{e^{(cx\pm d)}}{c^{m+n}}\left[c^m(ax^m\pm b)(-1)^0\binom{m}{0} + na(cx)^{m-1}(-1)^1\binom{m}{1} + n(n+1)a(cx)^{m-2}(-1)^2\binom{m}{2} + n(n+1)(n+2)a(cx)^{m-3}(-1)^3\binom{m}{3} + \cdots + n(n+1)(n+2)(n+3)\cdots\{n+(m-1)\}a(cx)^{m-m}(-1)^m\binom{m}{m}\right] + \sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!}
\end{array} \tag{125}$$

On rewriting the solutions of n^{th} integral from equations (102), (107), (112) & (117) in form of Pascal's triangle as shown above in equation (125), a beautiful pattern generates. Hence, on comparing all the elements of the $m^{th} Row$ of Pascal's triangle, I have concluded that these elements are the coefficients of the generalized expression for the n^{th} Integral of form 5: $(ax^m \pm b)e^{(cx\pm d)}$ as discussed below.

5. $D^0 = (ax^m \pm b)e^{(cx\pm d)}$

Solution=

$$\frac{1}{D^n} = \iiint \underbrace{\cdots\cdots}_{n-times} \int (ax^m \pm b)e^{(cx\pm d)}(dx)^n = \frac{e^{(cx\pm d)}}{c^{m+n}}\left[c^m(ax^m\pm b)(-1)^0\binom{m}{0} + na(cx)^{m-1}(-1)^1 \cdot \binom{m}{1} + n(n+1)a(cx)^{m-2}(-1)^2\binom{m}{2} + n(n+1)(n+2)a(cx)^{m-3}(-1)^3\binom{m}{3} + \cdots + n(n+1)(n+2)\cdot (n+3)\cdots\{n+(m-1)\}a(cx)^{m-m}(-1)^m\binom{m}{m}\right] + \sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!} \tag{126}$$

Here, from the equation (126), we can also determine the coefficient of any x^q, where $q = (m-p)$ and $q \le m$.

For the any term of x^q where $q = (m-p)$

If $0 \le p \le 2$, then the solution will be:

$$p = 0 \Rightarrow \frac{e^{(cx\pm d)}}{c^{(m+n)}}\left[c^q(ax^m \pm b)(-1)^0\binom{m}{0}\right] \tag{127}$$

$$p = 1 \Rightarrow \frac{e^{(cx\pm d)}}{c^{(m+n)}}\left[na(cx)^q(-1)^1\binom{m}{1}\right] \tag{128}$$

$$p = 2 \Rightarrow \frac{e^{(cx\pm d)}}{c^{(m+n)}}\left[n(n+1)a(cx)^q(-1)^2\binom{m}{2}\right] \tag{129}$$

If $3 \le p \le m$, then the solution will be:

$$3 \le p \le m \Rightarrow \frac{e^{(cx\pm d)}}{c^{(m+n)}}\left[n(n+1)(n+2)(n+3)\cdots\{n+(p-1)\}a(cx)^q(-1)^p\binom{m}{p}\right] \tag{130}$$

Aesthetic form:

$$3 \le p \le m \Rightarrow \frac{e^{(cx\pm d)}}{c^{(m+n)}}\left[\prod_{i=0}^{(p-1)}(n+i)a(cx)^q(-1)^p\binom{m}{p}\right] \tag{131}$$

Integral of form 6- Initially, I have found the expression for the n^{th} integral of these integrands $(ax^0 \pm b)\sin(cx \pm d)$, $(ax^1 \pm b)\sin(cx \pm d)$, $(ax^2 \pm b)\sin(cx \pm d)$, & $(ax^3 \pm b)\sin(cx \pm d)$ that are solved below-

$$D^0 = (ax^0 \pm b)\sin(cx \pm d)$$

Solution=

$$\frac{1}{D^1} = \int (ax^0 \pm b)\sin(cx \pm d)dx = \frac{(-1)^1}{c^{0+1}}\left[(ax^0 \pm b)\sin\left\{\frac{\pi}{2} + (cx \pm d)\right\}\right] + \sum_{i=0}^{0}\frac{C_{i+1}x^i}{i!} \tag{132}$$

$$\frac{1}{D^2} = \iint (ax^0 \pm b)\sin(cx \pm d)(dx)^2 = \frac{(-1)^2}{c^{0+2}}\left[(ax^0 \pm b)\sin\left\{\frac{2\pi}{2} + (cx \pm d)\right\}\right] + \sum_{i=0}^{1}\frac{C_{i+1}x^i}{i!} \tag{133}$$

$$\frac{1}{D^3} = \iiint (ax^0 \pm b)\sin(cx \pm d)(dx)^3 = \frac{(-1)^3}{c^{0+3}}\left[(ax^0 \pm b)\sin\left\{\frac{3\pi}{2} + (cx \pm d)\right\}\right] + \sum_{i=0}^{2}\frac{C_{i+1}x^i}{i!} \tag{134}$$

$\vdots$

$$\frac{1}{D^n} = \iiint \underbrace{\cdots\cdots}_{n-times}\int (ax^0 \pm b)\sin(cx \pm d)(dx)^n = \frac{(-1)^n}{c^{0+n}}\left[(ax^0 \pm b)\sin\left\{\frac{n\pi}{2} + (cx \pm d)\right\}\right] + \sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!} \tag{135}$$

Combination form:

$$\frac{1}{D^n} = \iiint \underbrace{\cdots\cdots}_{n-times}\int (ax^0 \pm b)\sin(cx \pm d)(dx)^n = \frac{(-1)^n}{c^{0+n}}\left[c^0(ax^0 \pm b)\binom{0}{0}\sin\left\{\frac{n\pi}{2} + (cx \pm d)\right\}\right] + \sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!} \tag{136}$$

$$D^0 = (ax^1 \pm b)\sin(cx \pm d)$$

Solution=

$$\frac{1}{D^1} = \int (ax^1 \pm b)\sin(cx \pm d)dx = \frac{(-1)^1}{c^{1+1}}\left[c^1(ax^1 \pm b)\sin\left\{\frac{\pi}{2} + (cx \pm d)\right\} + 1a \cdot \sin\left\{\frac{2\pi}{2} + (cx \pm d)\right\}\right] + \sum_{i=0}^{0}\frac{C_{i+1}x^i}{i!} \tag{137}$$

$$\frac{1}{D^2} = \iint (ax^1 \pm b)\sin(cx \pm d)(dx)^2 = \frac{(-1)^2}{c^{1+2}}\left[c^1(ax^1 \pm b)\sin\left\{\frac{2\pi}{2} + (cx \pm d)\right\} + 2a \cdot \sin\left\{\frac{3\pi}{2} + (cx \pm d)\right\}\right] + \sum_{i=0}^{1}\frac{C_{i+1}x^i}{i!} \tag{138}$$

$$\frac{1}{D^3} = \iiint (ax^1 \pm b)\sin(cx \pm d)(dx)^3 = \frac{(-1)^3}{c^{1+3}}\left[c^1(ax^1 \pm b)\sin\left\{\frac{3\pi}{2} + (cx \pm d)\right\} + 3a \cdot \sin\left\{\frac{4\pi}{2} + (cx \pm d)\right\}\right] + \sum_{i=0}^{2}\frac{C_{i+1}x^i}{i!} \tag{139}$$

$\vdots$

$$\frac{1}{D^n} = \iiint \underbrace{\cdots\cdots}_{n-times}\int (ax^1 \pm b)\sin(cx \pm d)(dx)^n = \frac{(-1)^n}{c^{1+n}}\left[c^1(ax^1 \pm b)\sin\left\{\frac{n\pi}{2} + (cx \pm d)\right\} + na \cdot \sin\left\{\frac{(n+1)\pi}{2} + (cx \pm d)\right\}\right] + \sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!} \tag{140}$$

Combination form:

$$\frac{1}{D^n}=\iiint\underbrace{\cdots\cdots}_{n-times}\int(ax^1\pm b)\sin(cx\pm d)(dx)^n=\frac{(-1)^n}{c^{1+n}}\left[c^1(ax^1\pm b)\binom{1}{0}\sin\left\{\frac{n\pi}{2}+(cx\pm d)\right\}+ na(cx)^0\binom{1}{1}\sin\left\{\frac{(n+1)\pi}{2}+(cx\pm d)\right\}\right]+\sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!} \tag{141}$$

