Some discussion on properties of Wronskian in boundary value problem

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Abstract — *In this paper we have discussed some properties of Wronskian and explained these properties with suitable examples. In the present work we also established some new properties of Wronskian and verified the result with examples.*

Keywords — *Wronskian, Properties of Wronskian.*

1. Introduction

Differential equation is very useful in the field of engineering, physics, chemistry, economics etc. Boundary value problem plays an important role in several branches, having a physical differential equation together with a set of additional constraints called the boundary condition.

Example: A boundary value problem is given by,
y'' + 8y = 0, subject to the condition, y (0) = 3 and y $(\frac{\pi}{2})$ = 0.

Ch. equation is, $r^2 + 8 = 0$

⇨ $r = 0 + i2\sqrt{2}$

⇨ $y = c_1 \text{Cos}\,(2\sqrt{2}\,t) + c_2 \text{ Sin}\,(2\sqrt{2}\,t)$

using the boundary condition, y(0) = 3 and y $(\frac{\pi}{2})$ = 0.

⇨ $3 = c_1 . 1 + c_2 . 0$

⇨ $c_1 = 3$

And, $0 = 3 \text{ Cos}\,(\sqrt{2}\,\pi) + c_2 \text{ Sin}\,(\sqrt{2}\,\pi)$

⇨ $c_2 = -\frac{3 \text{ Cos}(\sqrt{2}\,\pi)}{\text{Sin}\,(\sqrt{2}\,\pi)} = -3 \text{ Cot}(\sqrt{2}\,\pi)$

Hence, $y = 3\text{Cos}\,(2\sqrt{2}\,t) - 3 \text{ Cot}(\sqrt{2}\,\pi) \text{ Sin}\,(2\sqrt{2}\,t)$.

Wronskian:

In the field of Mathematics the term Wronkian is a determinant introduced by Jozef Hoene-Wronski (1776). It is required in the field of differential equation where it helps to find out the linear independence in the set of solution [1] & [2]. The properties and the solution of Wronskian differential equation was studied in [3] & [4].

Let a linear homogeneous equation of the form, $y'' + p(t)\,y' + q(t)\,y = 0$,
Let two solution of this equation are u and v, So Wronskian of this equation can be written as
$W[u, v] = uv' - vu'$. 1.1

(a) If u is a constant multiple of v then W [u, v] is identically zero. Then u and v are linearly dependent.
(b) If u and v agree at some point t_o and their derivative also exist at t_o, then W[u, v] vanishes at t_o, that is if u and v are two solution of same initial value problem then their Wronskian vanishes at t_o.

Let f(t) and g(t) be two differential function then they are linearly dependent, if there are non-zero constants c_1 and c_2 with, c_1f(t) + c_1g(t) =0, for all t, otherwise they are called linearly independent.
Example**:** The functions, f(t) =10$t^2 + t^3$ and g(t) = - t^4 are linearly independent.
If the function f(t) and g(t) are linearly dependent then there would be a nonzero constant c_1 and c_2 such that,
c_1 f(t) + c_2g(t) = 0.
c_1 $(10t^2 + t^3)$+ $c_2(-t^4)$ = 0, for all t,
When t= - 1, then,
$9c_1$ - c_2= 0 (1)
When t= - 2, then,
$2c_1$ - c_2 = 0 (2)
Equations (1) and (2) are the system of linear equations. Now the determinant of the corresponding coefficient matrix is,
$\begin{vmatrix} 9 & -1 \\ 2 & -1 \end{vmatrix}$ = - 9 + 2 = -7 $\neq$ 0.
Since the determinant is nonzero, the only solution is the trivial solution,
That is, $c_1 = c_2$.
Hence the given two functions are linearly independent.

2. Theorems

2.1 Theorem

W [Φ_1, Φ_2] = - W [Φ_2, Φ_1]

Example: Φ_1= tanx and Φ_2 = Cosecx, then
W [Φ_1, Φ_2] = $\begin{vmatrix} tanx & Cosxec \\ Sec^2x & -CosecxCotx \end{vmatrix}$
= - tanx. Cosecx.Cotx – Cosecx. Sec^2x
= - Cosecx (1 + Sec^2x), And,
W [Φ_2, Φ_1] = $\begin{vmatrix} Cosecx & tanx \\ -Cosecx\,Cotx & Sec^2x \end{vmatrix}$
= Cosecx. Sec^2x + tanx. $Cosecx$. Cotx
= Cosecx (1 + Sec^2x),
= - W[Φ_1, Φ_2]
So, W [Φ_1, Φ_2] = - W [Φ_2, Φ_1]

2.2 Theorem

W [αΦ_1, βΦ_2] = αβ W [Φ_1, Φ_2], where α and β are constant,

Example: Φ_1= a^x and Φ_2= e^x, and α = 5 and β = 7, then
W [αΦ_1, βΦ_2] = W [5a^x , 7e^x]
= $\begin{vmatrix} 5a^x & 7e^x \\ 5a^x loga & 7e^x \end{vmatrix}$
= 5 × 7 $\begin{vmatrix} a^x & e^x \\ a^x loga & e^x \end{vmatrix}$ = 5 × 7 W [Φ_1, Φ_2].

