# Total Neighborhood Prime Labeling of Cycle Related Graphs

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# Abstract

Let G = (V, E) be a graph with p vertices and q edges. A bijection  $f:V(G) \cup E(G) \rightarrow \{1,2,3,...,p+q\}$  is said to be total neighborhood prime labeling if it satisfies that for each vertex of degree at least two, the gcd of labeling on the neighborhood vertices is one and for each vertex of degree at least two, the gcd of labeling on the induced edges is one. In this paper we investigate total neighborhood prime labeling of cycle  $C_n$  with a chord, switching of any vertex in a cycle  $C_n$  path union of finite number of copies of the cycle  $C_n$  and the graph obtained by joining two copies of a cycle  $C_n$  by a path  $P_k$ .

Keywords: Neighborhood prime labeling, total neighborhood prime labeling, vertex switching, path union

# I. Introduction

The prime labeling concept was initiated by Roger Entringer and was introduced in 1980's by Tout et al [1]. S K Patel and N P Shrimali [2] introduced the neighborhood prime labeling and proved many results on it. Total neighborhood prime labelingwere introduced in [3] and proved paths and cycles are total neighborhood prime graphs. In [4] proved total neighbourhood prime labeling of disjoint union of paths, disjoint union of sunlets, disjoint union of wheels, graph obtained by one copy of path  $P_n$  and n copies of  $K_{1,m}$  and joining i<sup>th</sup> vertex of  $P_n$  with an edge to fix vertex in the i<sup>th</sup> copy of  $K_{1,m}$ , corona product of cycle with m copies of  $K_1$  and, subdivision of bistar.

In this paper we consider finite graphs which are simple, connected and undirected. For notations and definitions in graph theory we follow [5] and Gallian [6] for all terminology regarding prime and neighborhood prime labelings. We follow [7] for number theoretical results.

Total neighbourhood prime labeling of cycle  $C_n$  was discussed in [3]. In this paper we investigate total neighborhood prime labeling of cycle  $C_n$  with a chord for all  $n \ge 4$ , switching of any vertex in a cycle  $C_n$ , path union of finite number of copies of cycle  $C_n$  and the graph obtained by joining two copies of a cycle  $C_n$  by a path  $P_k$ . In this paper there are three sections in which section I deals with introduction of total neighborhood prime labeling. Section II is about preliminaries which contains some definitions. Section III includes main results regarding total neighborhood prime labeling.

# **II. Preliminaries**

# **Definition 1.**

Let G = (V (G), E (G)) be a graph with p vertices. A bijection  $f: V(G) \rightarrow \{1, 2, 3, ..., p\}$  is called prime labeling if for each edge e=uv, gcd (f (u), f (v)) = 1. A graph which admits prime labeling is called a Prime graph.

# Definition 2.

Let G = (V (G), E (G)) be a graph with p vertices. A bijection  $f: V(G) \rightarrow \{1, 2, 3, ..., p\}$  is called neighborhood prime labeling if for each vertex  $v \in V(G)$  with deg (v) >1, gcd {f (u):  $u \in N(v)$ } = 1. A graph which admits neighborhood prime labeling is called a Neighborhood prime graph.

For a vertex  $v \in V(G)$ , the set of all vertices in G which are adjacent to v is called the neighborhood of v and is denoted by N (v). If every vertex is of degree at most one in a graph G, then it is neighborhood prime graph. In [8] proved that, every cycle  $C_n$  with a chord is neighborhood prime for all  $n \ge 4$ , the graph obtained by switching of any vertex in a cycle  $C_n$  is neighborhood prime, the graph obtained by the path union of finite number of

copies of cycle  $C_n$  is neighborhood prime and the graph obtained by joining two copies of cycle  $C_n$  by a path  $P_k$  is neighborhood prime.

# **Definition 3.**

Let G = (V (G), E (G)) be a graph with p vertices and q edges. A bijection  $f: V(G) \cup E(G) \rightarrow \{1, 2, 3, ..., p + q\}$  is said to be total neighborhood prime labeling if it satisfies the following two conditions:

- 1. For each vertex of degree at least two, the gcd of labeling on its neighborhood vertices is one;
- 2. For each vertex of degree at least two, the gcd of labeling on the induced edges is one.

A graph which admits total neighborhood prime labeling is called Total neighborhood prime graph. The cycle  $C_4$  is a total neighborhood prime graph but the cycle  $C_3$  is not a total neighborhood prime graph. The path  $P_n$  is a total neighborhood prime graph and the cycle  $C_n$  is a total neighborhood prime graph if n is even and  $n \neq 2(mod4)$ .