$D^0=(ax^2\pm b)\sin(cx\pm d)$

Solution=

$$\frac{1}{D^1}=\int(ax^2\pm b)\sin(cx\pm d)dx=\frac{(-1)^1}{c^{2+1}}\left[c^2(ax^2\pm b)\sin\left\{\frac{\pi}{2}+(cx\pm d)\right\}+2acx\cdot \sin\left\{\frac{2\pi}{2}+(cx\pm d)\right\}+2a\sin\left\{\frac{3\pi}{2}+(cx\pm d)\right\}\right]+\sum_{i=0}^{0}\frac{C_{i+1}x^i}{i!} \tag{142}$$

$$\frac{1}{D^2}=\iint(ax^2\pm b)\sin(cx\pm d)(dx)^2=\frac{(-1)^2}{c^{2+2}}\left[c^2(ax^2\pm b)\sin\left\{\frac{2\pi}{2}+(cx\pm d)\right\}+4acx\cdot \sin\left\{\frac{3\pi}{2}+(cx\pm d)\right\}+6a\sin\left\{\frac{4\pi}{2}+(cx\pm d)\right\}\right]+\sum_{i=0}^{1}\frac{C_{i+1}x^i}{i!} \tag{143}$$

$$\frac{1}{D^3}=\iiint(ax^2\pm b)\sin(cx\pm d)(dx)^3=\frac{(-1)^3}{c^{2+3}}\left[c^2(ax^2\pm b)\sin\left\{\frac{3\pi}{2}+(cx\pm d)\right\}+6acx\cdot \sin\left\{\frac{4\pi}{2}+(cx\pm d)\right\}+12a\sin\left\{\frac{5\pi}{2}+(cx\pm d)\right\}\right]+\sum_{i=0}^{2}\frac{C_{i+1}x^i}{i!} \tag{144}$$

$\vdots$

$$\frac{1}{D^n}=\iiint\underbrace{\cdots\cdots}_{n-times}\int(ax^2\pm b)\sin(cx\pm d)(dx)^n=\frac{(-1)^n}{c^{2+n}}\left[c^2(ax^2\pm b)\sin\left\{\frac{n\pi}{2}+(cx\pm d)\right\}+ 2nacx\sin\left\{\frac{(n+1)\pi}{2}+(cx\pm d)\right\}+n(n+1)a\sin\left\{\frac{(n+2)\pi}{2}+(cx\pm d)\right\}\right]+\sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!} \tag{145}$$

Combination form:

$$\frac{1}{D^n}=\iiint\underbrace{\cdots\cdots}_{n-times}\int(ax^2\pm b)\sin(cx\pm d)(dx)^n=\frac{(-1)^n}{c^{2+n}}\left[c^2(ax^2\pm b)\binom{2}{0}\sin\left\{\frac{n\pi}{2}+(cx\pm d)\right\}+ na(cx)^1\binom{2}{1}\sin\left\{\frac{(n+1)\pi}{2}+(cx\pm d)\right\}+n(n+1)a(cx)^0\binom{2}{2}\sin\left\{\frac{(n+2)\pi}{2}+(cx\pm d)\right\}\right]+ \sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!} \tag{146}$$

$D^0=(ax^3\pm b)\sin(cx\pm d)$

Solution=

$$\frac{1}{D^1}=\int(ax^3\pm b)\sin(cx\pm d)dx=\frac{(-1)^1}{c^{3+1}}\left[c^3(ax^3\pm b)\sin\left\{\frac{\pi}{2}+(cx\pm d)\right\}+3a(cx)^2\cdot \sin\left\{\frac{2\pi}{2}+(cx\pm d)\right\}+6acx\sin\left\{\frac{3\pi}{2}+(cx\pm d)\right\}+6a\sin\left\{\frac{4\pi}{2}+(cx\pm d)\right\}\right]+C\sum_{i=0}^{0}\frac{x^i}{i!} \tag{147}$$

$$\frac{1}{D^2}=\iint (ax^3\pm b)\sin(cx\pm d)(dx)^2=\frac{(-1)^2}{c^{3+2}}\left[c^3(ax^3\pm b)\sin\left\{\frac{2\pi}{2}+(cx\pm d)\right\}+6a(cx)^2\cdot\right.$$
$$\left.\sin\left\{\frac{3\pi}{2}+(cx\pm d)\right\}+18acx\sin\left\{\frac{4\pi}{2}+(cx\pm d)\right\}+24a\sin\left\{\frac{5\pi}{2}+(cx\pm d)\right\}\right]+\sum_{i=0}^{1}\frac{C_{i+1}x^i}{i!} \tag{148}$$

$$\frac{1}{D^3}=\iiint (ax^3\pm b)\sin(cx\pm d)(dx)^3=\frac{(-1)^3}{c^{3+3}}\left[c^3(ax^3\pm b)\sin\left\{\frac{3\pi}{2}+(cx\pm d)\right\}+9a(cx)^2\cdot\right.$$
$$\left.\sin\left\{\frac{4\pi}{2}+(cx\pm d)\right\}+36acx\sin\left\{\frac{5\pi}{2}+(cx\pm d)\right\}+60a\sin\left\{\frac{6\pi}{2}+(cx\pm d)\right\}\right]+\sum_{i=0}^{2}\frac{C_{i+1}x^i}{i!} \tag{149}$$

$\vdots$

$$\frac{1}{D^n}=\iiint\underbrace{\cdots\cdots}_{n-times}\int (ax^3\pm b)\sin(cx\pm d)(dx)^n=\frac{(-1)^n}{c^{3+n}}\left[c^3(ax^3\pm b)\sin\left\{\frac{n\pi}{2}+(cx\pm d)\right\}+\right.$$
$$3na(cx)^2\sin\left\{\frac{(n+1)\pi}{2}+(cx\pm d)\right\}+3nacx(n+1)\sin\left\{\frac{(n+2)\pi}{2}+(cx\pm d)\right\}+n(n+1)\cdot \tag{150}$$
$$\left.(n+2)a\sin\left\{\frac{(n+3)\pi}{2}+(cx\pm d)\right\}\right]+\sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!}$$

Combination form:

$$\frac{1}{D^n}=\iiint\underbrace{\cdots\cdots}_{n-times}\int (ax^3\pm b)\sin(cx\pm d)(dx)^n=\frac{(-1)^n}{c^{3+n}}\left[c^3(ax^3\pm b)\binom{3}{0}\sin\left\{\frac{n\pi}{2}+(cx\pm d)\right\}+\right.$$
$$na(cx)^2\binom{3}{1}\sin\left\{\frac{(n+1)\pi}{2}+(cx\pm d)\right\}+n(n+1)a(cx)^1\binom{3}{2}\sin\left\{\frac{(n+2)\pi}{2}+(cx\pm d)\right\}+ \tag{151}$$
$$\left.n(n+1)(n+2)a(cx)^0\binom{3}{3}\sin\left\{\frac{(n+3)\pi}{2}+(cx\pm d)\right\}\right]+\sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!}$$

Now to get the expression for the n^{th} Integral of $(ax^m\pm b)\sin(cx\pm d)$, I did some analysis in all the solutions of integrals that are solved above in equations (136), (141), (146) & (151). Let us consider a Pascal's triangle having rows starting from 0^{th} row to m^{th} row with elements in the form of binomial coefficients.

$$\begin{array}{ccccccccccccc}
 & & & 1 & & & 0^{th}\,Row & & & \binom{0}{0} & & & \\
 & & 1 & & 1 & & 1^{st}\,Row & & \binom{1}{0} & & \binom{1}{1} & & \\
 & 1 & & 2 & & 1 & 2^{nd}\,Row & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & \\
1 & & 3 & & 3 & & 1 \quad 3^{rd}\,Row & \binom{3}{0} & \binom{3}{1} & & \binom{3}{2} & \binom{3}{3} & \\
\vdots & & \vdots & & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\binom{m}{0} & \binom{m}{1} & \binom{m}{2} & \binom{m}{3} & \cdots & \binom{m}{m} & m^{th}\,Row & \binom{m}{0} & \binom{m}{1} & \binom{m}{2} & \binom{m}{3} & \cdots & \binom{m}{m}
\end{array} \tag{152}$$

Now in equation (152), I have introduced a sequence diagonally:
$1,\ n,\ n(n+1),\ n(n+1)(n+2),\ \cdots,\ n(n+1)(n+2)(n+3)\cdots\{n+(m-1)\}$ and we get-

$$\begin{array}{ccccccccccc}
 & & & & \swarrow 1 & & & & & & \\
 & & & \binom{0}{0} & & \swarrow n & & & & & \\
 & & \binom{1}{0} & & \binom{1}{1} & & \swarrow n(n+1) & & & & \\
 & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \swarrow n(n+1)(n+2) & & & \\
\binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & & & \\
\cdot^{\cdot^{\cdot}} \; \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \ddots & \swarrow n(n+1)(n+2)(n+3)\cdots\{n+(m-1)\} & \\
\binom{m}{0} & \binom{m}{1} & \binom{m}{2} & \binom{m}{3} & \cdots & & \binom{m}{m} & & & &
\end{array} \tag{153}$$