2.3 Theorem

Let Φ_1 and Φ_2 be any two differential function, then
W [Φ_1+ α, Φ_2 + α] = W [Φ_1, Φ_2] + α$\frac{d}{dx}$[Φ_2 - Φ_1] where α is a constant.

Example: Let Φ_1= Sin3x, Φ_2 = Cos5x, and let α = 7, W [Sin3x + 7, Cos5x + 7] = $\begin{vmatrix} \text{Sin}3x+7 & \text{Cos}5x+7 \\ 3\,Cos3x & -5Sin5x \end{vmatrix}$

= $(\text{Sin}3x+7)(-5Sin5x) - (3\,Cos3x)(\text{Cos}5x+7)$

= $(-5\text{Sin}3x)(\text{Sin}3x) - (3\,Cos3x)(\text{Cos}5x) + 7[-5Sin5x - 3\,Cos3x]$

= $W[\Phi_1, \Phi_2] + 7\frac{d}{dx}[\Phi_2 - \Phi_1]$.

2.4 Theorem

Let Φ_1 and Φ_2 be any two differential function, then

$\Phi_1 W[\frac{\Phi_2}{\Phi_1}, \Phi_1] + \Phi_2 W[\frac{\Phi_1}{\Phi_2}, \Phi_2] = \frac{d}{dt}(\Phi_1, \Phi_2)$.

Example: Let us consider two differential function, Φ_1 and Φ_2 such that

$\Phi_1 = 2\,Sin^2\text{t}$ and $\Phi_2 = e^{2t}$,

$\Phi_1 W[\frac{\Phi_2}{\Phi_1}, \Phi_1] = 2\,Sin^2\text{t}\; W\left[\frac{e^{2t}}{2\,Sin^2 t}, \; 2Sin^2\text{t}\right]$

$= 2\,Sin^2\text{t}\begin{vmatrix} \frac{e^{2t}}{2\,Sin^2 t} & 2Sin^2\text{t} \\ \frac{2e^{2t}\,2Sin^2 t - e^{2t}\,4Sint\ Cost}{4\,Sin^4 t} & 4Sint\ Cost \end{vmatrix}$

$= 8e^{2t}\ Sint\ Cost - 4\,e^{2t}Sin^2\text{t}$ (3)

And, $\Phi_2 W[\frac{\Phi_1}{\Phi_2}, \Phi_2] = e^{2t}\ W\left[\frac{2\,Sin^2 t}{e^{2t}}, \; e^{2t}\right]$

$= e^{2t}\begin{vmatrix} \frac{2\,Sin^2 t}{e^{2t}} & e^{2t} \\ \frac{4Sint\ Cost.e^{2t} - 2\,Sin^2 t.\ 2e^{2t}}{e^{4t}} & 2e^{2t} \end{vmatrix}$

$= e^{2t}(\,8Sin^2\text{t} - 4\ Sint\ Cost\,)$

$= 8e^{2t}Sin^2\text{t} - 4e^{2t}\ Sint\ Cost$ (4)

Adding (3) and (4) we get,

$\Phi_1 W[\frac{\Phi_2}{\Phi_1}, \Phi_1] + \Phi_2 W[\frac{\Phi_1}{\Phi_2}, \Phi_2] = 4\,e^{2t}Sint\ Cost + 4\,e^{2t}\ 2\,Sin^2\text{t}$

$= \frac{d}{dt}[(2Sin^2\text{t})(e)^{2t}$

$= \frac{d}{dt}[\Phi_1, \Phi_2]$.