# **Definition 4.**

A vertex switching  $G_v$  of a graph G is obtained by taking a vertex v of G, removing all the edges incident with v and adding edges joining v to every vertex which are not adjacent to G.

# **Definition 5.**

Let  $G_1, G_2, ..., G_n, n \ge 2$  be n copies of a fixed graph G. The graph obtained by adding an edge between  $G_i$  and  $G_{i+1}$  for i = 1, 2, ..., n-1 is called the path union of G.

#### **III. Main Results**

# Theorem 1.

Every cycle  $C_n$  with a chord is total neighborhood prime graph for all  $n \ge 4$ .

## Proof.

Let G be the graph obtained by joining two non-adjacent vertices of cycle  $C_n$  with a chord for all  $n \ge 4$ .Let {  $v_1, v_2, \ldots, v_n$ } be the vertex set of G. Let  $\{e_1, e_2, \ldots, e_{n-2}\}$  are the vertices in the cycle of G and e be the edge connecting two non-adjacent vertices of the cycle  $C_n$ . Let the number of vertices of G be n and the number of edges of G be n+1.

(1) If  $n \not\equiv 2 \pmod{4}$ , define a function  $f: V(G) \cup E(G) \rightarrow \{1, 2, 3, \dots, 2n + 1\}$  as follows:

Case (1). If n is odd, let

$$f(v_{2j-1}) = \frac{n-1}{2} + j, 1 \le j \le \frac{n+1}{2}$$
$$f(v_{2j}) = j, 1 \le j \le \frac{n-1}{2}$$
$$f(e_i) = f(v_n) + i, 1 \le i \le n$$
$$f(e) = f(e_n) + 1$$

Case (2). If n is even, let

$$f(v_{2j-1}) = \frac{n}{2} + j, 1 \le j \le \frac{n}{2}$$
$$f(v_{2j}) = j, 1 \le j \le \frac{n}{2}$$
$$f(e_i) = f(v_n) + i, 1 \le i \le n$$
$$f(e) = f(e_n) + 1$$

The neighborhood vertices of each vertex  $v_i$  except  $v_n$  is  $\{v_{i-1}, v_{i+1}\}$  and they are consecutive integers, so it is neighborhood prime. The neighborhood vertices of  $v_n$  is  $\{v_{n-1}, v_1\}$  and the corresponding labels are consecutive integers  $\frac{n-1}{2}$  and  $\frac{n+1}{2}$  if n is odd, n and  $\frac{n}{2} + 1$  if n is even. Select the vertex  $v_i$  and join this to any vertex of G which is not adjacent to  $v_i$ . Then the gcd of labeling of the neighborhood vertices of each vertex is one. Also for each vertex of G, the gcd of labeling on the induced edges is one. So G is a total neighborhood prime graph.

(2) If  $n \equiv 2 \pmod{4}$ , the labeling of the same function shows that there exists at least

one vertex vi whose neighborhood set is not prime. Now join with a chord to the vertex

v<sub>i</sub> which is not adjacent and relatively prime to v<sub>i</sub> and e be the edge connecting v<sub>i</sub> and

v<sub>j</sub>. Then G is a neighborhood prime graph.

#### **Corollory 1.**

Every cycle  $C_n$ ,  $n \in N$  with n - 3 chords from a vertex is total neighbourhood prime graph if  $n \ge 5$ .

#### Proof.

Let G be a graph such that  $G = C_n$ , n > 5. Let  $\{v_1, v_2, ..., v_n\}$  be the vertexset of G.Choose an arbitrary vertex  $v_i$  and joining  $v_i$  to all the vertices which are not adjacent to  $v_i$ . Then there are n-3 chords to  $v_i$  and from Theorem 1,G admits neighborhood prime labeling.

#### Theorem 2.

The graph obtained by switching of any vertex in a cycle  $C_n$  is total neighbourhood prime graph for every  $n \in N$ .

#### Proof.

Let  $G = C_n$  and  $v_1, v_2, ..., v_n$  be the successive vertices of  $C_n$ . Let  $G_v^*$  denotes the vertex switching of G with respect to the vertex  $v_k$ . Here  $|V(G_v^*)| = n$  and  $|E(G_v^*)| = 2n - 5$ .