After introducing this sequence in equation (153), I have multiplied the entire diagonal elements by the respective terms of this sequence and we get-

$$\begin{array}{ccccccc}
 & & & 1\binom{0}{0} & & & \\
 & & 1\binom{1}{0} & & n\binom{1}{1} & & \\
 & 1\binom{2}{0} & & n\binom{2}{1} & & n(n+1)\binom{2}{2} & \\
1\binom{3}{0} & & n\binom{3}{1} & & n(n+1)\binom{3}{2} & & n(n+1)(n+2)\binom{3}{3} \\
\cdot^{\cdot^{\cdot}} \; \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \quad \ddots \\
1\binom{m}{0} & n\binom{m}{1} & n(n+1)\binom{m}{2} & n(n+1)(n+2)\binom{m}{3} & \cdots & & n(n+1)(n+2)(n+3)\cdots\{n+(m-1)\}\binom{m}{m}
\end{array} \tag{154}$$

Now we again in equation (154), I have introduced a sequence diagonally:

$c^m(ax^m \pm b),\ (cx)^{m-1},\ (cx)^{m-2},\ (cx)^{m-3},\ \cdots\ ,\ (cx)^{m-m}$ and we get-

$$\begin{array}{ccccccccc}
 & & & & \swarrow c^m(ax^m \pm b) & & & & \\
 & & & 1\binom{0}{0} & & \swarrow a(cx)^{m-1} & & & \\
 & & 1\binom{1}{0} & & n\binom{1}{1} & & \swarrow a(cx)^{m-2} & & \\
 & 1\binom{2}{0} & & n\binom{2}{1} & & n(n+1)\binom{2}{2} & & \swarrow a(cx)^{m-3} & \\
1\binom{3}{0} & & n\binom{3}{1} & & n(n+1)\binom{3}{2} & & n(n+1)(n+2)\binom{3}{3} & & \\
\cdot^{\cdot^{\cdot}} \; \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \swarrow a(cx)^{m-m} \\
1\binom{m}{0} & n\binom{m}{1} & n(n+1)\binom{m}{2} & n(n+1)(n+2)\binom{m}{3} & \cdots & & n(n+1)(n+2)(n+3)\cdots\{n+(m-1)\}\binom{m}{m} & &
\end{array} \tag{155}$$

After introducing this sequence in equation (155), I have multiplied the entire diagonal elements by the respective terms of this sequence and we get-

$$\begin{array}{ccccccc}
 & & & 1c^m(ax^m \pm b)\binom{0}{0} & & & \\
 & & 1c^m(ax^m \pm b)\binom{1}{0} & & na(cx)^{m-1}\binom{1}{1} & & \\
 & 1c^m(ax^m \pm b)\binom{2}{0} & & na(cx)^{m-1}\binom{2}{1} & & n(n+1)a(cx)^{m-2}\binom{2}{2} & \\
1c^m(ax^m \pm b)\binom{3}{0} & & na(cx)^{m-1}\binom{3}{1} & & n(n+1)a(cx)^{m-2}\binom{3}{2} & & n(n+1)(n+2)a(cx)^{m-3}\binom{3}{3} \\
\cdot^{\cdot^{\cdot}} \; \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \quad \ddots \\
1c^m(ax^m \pm b)\binom{m}{0} & na(cx)^{m-1}\binom{m}{1} & n(n+1)a(cx)^{m-2}\binom{m}{2} & n(n+1)(n+2)a(cx)^{m-3}\binom{m}{3} & \cdots & & n(n+1)(n+2)(n+3)\cdots\{n+(m-1)\}a(cx)^{m-m}\binom{m}{m}
\end{array} \tag{156}$$

Now in equation (156), we can observe that, on putting $m = 0,\ 1,\ 2,\ 3,\ \cdots\ ,\ m$ in $0^{th}\,Row,\ 1^{st}\,Row,\ 2^{nd}\,Row,\ 3^{rd}\,Row,\ \cdots\ ,\ m^{th}\,Row$ respectively. Then each row of Pascal's triangle is representing the coefficients of n^{th} indefinite integral of the following integrands: $(ax^0 \pm b)\sin(cx \pm d),\ (ax^1 \pm b)\sin(cx \pm d),\ (ax^2 \pm b)\sin(cx \pm d),\ (ax^3 \pm b)\sin(cx \pm d),\ \cdots\ ,$ $(ax^m \pm b)\sin(cx \pm d)$

$$\begin{gathered}
0^{th}\,Row \Rightarrow \frac{1}{D^n} = \iiint \underbrace{\cdots\cdots}_{n-times} \int (ax^0 \pm b)\sin(cx\pm d)(dx)^n = \frac{(-1)^n}{c^{0+n}}\left[c^0(ax^0\pm b)\binom{0}{0}\sin\left\{\frac{n\pi}{2}+(cx\pm d)\right\}\right]+\sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!} \\
1^{st}\,Row \Rightarrow \frac{1}{D^n} = \iiint \underbrace{\cdots\cdots}_{n-times} \int (ax^1 \pm b)\sin(cx\pm d)(dx)^n = \frac{(-1)^n}{c^{1+n}}\left[c^1(ax^1\pm b)\binom{1}{0}\sin\left\{\frac{n\pi}{2}+(cx\pm d)\right\}+na(cx)^0\binom{1}{1}\sin\left\{\frac{(n+1)\pi}{2}+(cx\pm d)\right\}\right]+\sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!} \\
2^{nd}\,Row \Rightarrow \frac{1}{D^n} = \iiint \underbrace{\cdots\cdots}_{n-times} \int (ax^2 \pm b)\sin(cx\pm d)(dx)^n = \frac{(-1)^n}{c^{2+n}}\left[c^2(ax^2\pm b)\binom{2}{0}\sin\left\{\frac{n\pi}{2}+(cx\pm d)\right\}+na(cx)^1\binom{2}{1}\sin\left\{\frac{(n+1)\pi}{2}+(cx\pm d)\right\}+n(n+1)a(cx)^0\binom{2}{2}\sin\left\{\frac{(n+2)\pi}{2}+(cx\pm d)\right\}\right]+\sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!} \\
3^{rd}\,Row \Rightarrow \frac{1}{D^n} = \iiint \underbrace{\cdots\cdots}_{n-times} \int (ax^3 \pm b)\sin(cx\pm d)(dx)^n = \frac{(-1)^n}{c^{3+n}}\left[c^3(ax^3\pm b)\binom{3}{0}\sin\left\{\frac{n\pi}{2}+(cx\pm d)\right\}+na(cx)^2\binom{3}{1}\sin\left\{\frac{(n+1)\pi}{2}+(cx\pm d)\right\}+n(n+1)a(cx)^1\binom{3}{2}\sin\left\{\frac{(n+2)\pi}{2}+(cx\pm d)\right\}+n(n+1)(n+2)a(cx)^0\binom{3}{3}\sin\left\{\frac{(n+3)\pi}{2}+(cx\pm d)\right\}\right]+\sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!} \\
\vdots \\
m^{th}\,Row \Rightarrow \frac{1}{D^n} = \iiint \underbrace{\cdots\cdots}_{n-times} \int (ax^m \pm b)\sin(cx\pm d)(dx)^n = \frac{(-1)^n}{c^{m+n}}\left[c^m(ax^m\pm b)\binom{m}{0}\sin\left\{\frac{n\pi}{2}+(cx\pm d)\right\}+na(cx)^{m-1}\binom{m}{1}\sin\left\{\frac{(n+1)\pi}{2}+(cx\pm d)\right\}+n(n+1)a(cx)^{m-2}\binom{m}{2}\sin\left\{\frac{(n+2)\pi}{2}+(cx\pm d)\right\}+n(n+1)(n+2)a(cx)^{m-3}\binom{m}{3}\sin\left\{\frac{(n+3)\pi}{2}+(cx\pm d)\right\}+\cdots+n(n+1)(n+2)(n+3)\cdots\{n+(m-1)\}a(cx)^{m-m}\binom{m}{m}\sin\left\{\frac{(n+m)\pi}{2}+(cx\pm d)\right\}\right]+\sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!}
\end{gathered} \tag{157}$$

On rewriting the solutions of n^{th} integral from equations (136), (141), (146) & (151) in the form of Pascal's triangle as shown above in equation (157), a beautiful pattern generates. Hence, on comparing all the elements of the $m^{th}Row$ of Pascal's triangle, I have concluded that these elements are the coefficients of the generalized expression for the n^{th} Integral of form 6: $(ax^m \pm b)\sin(cx \pm d)$ as discussed below.