*2.5 **Theorem.***

Let Φ_i and φ_j are the differential function of x for i= 1,2…n and j= 1,2…n,

then, $W[\prod_{i=1}^{n}\Phi_i,\ \prod_{j=1}^{n}\varphi_j] = \sum_{i=1}^{n}\{(\prod_{j=1, i\neq j}^{n}\Phi_j\varphi_j)\ W[\Phi_i,\ \varphi_i]\}$

(For, i= 1,2….,n and j=1,2,…..n)

Example: - For i= 1, 2 and j= 1, 2,

Let, Φ_1= Sint, Φ_2= Cost, $\varphi_1 = a^t$, $\varphi_2 = e^t$

$W[\prod_{i=1}^{2}\Phi_i,\ \prod_{j=1}^{2}\varphi_j] = W[\Phi_1\Phi_2,\ \varphi_1\varphi_2] = W[\text{Sint Cost},\ a^t e^t]$

$= \begin{vmatrix} \text{Sint Cost} & a^t e^t \\ \text{Cos}^2 t - Sin^2 t & a^t loga.e^t + a^t.e^t \end{vmatrix}$

$= a^t e^t\text{Sint Cost}\,(1 + \text{loga}) - a^t e^t\ \text{Cos2t}$ (5)

And, $\sum_{i=1}^{n}\{(\prod_{j=1, i\neq j}^{n}\Phi_j\varphi_j)\ W[\Phi_i,\ \varphi_j]\} = \Phi_1\varphi_1 W[\Phi_2,\ \varphi_2] + \Phi_2\varphi_2 W[\Phi_1\ \varphi_1]$

= Sint. a^t W[Cost, e^t] + Cost. e^t W[Sint, a^t]

$= a^t.\ \text{Sint}\begin{vmatrix} \text{Cost} & e^t \\ -Sint & e^t \end{vmatrix} + \text{Cost}.\ e^t\begin{vmatrix} \text{Sint} & a^t \\ \text{Cost} & a^t loga \end{vmatrix}$

$= a^t e^t \text{Sint Cost} (1 + \log a) - a^t e^t \text{ Cos2t}$ (6)

From (5) and (6) we conclude that,

$$W [\Phi_1 \Phi_2, \varphi_1 \varphi_2] = \Phi_1 \varphi_1 W [\Phi_2, \varphi_2] + \Phi_2, \varphi_2 W[\Phi_1 \varphi_1]$$

3. Important properties of Wronskian:

3.1 W [0, g(t)] = 0
3.2 W [f(t), f(t)] = 0
3.3 W [1, g(t)] = g'(t)
3.4 W [f(t), g(t) + h(t)] = W [f(t), g(t)] + W [f(t), h(t)]
3.5 W' [f(t), g(t)] = fg'' - f''g = W[f, g'] + W[f', g]

Example: (3.5)

Let, f (t) = 1 - t^3 and g(t) = 7 t^5

$$W [f(t), g(t)] = \begin{vmatrix} 1 - t^3 & 7 t^5 \\ -3t^2 & 35t^4 \end{vmatrix}$$
$$= 35 t^4 - 14 t^7$$

$$W'[f(t), g(t)] = \frac{d}{dt}(35 t^4 - 14 t^7)$$
$$= 140 t^3 - 98t^6$$
$$= (1 - t^3)(140 t^3) - (-6t)(7t^5)$$
$$= f g'' - f''g \qquad (7)$$

$$W [f, g'] + W[f', g] = \begin{vmatrix} f & g' \\ f' & g'' \end{vmatrix} + \begin{vmatrix} f' & g \\ f'' & g' \end{vmatrix}$$
$$= \begin{vmatrix} 1 - t^3 & 35 t^4 \\ -3t^2 & 140t^3 \end{vmatrix} + \begin{vmatrix} -3t^2 & 7t^5 \\ -6t & 35t^4 \end{vmatrix}$$
$$= 140 t^3 - 98t^6 \qquad (8)$$

From (7) and (8) we conclude that,

W' [f(t), g(t)] = fg'' - f''g = W[f, g'] + W[f', g]

3.6 For any constant, c

W [f(t), cg(t)] = c W[f(t), g(t)] = W[cf(t), g(t)]

3.7 W[f(t),g(t)] = - W[g(t), f(t)]

4. Some more important results:

4.1 W[f(t), c (g(t) + h(t))] = W [f(t), cg(t)] + W [f(t), ch(t)]

Example: Let, f(t) = a^t, g(t) = e^t , h(t) = t^3 and c = 7

$$W[f(t), c (g(t) + h(t))] = \begin{vmatrix} a^t & 7(e^t + t^3) \\ a^t \log a & 7(e^t + 3t^2) \end{vmatrix}$$
$$= 7a^t e^t(1 - \log a) + 7a^t t^2 (3 - t \log a) \qquad (9)$$

W [f(t), cg(t)] + W [f(t), ch(t)]

$$= \begin{vmatrix} a^t & 7e^t \\ a^t \log a & 7e^t \end{vmatrix} + \begin{vmatrix} a^t & 7t^3 \\ a^t \log a & 21t^2 \end{vmatrix}$$
$$= 7a^t e^t(1 - \log a) + 7a^t t^2 (3 - t \log a) \qquad (10)$$

From (9) and (10) we conclude that,

W[f(t), c (g(t) + h(t))] = W [f(t), cg(t)] + W [f(t), ch(t)]

4.2 W [f(t), g(t) + h(t) + s(t)] = W [f(t), g(t)] + W [f(t), h(t)] + W [f(t), s(t)]

Example: Let, f(t) = t^4, g(t)= t^3, h(t) = t^2 + 1, and s(t) = t +1.