Define a function  $f: V \cup E \rightarrow \{1, 2, 3, ..., 3n - 5\}$  as follows:

$$\begin{split} f(v_k) &= 1 \\ f(v_i) &= i+1, \ 1 \leq i \leq k-1 \\ f(v_{k+i}) &= f(v_{k-1})+i, \ 1 \leq i \leq n-k \\ f(e_j) &= n+j, \ 1 \leq j \leq 2n-5. \end{split}$$

Clearly f is a bijective map. Then for any vertex  $v_i$  other than  $v_k$ , then eighborhood vertices containing  $v_k$  and so the gcdof the label of vertices in  $N(v_i)$  is 1.

Also gcd  $\{f(u) : u \in N(v_k)\} = gcd (2,3,...,n) = 1$ . The gcd of labeling onincident edges of every vertex in  $G_v^*$  is one. Hence  $G_v^*$  is a total neighborhood primegraph.

# Theorem 3.

The ring sum of the cycle  $C_n$  and the star graph K <sub>1,n</sub> is total neighbourhood prime graph for every  $n \in N$ .

# Proof.

Let  $G = C_n \bigoplus K_{1,n}$  be the graph with ring sum of the cycle  $C_n$  and  $K_{1,n}$ .Let  $v_1, v_2, ..., v_n$  be the vertices of  $C_n$  and  $u = v_1, u_1, u_2, ..., u_n$  be the vertices of  $K_{1,n}$ . The corresponding edges in  $C_n$  are  $e_j = v_j v_{j+1}, 1 \le j \le n$ ,  $e_n = v_n v_1$  and the corresponding edges in  $K_{1,n}$  are  $e_k = uu_k, 1 \le k \le n$ . The total number of vertices and edges in G is 4n.

Define a function  $f: V \cup E \rightarrow \{1, 2, ..., 4n\}$  as follows.

 $f(v_i) = 2i - 1, 1 \le i \le n$  $f(u_i) = 2i, 1 \le i \le n$   $f(e_j) = 2n + j, \ 1 \le j \le n$ 

 $f(e_k) = 3n + k, \ 1 \le k \le n.$ 

Clearly f is a bijective map.

Here  $u_1, u_2, ..., u_n$  are pendant vertices. The labeling on the neighbourhood vertices of  $v_i$  ( $i \neq 1$ ) are alternate odd integers. They are relatively prime. Also, for each vertex of degree atleast two, the gcd of labeling on the incident edges is one. Hence G is a neighbourhood prime graph.

# Corollory 2.

The ring sum of G and the star graph  $K_{1,n}$  is total neighbourhood graph, where G is a cycle with one chord and chord form a triangle with two edges of the cycle, for every  $n \in N$ .

# **Corollory 3.**

Let G be a cycle  $C_n$  with (n-3) chords from a vertex. Then  $G \bigoplus K_{1,n}$  is total neighbourhood prime graph for every  $n \in N$ .

# Theorem 4.

The graph obtained by the path union of finite number of copies of cycle  $C_n$ ,  $n \in N$  is total neighborhood prime graph if n is even and  $n \not\cong 2 \pmod{4}$ .

# Proof.

Let  $G^*$  be the path union of cycle  $C_n$  and  $G_1, G_2, ..., G_k$  be k copies of cycle  $C_n$ . The vertices of  $G^*$  is nk and edges of  $G^*$  is (n + 1)k - 1. Let us denote the vertices of  $G^*$  be  $v_{ij}$ ,  $1 \le i \le n$ ,  $1 \le j \le k$  and the successive vertices of the graph  $G_r$  by  $v_{1r}, v_{2r}, ..., v_{nr}$ . Let  $e_{ij}$  be the edge joining consecutive vertices in the cycleof  $G_r$  and  $e_m = v_{1r}v_{1(r+1)}$  betheedgejoining  $G_r$  and  $G_{(r+1)}$  forr = 1, 2, ..., k-1.

Define a function  $f: V \cup E \rightarrow \{1, 2, ..., 2nk + k - 1\}$  as follows.

For  $1 \le j \le k$ :

$$f(v_{(2i-1)j}) = nj + i - \frac{n}{2}, 1 \le i \le \frac{n}{2}$$
$$f(v_{(2i)j}) = n(j-1) + i, 1 \le i \le \frac{n}{2}$$
$$f(e_{ij}) = n(k+j-1) + i, 1 \le i \le n$$
$$f(e_m) = 2nk + m, 1 \le m \le k - 1.$$

Clearly f is a bijection.

We claim that f is a neighborhood prime labeling.

