Confidence Intervals For The Common Variance Of Lognormal Distributions

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Abstract : *The aim of this paper is to investigate constructing confidence intervals for the common variance of lognormal distributions using four approaches, namely the generalized confidence intervals approach (GCI), the large sample approach (LS) and the adjusted method of variance estimates recovery approach (Adjusted MOVER) based on cox's method (AM-Cox) and based on Angus's conservative method (AM- Angus). The natural logarithm transformation was used to change the data to a normal distribution. The proposed intervals were evaluated focusing on coverage probability and average length using a Monte Carlo simulation. The results showed that the AM- Angus approach performs well in terms of coverage probability, but the average length was wide. Moreover, the coverage probability of the AM-Cox approach was closer to the nominal level more than the GCI approach. All approaches are illustrated using two real data examples.*

Keywords : *Common variance, Lognormal distribution, Generalized confidence interval, Large sample approach, Adjusted method of variance estimates recovery*

I. INTRODUCTION

Lognormal distribution is a continuous probability distribution of a random variable whose natural logarithm function is close to normal distributed. Sometimes, the lognormal is called the anti-lognormal distribution, because it is not the distribution of the logarithm of a normal variable, but is instead the anti-log of a normal variable (Johnson and Kotz [1]). The lognormal distribution has been used in a wide range of applications, such as environmental study, survival analysis, biostatistics and other statistical fields.

One of the main interests in statistical inferences is construction of the confidence intervals for the parameter of lognormal distribution. Thus, estimating the common parameter of several lognormal populations is one of the most interesting problems. There has been many papers investigating approaches for the construction of confidence intervals for lognormal distributions in terms of the common mean. For example, interval estimation of common lognormal mean of several populations was studied by Baklizi and Ebrahem [2], then Behboodian and Jafari [3] presented the generalized inference for the common mean of several lognormal populations, and then inferences on the common mean of several log-normal populations: The generalized variable approach was proposed by Tian and Wu [4] and Lin and Wang [5], they proposed a modified method on the means for several log-normal distributions. On the other hand, the construction of confidence intervals for lognormal distributions in terms of the common variance has not yet gained attention from other researchers. Thus, researchers of this paper have been motivated by ideas from previous papers about common mean to construct confidence intervals of common variance for lognormal distributions. Therefore, methods to construct confidence intervals for a common variance of lognormal distributions using the generalized confidence intervals approach (GCI), large sample approach (LS) and adjusted method of variance estimates recovery approach (adjusted MOVER) based on cox's method (AM-Cox) and Angus's conservative method (AM-Angus) are proposed. The concept of generalized confidence interval was introduced by Weerahandi [6] and since then these ideas have been frequently applied to develop confidence intervals for many common parameters and the results are all contented. Example works of Tian and Wu [4], Tian [7], Krishnamoorthy [8], Ye et al. [9] and Lin Shu-Hui and Jack C. Lee [10] pay attention to common mean. Tian [11] and C. K. NG [12] were interested in common coefficient. In addition, Tian and Wilding [13] was also interested in common correlation coefficient. The method of variance of estimates recovery (MOVER) was described by Zou and Donner [14]. Moreover, numerous applications see papers by Donner and Zou [15], Suwan and Niwitpong [16], Li et al. [17] and Sangnawakij and Niwitpong [18] have been studied. Thangjai and Niwitpong[19] proposed an adjusted method of variance estimates recovery approach (adjusted MOVER). This approach used the concept of the Mover to develop an approach to construct the confidence intervals from several populations; the result of this approach was satisfactory. Therefore, the object of this paper is to present four approaches which can construct the confidence intervals of common variance for the lognormal distributions obtained from the transformed data and then compare them for each situation. Based on the literature review, there has been no previous works focusing on construction of the confidence intervals of common variance for lognormal distributions the using proposed approaches.

The rest of the paper is structured as follows. Section 2 provides preliminaries of variance of lognormal distribution. Section 3 presents the four approaches developed and describes the computational procedures. Section 4 presents simulation results to evaluate performances of the four approaches on coverage probabilities and average lengths. Two real data sets are used to illustrate the proposed approaches in Section 5. The paper closes with a conclusion.

II. LOGNORMAL DISTRIBUTION AND PARAMETER OF INTEREST

Let $Y=(Y_1,Y_2,\ldots,Y_n)$ be independent and identically distributed as lognormal with parameters μ and σ^2. This is to say that the log-transformed variables $X_1=\log Y_1, X_2=\log Y_2,\ldots, X_n=\log Y_n$ are independent and identically distributed as normal, denoted here as $N(\mu,\sigma^2)$.

It is well known that the mean and variance of Y are

$$E(Y)= \alpha = \exp\left(\mu+\frac{\sigma^2}{2}\right) \tag{1}$$

$$\mathrm{var}(Y)= \beta^2 = \exp\left(2\mu+\sigma^2\right)\left\{\exp(\sigma^2)-1\right\} = \alpha^2\left\{\exp(\sigma^2)-1\right\}. \tag{2}$$

Consider k independent lognormal populations with a common variance η.

Let $Y_{i1},Y_{i2},\ldots,Y_{in_i}$ be a random sample from the i-th lognormal population. This is to say that the log-transformed variables $X_{il}=\log Y_{il} \sim N(\mu_i,\sigma_i^2)$, for $i=1,\ldots,k$, $l=1,\ldots,n_l$.