W [f(t), g(t) + h(t) + s(t)] = $\begin{vmatrix} t^4 & t^3+t^2+t+2 \\ 4t^3 & 3t^2+2t+1 \end{vmatrix}$

= $t^4(3t^2+2t+1)$ - $4t^3(t^3+t^2+t+2)$

= - t^6 - $2t^5$ - 3 t^4 - $8t^3$ (11)

W [f(t), g(t)] + W [f(t), h(t)] + W [f(t), s(t)] =

= $\begin{vmatrix} t^4 & t^3 \\ 4t^3 & 3t^2 \end{vmatrix}$ + $\begin{vmatrix} t^4 & t^2+1 \\ 4t^3 & 2t+1 \end{vmatrix}$ + $\begin{vmatrix} t^4 & t+1 \\ 4t^3 & 1 \end{vmatrix}$

= - t^6+ (-$2t^5$ - 4 t^3) + (- 3 t^4 - $4t^3$)

= - t^6 - $2t^5$ - 3 t^4 - $8t^3$ (12)

From (11) and (12) we conclude that,

W [f(t), g(t) + h(t) + s(t)] = W [f(t), g(t)] + W [f(t), h(t)] + W [f(t), s(t)]

4.3 W [f(t), g(t). h(t)] $\neq$ W [f(t), g(t)] . W [f(t), h(t)]

Example: Let, f(t) = t^5, g(t) = Sint, and h(t) = e^t

W [f(t), g(t). h(t)] = $\begin{vmatrix} t^5 & Sint.e^t \\ 5t^4 & Sint.e^t + Cost.e^t \end{vmatrix}$

=$t^5(Sint.e^t + Cost.e^t)$ - $Sint.e^t.\, 5t^4$

= t^4e^t[t $Cost$ + (t – 5) $Sint$] (13)

W [f(t), g(t)] . W [f(t), h(t)] = $\begin{vmatrix} t^5 & Sint \\ 5t^4 & Cost \end{vmatrix}$. $\begin{vmatrix} t^5 & e^t \\ 5t^4 & e^t \end{vmatrix}$

= $[t^5\, Cost - 5t^4 Sint]\, [t^5e^t - e^t 5t^4]$

= $t^4e^t.t^4$ [{tCost – 5Sint}{t – 5}]

= $t^8e^t[t^2$Cost – 5{$tCost$ + (t – 5) $Sint$}

= $t^{10}e^t$Cost - t^4.[t^4e^t{t $Cost$ + (t – 5) $Sint$}] (14)

From (13) and (14) we conclude that,

W [f(t), g(t). h(t)] $\neq$ W [f(t), g(t)] . W [f(t), h(t)]

4.4 W [f(t) + g(t), h(t) + s(t)] = W [f(t) + g(t), h(t)] + W [f(t) + g(t), s(t)]

Example: Let, f(t) = t^3 , g(t) = $Sint$, h(t) = t^5, s(t) = Cost,

W [f(t) + g(t), h(t) + s(t)] = $\begin{vmatrix} t^3 + Sint & t^5 + Cost \\ 3t^2 + Cost & 5t^4 - Sint \end{vmatrix}$

= $(t^3 + Sint)(5t^4 - Sint)$ – $(t^5 + Cost)(3t^2 + Cost)$

= $2t^7$ - t^3 $Sint$ (1- 5t) - t^2 Cost(3 + t^3) – 1. (15)

And, W [f(t) + g(t), h(t)] + W [f(t) + g(t), s(t)]

= $\begin{vmatrix} t^3 + Sint & t^5 \\ 3t^2 + Cost & 5t^4 \end{vmatrix}$ + $\begin{vmatrix} t^3 + Sint & Cost \\ 3t^2 + Cost & -Sint \end{vmatrix}$

= $2t^7$ - t^3 $Sint$ (1- 5t) - t^2 Cost(3 + t^3) – 1. (16)

From (15) and (16) we conclude that,

W [f(t) + g(t), h(t) + s(t)] = W [f(t) + g(t), h(t)] + W [f(t) + g(t), s(t)]

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