6. $D^0 = (ax^m \pm b)\sin(cx \pm d)$

Solution=

$$\begin{aligned}
\frac{1}{D^n} &= \iiint \underbrace{\cdots\cdots}_{n-times} \int (ax^m \pm b)\sin(cx\pm d)(dx)^n = \frac{(-1)^n}{c^{m+n}}\left[c^m(ax^m\pm b)\binom{m}{0}\sin\left\{\frac{n\pi}{2}+(cx\pm d)\right\}\right. \\
&+na(cx)^{m-1}\binom{m}{1}\sin\left\{\frac{(n+1)\pi}{2}+(cx\pm d)\right\}+n(n+1)a(cx)^{m-2}\binom{m}{2}\sin\left\{\frac{(n+2)\pi}{2}+(cx\pm d)\right\} \\
&+n(n+1)(n+2)a(cx)^{m-3}\binom{m}{3}\sin\left\{\frac{(n+3)\pi}{2}+(cx\pm d)\right\}+\cdots+n(n+1)(n+2)(n+3)\cdots \\
&\left.\{n+(m-1)\}a(cx)^{m-m}\binom{m}{m}\sin\left\{\frac{(n+m)\pi}{2}+(cx\pm d)\right\}\right]+\sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!}
\end{aligned} \tag{158}$$

Here, from the equation (158), we can also determine the coefficient of any x^q, where $q=(m-p)$ and $q \le m$.

For the any term of x^q where $q=(m-p)$

If $0 \le p \le 2$, then the solution will be:

$$p=0 \Rightarrow \frac{(-1)^n}{c^{(m+n)}}\left[c^q(ax^m\pm b)\binom{m}{0}\sin\left\{\frac{(n+0)\pi}{2}+(cx\pm d)\right\}\right] \tag{159}$$

$$p=1 \Rightarrow \frac{(-1)^n}{c^{(m+n)}}\left[na(cx)^q\binom{m}{1}\sin\left\{\frac{(n+1)\pi}{2}+(cx\pm d)\right\}\right] \tag{160}$$

$$p=2 \Rightarrow \frac{(-1)^n}{c^{(m+n)}}\left[n(n+1)a(cx)^q\binom{m}{2}\sin\left\{\frac{(n+2)\pi}{2}+(cx\pm d)\right\}\right] \tag{161}$$

If $3 \le p \le m$, then the solution will be:

$$\begin{aligned}
3 \le p \le m \Rightarrow \frac{(-1)^n}{c^{(m+n)}}&\left[n(n+1)(n+2)(n+3)\cdots\{n+(p-1)\}a(cx)^q\binom{m}{p}\cdot\right. \\
&\left.\sin\left\{\frac{(n+p)\pi}{2}+(cx\pm d)\right\}\right]
\end{aligned} \tag{162}$$

Aesthetic form:

$$3 \le p \le m \Rightarrow \frac{(-1)^n}{c^{(m+n)}}\left[\prod_{i=0}^{(p-1)}(n+i)a(cx)^q\binom{m}{p}\sin\left\{\frac{(n+p)\pi}{2}+(cx\pm d)\right\}\right] \tag{163}$$

Integral of form 7- Initially, I have found the expression for the n^{th} integral of these integrands $(ax^0 \pm b)\cos(cx \pm d)$, $(ax^1 \pm b)\cos(cx \pm d)$, $(ax^2 \pm b)\cos(cx \pm d)$, & $(ax^3 \pm b)\cos(cx \pm d)$ that are solved below-

$$D^0 = (ax^0 \pm b)\cos(cx \pm d)$$

Solution=

$$\frac{1}{D^1} = \int (ax^0 \pm b)\cos(cx \pm d)dx = \frac{(-1)^1}{c^{0+1}}\left[(ax^0 \pm b)\cos\left\{\frac{\pi}{2} + (cx \pm d)\right\}\right] + \sum_{i=0}^{0}\frac{C_{i+1}x^i}{i!} \tag{164}$$

$$\frac{1}{D^2} = \iint (ax^0 \pm b)\cos(cx \pm d)(dx)^2 = \frac{(-1)^2}{c^{0+2}}\left[(ax^0 \pm b)\cos\left\{\frac{2\pi}{2} + (cx \pm d)\right\}\right] + \sum_{i=0}^{1}\frac{C_{i+1}x^i}{i!} \tag{165}$$

$$\frac{1}{D^3} = \iiint (ax^0 \pm b)\cos(cx \pm d)(dx)^3 = \frac{(-1)^3}{c^{0+3}}\left[(ax^0 \pm b)\cos\left\{\frac{3\pi}{2} + (cx \pm d)\right\}\right] + \sum_{i=0}^{2}\frac{C_{i+1}x^i}{i!} \tag{166}$$

$\vdots$

$$\frac{1}{D^n} = \iiint\underbrace{\cdots\cdots}_{n-times}\int (ax^0 \pm b)\cos(cx \pm d)(dx)^n = \frac{(-1)^n}{c^{0+n}}\left[(ax^0 \pm b)\cos\left\{\frac{n\pi}{2} + (cx \pm d)\right\}\right] + \sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!} \tag{167}$$

Combination form:

$$\frac{1}{D^n} = \iiint\underbrace{\cdots\cdots}_{n-times}\int (ax^0 \pm b)\cos(cx \pm d)(dx)^n = \frac{(-1)^n}{c^{0+n}}\left[c^0(ax^0 \pm b)\binom{0}{0}\cos\left\{\frac{n\pi}{2} + (cx \pm d)\right\}\right] + \sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!} \tag{168}$$

$$D^0 = (ax^1 \pm b)\cos(cx \pm d)$$

Solution=

$$\frac{1}{D^1} = \int (ax^1 \pm b)\cos(cx \pm d)dx = \frac{(-1)^1}{c^{1+1}}\left[c(ax^1 \pm b)\cos\left\{\frac{\pi}{2} + (cx \pm d)\right\} + 1a \cdot \cos\left\{\frac{2\pi}{2} + (cx \pm d)\right\}\right] + \sum_{i=0}^{0}\frac{C_{i+1}x^i}{i!} \tag{169}$$

$$\frac{1}{D^2} = \iint (ax^1 \pm b)\cos(cx \pm d)(dx)^2 = \frac{(-1)^2}{c^{1+2}}\left[c(ax^1 \pm b)\cos\left\{\frac{2\pi}{2} + (cx \pm d)\right\} + 2a \cdot \cos\left\{\frac{3\pi}{2} + (cx \pm d)\right\}\right] + \sum_{i=0}^{1}\frac{C_{i+1}x^i}{i!} \tag{170}$$

$$\frac{1}{D^3} = \iiint (ax^1 \pm b)\cos(cx \pm d)(dx)^3 = \frac{(-1)^3}{c^{1+3}}\left[c(ax^1 \pm b)\cos\left\{\frac{3\pi}{2} + (cx \pm d)\right\} + 3a \cdot \cos\left\{\frac{4\pi}{2} + (cx \pm d)\right\}\right] + \sum_{i=0}^{2}\frac{C_{i+1}x^i}{i!} \tag{171}$$

$\vdots$

$$\frac{1}{D^n} = \iiint\underbrace{\cdots\cdots}_{n-times}\int (ax^1 \pm b)\cos(cx \pm d)(dx)^n = \frac{(-1)^n}{c^{1+n}}\left[c(ax^1 \pm b)\cos\left\{\frac{n\pi}{2} + (cx \pm d)\right\} + na \cdot\right.$$

$$\cos\left\{\frac{(n+1)\pi}{2}+(cx\pm d)\right\}\Bigg]+\sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!} \quad (172)$$