If  $v_{ir}$  is any vertex of G in the r<sup>th</sup> copy of the cycle  $C_n$  different from  $v_{1r}$ , then  $N(v_{ir}) = \{v_{(i-1)r}, v_{(i+1)r}\}$ . Since  $f(v_{(i-1)r})$  and  $f(v_{(i+1)r})$  are consecutiveintegers, gcdoflabeling of the neighbourhood vertices of  $v_{ir}$  is 1. Since  $N(v_{11}) = \{v_{n1}, v_{21}\}$  and  $f(v_{21}) = 1$ , the gcd of labeling of the neighbourhood vertices of  $v_{11}$  is 1. Now we consider  $v_{1r}, (1 \le r \le k)$ .

$$N(v_{1r}) \text{ are } n\left(r - \frac{3}{2}\right) + \frac{1}{2}, n\left(r + \frac{1}{2}\right) + \frac{1}{2}, n(r-1) + 2 \text{ and } n\left(r - \frac{1}{2}\right)$$
$$gcd(n\left(r - \frac{3}{2}\right) + \frac{1}{2}, n\left(r + \frac{1}{2}\right) + \frac{1}{2}, n(r-1) + 2, n\left(r - \frac{1}{2}\right))$$
$$= gcd(2n, n\left(r + \frac{1}{2}\right) + \frac{1}{2}, n(r-1) + 2, n\left(r - \frac{1}{2}\right))$$
$$= gcd(2n, n + \frac{1}{2}, n(r-1) + 2, n\left(r - \frac{1}{2}\right))$$
$$= gcd(2n, n + \frac{1}{2}, \frac{n}{2} - 2, n\left(r - \frac{1}{2}\right))$$

$$= \gcd(1,9,\frac{n}{2} - 2, n\left(r - \frac{1}{2}\right))$$
$$= 1.$$

Finally we consider  $v_{1k}$ . N( $v_{1k}$ ) are  $n\left(k-\frac{3}{2}\right)+\frac{1}{2}$ , n(k-1)+1 and  $n(k-\frac{1}{2})$ .

$$gcd \left(n\left(k-\frac{3}{2}\right)+\frac{1}{2}, n(k-1)+1, n(k-\frac{1}{2})\right)$$

$$= gcd \left(n-\frac{1}{2}, \frac{n}{2}-1, n(k-\frac{1}{2})\right)$$

$$= gcd \left(n-\frac{1}{2}, 2k+1, 2k-\frac{n}{2}\right)$$

$$= gcd \left(\frac{1}{2}+4k, 2k+1, 2k-\frac{n}{2}\right)$$

$$= gcd \left(2k-\frac{1}{2}, 2k+1, 2k-\frac{n}{2}\right)$$

$$= 1.$$

Also, for each vertex of degree at least two, the gcd of labeling on the incident edges is one.

Hence  $G^*$  is total neighborhood prime if  $n \not\cong 2 \pmod{4}$  and n is even.

# Corollory4.

Let G<sup>\*</sup> be the graph obtained by the path union of finite number of copies of cycle C<sub>n</sub>, then

1. If  $n \equiv 2 \pmod{4}$ ,  $G^*$  is not a total neighbourhood prime graph.

2. If n is odd,  $G^*$  is not total neighbourhood prime graph.

# Proof.

The cycle  $C_n$  is not total neighborhood prime if  $n \equiv 2 \pmod{4}$ . Thus  $G^*$  is not total neighborhood prime if  $n \equiv 2 \pmod{4}$ . The cycle  $C_n$  is not total neighborhood prime if n is odd. Thus  $G^*$  is not total neighborhood prime if n is odd.

# Theorem 5.

The graph obtained by joining two copies of cycle  $C_n$  by a path  $P_k$  istotal neighborhood prime graph if n is even and n  $\ncong 2(\text{mod}4)$ .

# Proof.

Let  $G^{**}$  be the graph obtained by joining two copies of cycle  $C_n$  by a path  $P_k$ . The vertices of  $G^{**}$  are 2n + k - 2and edges of  $G^{**}$  are 2n + k - 1. Let  $v_1, v_2, ..., v_n$  be the vertices of the first copy of cycle  $C_n$  and  $w_1, w_2, ..., w_n$  be the vertices of the second copy of cycle  $c_n$ . Let  $u_1, u_2, ..., u_k$  be the vertices of path  $P_k$  with  $v_1 = u_1$  and  $w_1 = u_k$ . Let  $e_{1i}$  $=v_iv_j$  be the edge joining the vertices in the first copy of cycle  $C_n$  and  $e_{2i} = w_iw_j$  be the edge joining the vertices in the second copy of cycle  $C_n$ . Let  $e_m, 1 \le m \le k - 1$  be the edge joining the vertices in the path  $P_k$ .