Combination form:

$$\frac{1}{D^n}=\underbrace{\iiint\cdots\cdots\int}_{n-times}(ax^1\pm b)\cos(cx\pm d)(dx)^n=\frac{(-1)^n}{c^{1+n}}\Bigg[c^1(ax^1\pm b)\binom{1}{0}\cos\left\{\frac{n\pi}{2}+(cx\pm d)\right\}+ na(cx)^0\binom{1}{1}\cos\left\{\frac{(n+1)\pi}{2}+(cx\pm d)\right\}\Bigg]+\sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!} \quad (173)$$

$$D^0=(ax^2\pm b)\cos(cx\pm d)$$

Solution=

$$\frac{1}{D^1}=\int(ax^2\pm b)\cos(cx\pm d)dx=\frac{(-1)^1}{c^{2+1}}\Bigg[c^2(ax^2\pm b)\cos\left\{\frac{\pi}{2}+(cx\pm d)\right\}+2acx\cdot \cos\left\{\frac{2\pi}{2}+(cx\pm d)\right\}+2a\cos\left\{\frac{3\pi}{2}+(cx\pm d)\right\}\Bigg]+\sum_{i=0}^{0}\frac{C_{i+1}x^i}{i!} \quad (174)$$

$$\frac{1}{D^2}=\iint(ax^2\pm b)\cos(cx\pm d)(dx)^2=\frac{(-1)^2}{c^{2+2}}\Bigg[c^2(ax^2\pm b)\cos\left\{\frac{2\pi}{2}+(cx\pm d)\right\}+4acx\cdot \cos\left\{\frac{3\pi}{2}+(cx\pm d)\right\}+6a\cos\left\{\frac{4\pi}{2}+(cx\pm d)\right\}\Bigg]+\sum_{i=0}^{1}\frac{C_{i+1}x^i}{i!} \quad (175)$$

$$\frac{1}{D^3}=\iiint(ax^2\pm b)\cos(cx\pm d)(dx)^3=\frac{(-1)^3}{c^{2+3}}\Bigg[c^2(ax^2\pm b)\cos\left\{\frac{3\pi}{2}+(cx\pm d)\right\}+6acx\cdot \cos\left\{\frac{4\pi}{2}+(cx\pm d)\right\}+12a\cos\left\{\frac{5\pi}{2}+(cx\pm d)\right\}\Bigg]+\sum_{i=0}^{2}\frac{C_{i+1}x^i}{i!} \quad (176)$$

$$\vdots$$

$$\frac{1}{D^n}=\underbrace{\iiint\cdots\cdots\int}_{n-times}(ax^2\pm b)\cos(cx\pm d)(dx)^n=\frac{(-1)^n}{c^{2+n}}\Bigg[c^2(ax^2\pm b)\cos\left\{\frac{n\pi}{2}+(cx\pm d)\right\}+ 2nacx\cos\left\{\frac{(n+1)\pi}{2}+(cx\pm d)\right\}+n(n+1)a\cos\left\{\frac{(n+2)\pi}{2}+(cx\pm d)\right\}\Bigg]+\sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!} \quad (177)$$

Combination form:

$$\frac{1}{D^n}=\underbrace{\iiint\cdots\cdots\int}_{n-times}(ax^2\pm b)\cos(cx\pm d)(dx)^n=\frac{(-1)^n}{c^{2+n}}\Bigg[c^2(ax^2\pm b)\binom{2}{0}\cos\left\{\frac{n\pi}{2}+(cx\pm d)\right\}+na(cx)^1 \binom{2}{1}\cos\left\{\frac{(n+1)\pi}{2}+(cx\pm d)\right\}+n(n+1)a(cx)^0\binom{2}{2}\cos\left\{\frac{(n+2)\pi}{2}+(cx\pm d)\right\}\Bigg]+\sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!} \quad (178)$$

$$D^0=(ax^3\pm b)\cos(cx\pm d)$$

Solution=

$$\frac{1}{D^1}=\int(ax^3\pm b)\cos(cx\pm d)dx=\frac{(-1)^1}{c^{3+1}}\Bigg[c^3(ax^3\pm b)\cos\left\{\frac{\pi}{2}+(cx\pm d)\right\}+3a(cx)^2\cdot \cos\left\{\frac{2\pi}{2}+(cx\pm d)\right\}+6acx\cos\left\{\frac{3\pi}{2}+(cx\pm d)\right\}+6a\cos\left\{\frac{4\pi}{2}+(cx\pm d)\right\}\Bigg]+\sum_{i=0}^{0}\frac{C_{i+1}x^i}{i!} \quad (179)$$

$$\frac{1}{D^2}=\iint(ax^3\pm b)\cos(cx\pm d)(dx)^2=\frac{(-1)^2}{c^{3+2}}\left[c^3(ax^3\pm b)\cos\left\{\frac{2\pi}{2}+(cx\pm d)\right\}+6a(cx)^2\cdot\right.$$
$$\left.\cos\left\{\frac{3\pi}{2}+(cx\pm d)\right\}+18acx\cos\left\{\frac{4\pi}{2}+(cx\pm d)\right\}+24a\cos\left\{\frac{5\pi}{2}+(cx\pm d)\right\}\right]+\sum_{i=0}^{1}\frac{C_{i+1}x^i}{i!} \tag{180}$$

$$\frac{1}{D^3}=\iiint(ax^3\pm b)\cos(cx\pm d)(dx)^3=\frac{(-1)^3}{c^{3+3}}\left[c^3(ax^3\pm b)\cos\left\{\frac{3\pi}{2}+(cx\pm d)\right\}+9a(cx)^2\cdot\right.$$
$$\left.\cos\left\{\frac{4\pi}{2}+(cx\pm d)\right\}+36acx\cos\left\{\frac{5\pi}{2}+(cx\pm d)\right\}+60a\cos\left\{\frac{6\pi}{2}+(cx\pm d)\right\}\right]+\sum_{i=0}^{2}\frac{C_{i+1}x^i}{i!} \tag{181}$$

$$\vdots$$

$$\frac{1}{D^n}=\iiint\underbrace{\cdots\cdots}_{n-times}\int(ax^3\pm b)\cos(cx\pm d)(dx)^n=\frac{(-1)^n}{c^{3+n}}\left[c^3(ax^3\pm b)\cos\left\{\frac{n\pi}{2}+(cx\pm d)\right\}+\right.$$
$$3na(cx)^2\cos\left\{\frac{(n+1)\pi}{2}+(cx\pm d)\right\}+3n(n+1)acx\cos\left\{\frac{(n+2)\pi}{2}+(cx\pm d)\right\}+n(n+1)\cdot \tag{182}$$
$$\left.(n+2)a\cos\left\{\frac{(n+3)\pi}{2}+(cx\pm d)\right\}\right]+\sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!}$$

Combination form:

$$\frac{1}{D^n}=\iiint\underbrace{\cdots\cdots}_{n-times}\int(ax^3\pm b)\cos(cx\pm d)(dx)^n=\frac{(-1)^n}{c^{3+n}}\left[c^3(ax^3\pm b)\binom{3}{0}\cos\left\{\frac{n\pi}{2}+(cx\pm d)\right\}+\right.$$
$$na(cx)^2\binom{3}{1}\cdot\cos\left\{\frac{(n+1)\pi}{2}+(cx\pm d)\right\}+n(n+1)a(cx)^1\binom{3}{2}\cos\left\{\frac{(n+2)\pi}{2}+(cx\pm d)\right\}+n\cdot \tag{183}$$
$$\left.(n+1)(n+2)a(cx)^0\binom{3}{3}\cos\left\{\frac{(n+3)\pi}{2}+(cx\pm d)\right\}\right]+\sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!}$$

Now to get the expression for the n^{th} Integral of $(ax^m\pm b)\cos(cx\pm d)$, I did some analysis in all the solutions of integrals that are solved above in equations (168), (173), (178) & (183).

Let us consider a Pascal's triangle having rows starting from 0^{th} row to m^{th} row with elements in the form of binomial coefficients.

$$\begin{array}{ccccccccc}
 & & & 1 & & & \\
 & & 1 & & 1 & & \\
 & 1 & & 2 & & 1 & \\
1 & & 3 & & 3 & & 1 \\
\vdots & & \vdots & & \vdots & & \vdots \\
\binom{m}{0} & \binom{m}{1} & \binom{m}{2} & \binom{m}{3} & \cdots & \binom{m}{m}
\end{array}
\quad
\begin{array}{c}
\overset{0^{th}}{Row} \\ \overset{1^{st}}{Row} \\ \overset{2^{nd}}{Row} \\ \overset{3^{rd}}{Row} \\ \vdots \\ \overset{m^{th}}{Row}
\end{array}
\quad
\begin{array}{ccccccc}
 & & & \binom{0}{0} & & & \\
 & & \binom{1}{0} & & \binom{1}{1} & & \\
 & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & \\
\binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\binom{m}{0} & \binom{m}{1} & \binom{m}{2} & \binom{m}{3} & \cdots & \binom{m}{m} &
\end{array} \tag{184}$$

Now in equation (184), I have introduced a sequence diagonally:

$1,\ n,\ n(n+1),\ n(n+1)(n+2),\ \cdots\ ,\ n(n+1)(n+2)(n+3)\cdots\{n+(m-1)\}$ and we get-

$$\begin{array}{ccccccccc}
 & & & & & \swarrow 1 & & & \\
 & & & & \binom{0}{0} & & \swarrow n & & \\
 & & & \binom{1}{0} & & \binom{1}{1} & & \swarrow n(n+1) & \\
 & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \swarrow n(n+1)(n+2) \\
 & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\
\iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \quad \swarrow n(n+1)(n+2)(n+3)\cdots\{n+(m-1)\} \\
\binom{m}{0} & \binom{m}{1} & \binom{m}{2} & \binom{m}{3} & \cdots & \binom{m}{m} & & &
\end{array} \tag{185}$$

After introducing this sequence in equation (185), I have multiplied the entire diagonal elements by the respective terms of this sequence and we get-

$$\begin{array}{cccccc}
1\binom{0}{0} & & & & & \\
1\binom{1}{0} & n\binom{1}{1} & & & & \\
1\binom{2}{0} & n\binom{2}{1} & n(n+1)\binom{2}{2} & & & \\
1\binom{3}{0} & n\binom{3}{1} & n(n+1)\binom{3}{2} & n(n+1)(n+2)\binom{3}{3} & & \\
\vdots & \vdots & \vdots & \vdots & \ddots & \\
1\binom{m}{0} & n\binom{m}{1} & n(n+1)\binom{m}{2} & n(n+1)(n+2)\binom{m}{3} & \cdots & n(n+1)(n+2)(n+3)\cdots\{n+(m-1)\}\binom{m}{m}
\end{array} \tag{186}$$

Now we again in equation (186), I have introduced a sequence diagonally:

$c^m(ax^m \pm b),\ (cx)^{m-1},\ (cx)^{m-2},\ (cx)^{m-3},\ \cdots\ ,\ (cx)^{m-m}$ and we get-

$$\begin{array}{cccccc}
\swarrow c^m(ax^m \pm b) & & & & & \\
1\binom{0}{0} & \swarrow a(cx)^{m-1} & & & & \\
1\binom{1}{0} & n\binom{1}{1} & \swarrow a(cx)^{m-2} & & & \\
1\binom{2}{0} & n\binom{2}{1} & n(n+1)\binom{2}{2} & \swarrow a(cx)^{m-3} & & \\
1\binom{3}{0} & n\binom{3}{1} & n(n+1)\binom{3}{2} & n(n+1)(n+2)\binom{3}{3} & & \\
\vdots & \vdots & \vdots & \vdots & \ddots & \swarrow a(cx)^{m-m} \\
1\binom{m}{0} & n\binom{m}{1} & n(n+1)\binom{m}{2} & n(n+1)(n+2)\binom{m}{3} & \cdots & n(n+1)(n+2)(n+3)\cdots\{n+(m-1)\}\binom{m}{m}
\end{array} \tag{187}$$

After introducing this sequence in equation (187), I have multiplied the entire diagonal elements by the respective terms of this sequence and we get-

$$\begin{array}{cccccc}
1c^m(ax^m \pm b)\binom{0}{0} & & & & & \\
1c^m(ax^m \pm b)\binom{1}{0} & na(cx)^{m-1}\binom{1}{1} & & & & \\
1c^m(ax^m \pm b)\binom{2}{0} & na(cx)^{m-1}\binom{2}{1} & n(n+1)a(cx)^{m-2}\binom{2}{2} & & & \\
1c^m(ax^m \pm b)\binom{3}{0} & na(cx)^{m-1}\binom{3}{1} & n(n+1)a(cx)^{m-2}\binom{3}{2} & n(n+1)(n+2)a(cx)^{m-3}\binom{3}{3} & & \\
\vdots & \vdots & \vdots & \vdots & \ddots & \\
1c^m(ax^m \pm b)\binom{m}{0} & na(cx)^{m-1}\binom{m}{1} & n(n+1)a(cx)^{m-2}\binom{m}{2} & n(n+1)(n+2)a(cx)^{m-3}\binom{m}{3} & \cdots & n(n+1)(n+2)(n+3)\cdots\{n+(m-1)\}a(cx)^{m-m}\binom{m}{m}
\end{array} \tag{188}$$

Now in equation (188), we can observe that, on putting $m = 0,\ 1,\ 2,\ 3,\ \cdots\ ,\ m$ in $0^{th}\,Row,\ 1^{st}\,Row,\ 2^{nd}\,Row,\ 3^{rd}\,Row,\ \cdots\ ,\ m^{th}\,Row$ respectively. Then each row of Pascal's triangle is representing the coefficients of n^{th} indefinite integral of following integrands: $(ax^0 \pm b)\cos(cx \pm d),\ (ax^1 \pm b)\cos(cx \pm d),\ (ax^2 \pm b)\cos(cx \pm d),\ (ax^3 \pm b)\cos(cx \pm d),\ \cdots\ ,$ $(ax^m \pm b)\cos(cx \pm d)$

$$
\begin{aligned}
&0^{th}\,Row \Rightarrow \frac{1}{D^n} = \iiint \underbrace{\cdots\cdots}_{n-times} \int (ax^0 \pm b)\cos(cx \pm d)(dx)^n = \frac{(-1)^n}{c^{0+n}}\left[c^0(ax^0 \pm b)\binom{0}{0}\cos\left\{\frac{n\pi}{2}+(cx \pm d)\right\}\right] + \sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!} \\
&1^{st}\,Row \Rightarrow \frac{1}{D^n} = \iiint \underbrace{\cdots\cdots}_{n-times} \int (ax^1 \pm b)\cos(cx \pm d)(dx)^n = \frac{(-1)^n}{c^{1+n}}\left[c^1(ax^1 \pm b)\binom{1}{0}\cos\left\{\frac{n\pi}{2}+(cx \pm d)\right\} + na(cx)^0\binom{1}{1}\cos\left\{\frac{(n+1)\pi}{2}+(cx \pm d)\right\}\right] + \sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!} \\
&2^{nd}\,Row \Rightarrow \frac{1}{D^n} = \iiint \underbrace{\cdots\cdots}_{n-times} \int (ax^2 \pm b)\cos(cx \pm d)(dx)^n = \frac{(-1)^n}{c^{2+n}}\left[c^2(ax^2 \pm b)\binom{2}{0}\cos\left\{\frac{n\pi}{2}+(cx \pm d)\right\} + na(cx)^1\binom{2}{1}\cos\left\{\frac{(n+1)\pi}{2}+(cx \pm d)\right\} + n(n+1)a(cx)^0\binom{2}{2}\cos\left\{\frac{(n+2)\pi}{2}+(cx \pm d)\right\}\right] + \sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!} \\
&3^{rd}\,Row \Rightarrow \frac{1}{D^n} = \iiint \underbrace{\cdots\cdots}_{n-times} \int (ax^3 \pm b)\cos(cx \pm d)(dx)^n = \frac{(-1)^n}{c^{3+n}}\left[c^3(ax^3 \pm b)\binom{3}{0}\cos\left\{\frac{n\pi}{2}+(cx \pm d)\right\} + na(cx)^2\binom{3}{1}\cos\left\{\frac{(n+1)\pi}{2}+(cx \pm d)\right\} + n(n+1)a(cx)^1\binom{3}{2}\cos\left\{\frac{(n+2)\pi}{2}+(cx \pm d)\right\} + n(n+1)(n+2)a(cx)^0\binom{3}{3}\cos\left\{\frac{(n+3)\pi}{2}+(cx \pm d)\right\}\right] + \sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!} \\
&\vdots \\
&m^{th}\,Row \Rightarrow \frac{1}{D^n} = \iiint \underbrace{\cdots\cdots}_{n-times} \int (ax^m \pm b)\cos(cx \pm d)(dx)^n = \frac{(-1)^n}{c^{m+n}}\left[c^m(ax^m \pm b)\binom{m}{0}\cos\left\{\frac{n\pi}{2}+(cx \pm d)\right\} + na(cx)^{m-1}\binom{m}{1}\cos\left\{\frac{(n+1)\pi}{2}+(cx \pm d)\right\} + n(n+1)a(cx)^{m-2}\binom{m}{2}\cos\left\{\frac{(n+2)\pi}{2}+(cx \pm d)\right\} + n(n+1)(n+2)a(cx)^{m-3}\binom{m}{3}\cos\left\{\frac{(n+3)\pi}{2}+(cx \pm d)\right\} + \cdots + n(n+1)(n+2)(n+3)\cdots\{n+(m-1)\}a(cx)^{m-m}\binom{m}{m}\cos\left\{\frac{(n+m)\pi}{2}+(cx \pm d)\right\}\right] + \sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!}
\end{aligned} \tag{189}
$$

On rewriting the solutions of n^{th} integral from equations (168), (173), (178) & (183) in the form of Pascal's Triangle as shown above in equation (189), a beautiful pattern generates. Hence, on comparing all the elements of the $m^{th}Row$ of Pascal's triangle, I have concluded that these are the coefficients of the generalized expression for the n^{th} Integral of form 7: $(ax^m \pm b)\cos(cx \pm d)$ as discussed below.