Define a bijective function  $f: V \cup E \rightarrow \{1, 2, 3, ..., 4n+2k-3\}$  as follows.

The labeling on C<sub>n</sub> are:

$$f(v_{2i-1}) = \frac{n}{2} + i, 1 \le i \le \frac{n}{2}$$
$$f(v_{2i}) = i, 1 \le i \le \frac{n}{2}$$
$$f(w_{2i-1}) = \frac{3n}{2} + i, 1 \le i \le \frac{n}{2}$$

$$f(w_{2i}) = n + i, 1 \le i \le \frac{n}{2}$$

The labeling on Pk are:

Case 1: If k is odd:

$$f(u_{2i}) = 2n + \frac{k-3}{2} + i, 1 \le i \le \frac{k-1}{2}$$
$$f(u_{2i+1}) = 2n + i, 1 \le i \le \frac{k-3}{2}$$

Case 2: If k is even:

$$f(u_{2i}) = 2n + \frac{k-2}{2} + i, 1 \le i \le \frac{k-2}{2}$$
$$f(u_{2i+1}) = 2n + i, 1 \le i \le \frac{k-2}{2}$$

The labeling on the edges are

$$\begin{split} f(e_{1i}) &= (2n+k-2)+i, \ 1 \leq i \leq n \\ f(e_{2i}) &= (3n+k-2)+i, \ 1 \leq i \leq n \\ f(e_m) &= (4n+k-2)+m, \ 1 \leq m \leq k-1. \end{split}$$

We claim that f is a neighborhood prime labeling.

If  $v_i$  is any vertex of G in the first copy of the cycle  $C_n$  different from  $v_1$ , then  $N(v_i) = \{v_{i-1}, v_{i+1}\}$ . Since  $f(v_{i-1})$  and  $f(v_{i+1})$  are consecutive integers, gcd of the label of the vertices is 1. Also  $N(v_1)$  contains the vertex  $v_2$  and  $f(v_2) = 1$ , gcd of the label of vertices in  $N(v_1)$  is 1.

If  $w_i$  is any vertex of G in the second copy of the cycle  $C_n$  different from  $w_1$ , then  $N(w_i) = \{w_{i-1}, w_{i+1}\}$ . Since f  $(w_{i-1})$  and f  $(w_{i+1})$  are consecutive integers, gcd of the label of the vertices is 1. Now, consider w1:

N(w<sub>1</sub>) are {w<sub>2</sub>,w<sub>n</sub>,u<sub>k-1</sub>} and the labelings are  $f(w_2) = n+1$ ,  $f(w_n) = \frac{3n}{2}$ .

 $gcd(n+1,\frac{3n}{2})$ 

 $= \gcd(n + 1, 4n + 1)$ 

= gcd (n + 1,3n)

= gcd (n + 1,3)

= 1.

Since  $n \not\cong 2 \pmod{4}, n + 1$  is not a multiple of 3.

Finally, if  $u_i$  is any vertex of G in the path  $P_k$  different from  $u_i$  and  $u_k$ , then

 $N(u_i) = \{u_{i-1}, u_{i+1}\}$ . Since  $f(u_{i-1})$  and  $f(u_{i+1})$  are consecutive integers, gcd of the label of vertices of  $N(u_i)$  is 1. Also, for each vertex of degree atleast two, the gcd of labeling on the incident edges is one.  $G^{**}$  is total neighborhood primegraph, if  $n \not\cong 2(\text{mod}4)$ .

#### **Corollory 5.**

Let  $G^{**}$  be the graph obtained by joining two copies of cycle  $C_n$  by a path  $P_k$ , then

1. If  $n \equiv 2 \pmod{4}$ ,  $G^{**}$  is not total neighbourhood prime graph.

2. If n is odd,  $G^{**}$  is not total neighbourhood prime graph.

## Proof.

The cycle  $C_n$  is not total neighborhood prime if  $n \equiv 2 \pmod{4}$ . Thus  $G^{**}$  is not total neighborhood prime if  $n \equiv 2 \pmod{4}$ . The cycle  $C_n$  is not total neighborhood prime if n is odd. Thus  $G^{**}$  is not total neighborhood prime if n is odd.

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