7. $D^0 = (ax^m \pm b)\cos(cx \pm d)$

Solution=

$$
\begin{aligned}
&\frac{1}{D^n} = \iiint \underbrace{\cdots\cdots}_{n-times} \int (ax^m \pm b)\cos(cx \pm d)(dx)^n = \frac{(-1)^n}{c^{m+n}}\left[c^m(ax^m \pm b)\binom{m}{0}\cdot\right. \\
&\cos\left\{\frac{n\pi}{2}+(cx \pm d)\right\} + na(cx)^{m-1}\binom{m}{1}\cos\left\{\frac{(n+1)\pi}{2}+(cx \pm d)\right\} + n(n+1)a(cx)^{m-2}\binom{m}{2}\cdot \\
&\cos\left\{\frac{(n+2)\pi}{2}+(cx \pm d)\right\} + n(n+1)(n+2)a(cx)^{m-3}\binom{m}{3}\cos\left\{\frac{(n+3)\pi}{2}+(cx \pm d)\right\} + \cdots + n\cdot \\
&\left.(n+1)(n+2)(n+3)\cdots\{n+(m-1)\}a(cx)^{m-m}\binom{m}{m}\cos\left\{\frac{(n+m)\pi}{2}+(cx \pm d)\right\}\right] + \sum_{i=0}^{n-1}\frac{C_{i+1}x^i}{i!}
\end{aligned} \tag{190}
$$

Here, from the equation (190), we can also determine the coefficient of any x^q, where $q = (m-p)$ and $q \le m$.

For the any term of x^q where $q = (m-p)$

If $0 \le p \le 2$, then the solution will be:

$$p = 0 \Rightarrow \frac{(-1)^n}{c^{(m+n)}}\left[c^q(ax^m \pm b)\binom{m}{0}\cos\left\{\frac{(n+0)\pi}{2}+(cx \pm d)\right\}\right] \tag{191}$$

$$p = 1 \Rightarrow \frac{(-1)^n}{c^{(m+n)}}\left[na(cx)^q\binom{m}{1}\cos\left\{\frac{(n+1)\pi}{2}+(cx \pm d)\right\}\right] \tag{192}$$

$$p = 2 \Rightarrow \frac{(-1)^n}{c^{(m+n)}}\left[n(n+1)a(cx)^q\binom{m}{2}\cos\left\{\frac{(n+2)\pi}{2}+(cx \pm d)\right\}\right] \tag{193}$$

If $3 \le p \le m$, then the solution will be:

$$
\begin{aligned}
&3 \le p \le m \Rightarrow \frac{(-1)^n}{c^{(m+n)}}\left[n(n+1)(n+2)(n+3)\cdots\{n+(p-1)\}a(cx)^q\binom{m}{p}\cdot\right. \\
&\left.\cos\left\{\frac{(n+p)\pi}{2}+(cx \pm d)\right\}\right]
\end{aligned} \tag{194}
$$

Aesthetic form:

$$3 \le p \le m \Rightarrow \frac{(-1)^n}{c^{(m+n)}}\left[\prod_{i=0}^{(p-1)}(n+i)a(cx)^q\binom{m}{p}\cos\left\{\frac{(n+p)\pi}{2}+(cx \pm d)\right\}\right] \tag{195}$$

V. Discussion

For the Integrals of Type-I:

Integral of form 1- We can clearly see the n^{th} integral of a non-transcendental function with any rational power is evaluated as per the different values of p/q are shown in equations (33), (35), (37) & (39) and when $p/q=-m$ the integral is solved easily until $n<m$ as shown in case 1.1 in equation (41) and when $n=m$, I have used the case 1.2 in which logarithmic form appears as shown in equation (43). After performing more successive integration such that $n>m$, then I have defined p such that $p=n-m$ for this I have used case 1.3 then the solution of this integration will generate a harmonic series of p terms as shown in equation (45). Since the partial sum of the harmonic series can be easily evaluated for large values of p by using Euler-Mascheroni formula.

Similarly, for **integral of form 2-** Initially, I have split the integral into two by using linearity property of the operator D as shown in equations (8) & (9) and then, we will perform the same procedure and steps which are discussed above to evaluate the generalized solution of the integral of form 2.

For the Integrals of Type-II:

Integral of form 3- We can clearly see the n^{th} integral of the product of non-transcendental and transcendental function i.e. the product of algebraic and logarithmic function having the same coefficients with any rational number p/q and r/s as an exponent respectively. We can easily evaluate the integrals by substitution of $ax\pm b=t$ and rearranging the terms after every integral, in such a manner to form a sequence that is easy to achieve generalized expression for n^{th} integral as shown in equation (67). The solution of integration will be changed as per the different values of p/q and r/s are shown in equations (69), (71) & (73) and when $p/q=-m$ the integral is done easily until $n<m$ as shown in case 3.1 of equation (75) and when $n=m$, I have used case 3.2 in which integral of $\dfrac{\log_e(ax\pm b)}{(ax\pm b)}$ form appears which is equals to $\dfrac{\log_e{}^2(ax\pm b)}{2a}$ and here I have taken this a as common from denominator which is shown in equation (77). After performing more successive integration such that $n>m$, then I have defined p such that $p=n-m$, for this I have used case 3.3 and here again $\dfrac{1}{D^p}\cdot\dfrac{\log_e{}^2(ax\pm b)}{2}$ appears i.e. p^{th} integral of $\dfrac{\log_e{}^2(ax\pm b)}{2}$ which is calculated manually and the solution of this integration will also generate a harmonic series of p terms as shown in equation (79). Since the partial sum of the harmonic series can be easily evaluated for large values of p by using Euler-Mascheroni formula.

Similarly, for **integral of form 4-** Initially, I have split the integral into two by using linearity property of operator D as shown in equations (8) & (9) and then, we will perform the same procedure and steps which are discussed above to evaluate the generalized solution of the integral of form 3.

For the Integrals of Type-III:

Integral of form 5- Initially, I have considered this integrand as the increasing integral exponent of x i.e. $(ax^0\pm b)e^{(cx\pm d)}$, $(ax^1\pm b)e^{(cx\pm d)}$, $(ax^2\pm b)e^{(cx\pm d)}$, & $(ax^3\pm b)e^{(cx\pm d)}$. On evaluating these integrals one by one, at every integral- I have arrange all the terms of the solution in descending power of x. Now taking L.C.M. of all terms $e^{(cx\pm d)}$ is taken as common from numerator and c is taken as common from the denominator. On performing the same operation repeatedly, we will get the expression for n^{th} integrals having $\dfrac{e^{(cx\pm d)}}{c^{0+n}}$, $\dfrac{e^{(cx\pm d)}}{c^{1+n}}$, $\dfrac{e^{(cx\pm d)}}{c^{2+n}}$, & $\dfrac{e^{(cx\pm d)}}{c^{3+n}}$ all are taken out as common from the square brackets as shown in equations (101), (106), (111) & (116) respectively. The numbers are present in these four results of equations (101), (106), (111) & (116), are written in the form of Binomial coefficients as shown in equations (102), (107),

(112) & (117). Now to get the expression for the n^{th} Integral of $(ax^m \pm b)e^{(cx\pm d)}$, I did some analysis in all the solutions of integrals discussed in equations (102), (107), (112) & (117). I have considered a generalized Pascal's triangle as shown in equation (118) which starts from 0^{th} row to m^{th} row and 0^{th} column to m^{th} column with elements in the form of binomial coefficients. In equation (118), I have introduced a sequence i.e. $1,\ n,\ n(n+1),\ n(n+1)(n+2),\ \cdots\ ,\ n(n+1)(n+2)(n+3)\cdots\{n+(m-1)\}$. Here, the entire diagonal elements of Pascal's triangle are multiplied by the respective terms of this sequence as shown in equation (119). After multiplying the entire diagonal elements by the respective terms of this sequence, we get equation (120). From equation (120), first five diagonals of Pascal's triangle represents, 1's, Natural numbers, Triangular numbers, Tetrahedral numbers and Pentatope numbers respectively. Now in equation (120), I have introduced a sequence i.e. $c^m(ax^m \pm b),\ (cx)^{m-1},\ (cx)^{m-2},\ (cx)^{m-3},\ \cdots\ ,\ (cx)^{m-m}$. Here, the entire diagonal elements of Pascal's triangle are multiplied by the respective terms of this sequence as shown in equation (121). After multiplying the entire diagonal elements by the respective terms of this sequence, we get equation (122). Now again in equation (122), I have introduced a sequence i.e. $(-1)^0,\ (-1)^1,\ (-1)^2,\ (-1)^3,\ \cdots\ ,\ (-1)^m$. Here, the entire diagonal elements of Pascal's triangle are multiplied by the respective terms of this sequence as shown in equation (123). After multiplying the entire diagonal elements by the respective terms of this sequence, it forms an alternating series in which every even term will become negative and every odd term will become positive in every row of Pascal's triangle, we get equation (124). Now in equation (124), we can observe that, on putting $m = 0,\ 1,\ 2,\ 3,\ \cdots\ ,\ m$ in $0^{th}\,Row,\ 1^{st}\,Row,\ 2^{nd}\,Row,\ 3^{rd}\,Row,\ \cdots\ ,\ m^{th}\,Row$ respectively. Then each row of Pascal's triangle is representing the coefficients of n^{th} indefinite integral of the following integrands: $(ax^0 \pm b)e^{(cx\pm d)},\ (ax^1 \pm b)e^{(cx\pm d)},\ (ax^2 \pm b)e^{(cx\pm d)},\ (ax^3 \pm b)e^{(cx\pm d)},\ \cdots\ ,\ (ax^m \pm b)e^{(cx\pm d)}$ respectively.

On rewriting the solutions of n^{th} integral from equations (102), (107), (112) & (117) in form of Pascal's triangle as shown above in equation (125), a beautiful pattern generates. Therefore, on comparing all the elements of the $m^{th}Row$ of Pascal's triangle, I have concluded that these elements are the coefficients of the generalized expression for the n^{th} Integral of $(ax^m \pm b)e^{(cx\pm d)}$. Hence, it is the generalized solution for the integral of form 5 as shown in equation (126).

We can also find coefficient of x^q directly by using the formula as shown in equations (127), (128), (129) & (130). For calculating coefficient of x^q, we express q as an exponent of x in terms of m and p.

Here-

$q = (m - p)$ where $q \le m$.

m - is the exponent of x that is given in the integrand.

p - is the value which is calculated by putting the values of q & m.

Now after expressing the exponent of x in terms of q, we have two cases are raised as per the range of p:

First case- if p is lies in the range $0 \le p \le 2$ i.e. $p = 0, 1\ \&\ 2$ so we use formula as shown in equations (127), (128) & (129).

Second case- if p is lies in the range $3 \le p \le m$ i.e. $p = 3,\ 4,\ 5,\ \cdots\ ,\ m$ so we use formula as shown in equation (130).

Integral of form 6- Initially, I have considered this integrand as the increasing integral exponent of x i.e. $(ax^0 \pm b)\sin(cx \pm d),\ (ax^1 \pm b)\sin(cx \pm d),\ (ax^2 \pm b)\sin(cx \pm d),\ \&\ (ax^3 \pm b)\sin(cx \pm d)$. Now on evaluating these given integrals, I have express $\int \sin(cx \pm d)dx = \frac{(-1)^1}{c}\sin\left\{\frac{\pi}{2} + (cx \pm d)\right\} + \sum_{i=0}^{0}\frac{C_{i+1}x^i}{i!}$ by adding $\frac{\pi}{2}$ in theta after performing every integral. So we get the solution in the terms of $\sin\{f(x)\}$ with a negative sign as shown above. Now arrange all the terms of the solution in descending power of x and taking the L.C.M. of all the terms, $(-1)^1$ is taken as common from numerator and c is taken as common from denominator. On performing the same operation repeatedly, we will get the expression for n^{th} integrals having $\frac{(-1)^n}{c^{0+n}}, \frac{(-1)^n}{c^{1+n}}, \frac{(-1)^n}{c^{2+n}}, \ \& \ \frac{(-1)^n}{c^{3+n}}$ all are taken out as common from the square brackets as shown in equations (135), (140), (145) & (150) respectively. On comparing the results of equations (135), (140), (145) & (150), we also observe the sequence: $\frac{n\pi}{2}, \frac{(n+1)\pi}{2}, \frac{(n+2)\pi}{2}, \ \& \ \frac{(n+3)\pi}{2}$ that are added in theta to get solution in terms of $\sin\{f(x)\}$. Now the numbers are present in these four results of equations (135), (140), (145) & (150), are written in the form of Binomial coefficients. as shown in equations (136), (141), (146) & (151). Now to get the expression for the n^{th} Integral of $(ax^m \pm b)\sin(cx \pm d)$, I did some analysis in all the solutions of integrals in equations (136), (141), (146) & (151). I have considered a generalized Pascal's triangle as shown in the equation (152) which starts from 0^{th} row to m^{th} row and 0^{th} column to m^{th} column with element in the form of binomial coefficients. In equation (152), I have introduced a sequence i.e. $1, n, n(n+1), n(n+1)(n+2), \cdots, n(n+1)(n+2)(n+3)\cdots\{n+(m-1)\}$. Here, the entire diagonal elements of Pascal's triangle are multiplied by the respective terms of this sequence as shown in equation (153). After multiplying the entire diagonal elements by the respective terms of this sequence, we get equation (154). From equation (154), first five diagonals of Pascal's triangle represents: 1's, Natural numbers, Triangular numbers, Tetrahedral numbers and Pentatope numbers respectively. Now in equation (154), I have introduced a sequence i.e. $c^m(ax^m \pm b), (cx)^{m-1}, (cx)^{m-2}, (cx)^{m-3}, \cdots, (cx)^{m-m}$. Here the entire diagonal elements of Pascal's triangle are multiplied by the respective terms of this sequence as shown in equation (155). After multiplying the entire diagonal elements by the respective terms of this sequence, we get our required result as shown in equation (156). Now in equation (156), we can observe that, on putting $m = 0, 1, 2, 3, \cdots, m$ in $0^{th}Row, 1^{st}Row, 2^{nd}Row, 3^{rd}Row, \cdots, m^{th}Row$ respectively. Then each row of Pascal's triangle is representing the coefficients of n^{th} indefinite integral of following integrands: $(ax^0 \pm b)\sin(cx \pm d), (ax^1 \pm b)\sin(cx \pm d), (ax^2 \pm b)\sin(cx \pm d), (ax^3 \pm b)\sin(cx \pm d), \cdots,$ $(ax^m \pm b)\sin(cx \pm d)$ respectively. On rewriting the solutions of n^{th} integral from equations (136), (141), (146) & (151) in the form of Pascal's triangle as shown above in equation (157), a beautiful pattern generates. Therefore, on comparing all the elements of the $m^{th}Row$ of Pascal's triangle, I have concluded that these elements are the coefficients of the generalized expression for the n^{th} Integral of $(ax^m \pm b)\sin(cx \pm d)$. Hence, it is the generalized solution for the integral of form 6 as shown in equation (158).

We can also find coefficient of x^q directly by using the formula as shown in equations (159), (160), (161) & (162). For calculating coefficient of x^q, we express q as an exponent of x in terms of m and p.
Here-
$q = (m - p)$ where $q \le m$.
m - is the exponent of x that is given in the integrand.
p - is the value which is calculated by putting the values of q and m.
Now after expressing the exponent of x in terms of q, we have two cases are raised as per the range of p:
First case- if p is lies in the range $0 \le p \le 2$ i.e. $p = 0, 1 \ \& \ 2$ so we use the formula as shown in equations (159), (160) & (161).
Second case- if p is lies in the range $3 \le p \le m$ i.e. $p = 3, 4, 5, \cdots, m$ so we use the formula as shown in equation (162).

Similarly, for **the integral of form 7-** In order to obtain the generalized solution for the integral of form 7, we will perform the same procedure and steps which are discussed above to evaluate the generalized solution of the integral of form 6.

VI. Conclusion

In this paper, I have derived the generalized expression for the n^{th} indefinite integration of transcendental and non-transcendental functions with a higher exponent. It eliminates the entire long procedure of successive integration and generates the compact form of the final solution in just a single step. These derived expressions shortcut the entire process of complex calculations and also remove the computational errors from the solution. The tedious and troubling repeated integration of a function having higher powers will also become easy to solve for students and researchers. In the generalized expression for the integrals which are categorized in integrals of Type-III, the divine beautiful symmetry in the derived expressions is revealed and it generates the key to achieve the solution of the n^{th} integration of periodic functions which was previously impossible to evaluate analytically. The coefficients of these generalized expressions are expressed in the terms of binomial coefficients and derived by the rows of Pascal's triangle. I have also expressed these generalized expressions in the aesthetic form of sequence and series that produces the multi-dimensional recurrence relation which shows the mathematical beauty of the derived formulae in this paper. This research paper shows the relationship between the number theory and repeated integration of periodic functions. In the future, these generalized expressions will be very applicable in various fields of computer science to build the more efficient calculators for fractional calculus and in the area of signals and systems, these formulae will be very useful for solving the n^{th} integration of various integral transforms. These expressions can also be used to obtain the compact form of the general solution for Simple Harmonic Motion (S.H.M.) differential equations of a physical system such as a system of N-coupled oscillators and also for the particular integral of n^{th} order differential equation with constant coefficients. In quantum mechanics, these expressions will also helpful for solving the harmonic wave function of the particle in the n-dimensional space i.e. Hilbert Space. Hence, using these formulas and expressions which are discussed in this paper, are very useful for the mathematicians and physicists to examine and explore the more applications of successive integration.

Declaration

The work presented in this paper has been proposed, solved and typed by the author (myself). No other person has contributed in any manner to this research work.

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