This chapter develops a confidence interval for the common variance. The following parameter of interest for common variance

Thus, the common variance is $$\eta=\exp\left(2\mu+\sigma^2\right)\left\{\exp(\sigma^2)-1\right\}. \tag{3}$$

The common log-variance is $$\theta=\log\eta=2\mu+\sigma^2+\log\left\{\exp(\sigma^2)-1\right\}. \tag{4}$$

Let $\bar{X}_i$ and S_i^2 denote the sample mean and variance for log-transformed data X_i for the i-th sample respectively. Let $\bar{x}_i$ and s_i^2 denote the observed sample mean and variance respectively.

The estimator of θ has the following form

$$\hat{\theta}_i = 2\bar{X}_i+S_i^2+\log\left\{\exp(S_i^2)-1\right\} \quad , \ \hat{\mu}_i=\bar{X}_i,\ \hat{\sigma}_i^2=S_i^2. \tag{5}$$

The variance of $\hat{\theta}_i$ is $$Var(\hat{\theta}_i)=\frac{4\sigma_i^2}{n_i}+\frac{2\sigma_i^4}{n_i-1}+\left[\left(\frac{\exp(\sigma_i^2)}{\exp(\sigma_i^2)-1}\right)^2\left(\frac{2\sigma_i^4}{n_i-1}\right)\right]. \tag{6}$$

The estimator and variance of estimator are used to construct confidence intervals for the common variance of lognormal distributions in the next sections.

LEMMA 1 . For a function g and random variable X we have

$$E[\mathrm{g(X)}] \approx g[E(\mathrm{X})] \quad \text{and}$$

$$\mathrm{var[g(X)]} \approx \mathrm{var(X)\{g'[E(X)]\}^2}.$$

In particular $$\mathrm{var[\log(X)]} \approx \frac{1}{\{\mathrm{E(X)}\}^2}\mathrm{var(X)}.$$

Proof. The first two terms of the Taylor series for g are:

$$g(\mathrm{x}) = \mathrm{g(a)+(x-a)\,g'(a)+\ldots.}$$

Setting $a=E(\mathrm{X})$

$$g(\mathrm{x}) = \mathrm{g}[E(\mathrm{X})]+[\mathrm{x}-E(\mathrm{X})]\mathrm{g}'[E(\mathrm{X})] \ . \tag{*}$$

Taking expectations on both sides of $(*)$ gives

$$E[\mathrm{g(X)}] \approx g[E(\mathrm{X})].$$

Squaring both sides of $(*)$ and taking the expectation gives

$$[g(\mathrm{x})]^2 \approx \mathrm{g}[E(\mathrm{X})]^2+[\mathrm{x}-E(\mathrm{X})]^2\{\mathrm{g}'[E(\mathrm{X})]\}^2+2g[\mathrm{E(X)}][\mathrm{x}-E(\mathrm{X})]\mathrm{g}'[E(\mathrm{X})]$$

$$\mathrm{E}\{[g(\mathrm{x})]^2\} \approx \{\mathrm{g}[E(\mathrm{X})]\}^2+\mathrm{var(X)}\{\mathrm{g}'[E(\mathrm{X})]\}^2$$

So $$\mathrm{var[g(X)]} = E[\mathrm{g(X)^2}]-E[\mathrm{g(X)}]^2$$

$$\text{var}[g(X)] \approx \text{var}(X)\{g'[E(X)]\}^2 .$$

The following lemma plays an important role in the derivation of the confidence intervals.

THEOREM 1 Let $X_{il} = \log Y_{il} \sim N(\mu_i, \sigma_i^2)$, $i = 1,2,\ldots,k$, $l = 1,2,\ldots,n_i$ where μ_i, σ_i^2 are respectively population mean and population variance of X_i. Then the estimator of θ is $\hat{\theta}_i = 2\bar{X}_i + S_i^2 + \log\{\exp(S_i^2)-1\}$, the variance of $\hat{\theta}_i$ is $Var(\hat{\theta}_i) = \frac{4\sigma_i^2}{n_i} + \frac{2\sigma_i^4}{n_i-1} + \left[\left(\frac{\exp(\sigma_i^2)}{\exp(\sigma_i^2)-1}\right)^2\left(\frac{2\sigma_i^4}{n_i-1}\right)\right]$.

Proof. According to LEMMA 1

$$\text{var}[\log(X)] \approx \frac{1}{\{E(X)\}^2}\text{var}(X).$$

Hence,

$$Var[\log\{\exp(S_i^2)-1\}] \approx \frac{1}{\{E[\exp(S_i^2)-1]\}^2} Var[\exp(S_i^2)-1].$$

Let $g(\sigma_i^2) = \exp(\sigma_i^2)-1$, $g'(\sigma_i^2) = \exp(\sigma_i^2)$ exists and is not 0. Using the Delta method, one can obtain the following approximate expression for the mean and variance of $\exp(S_i^2)-1$;

$$E[\exp(S_i^2)-1] \approx g(\sigma_i^2) = \exp(\sigma_i^2)-1,$$

$$Var[\exp(S_i^2)-1] \approx \{g'(\sigma_i^2)\}^2 Var(S_i^2) = \{\exp(\sigma_i^2)\}^2\left(\frac{2\sigma_i^4}{n_i-1}\right).$$

So,

$$Var\left[\log\left\{\exp(S_i^2)-1\right\}\right] \approx \left(\frac{\exp(\sigma_i^2)}{\exp(\sigma_i^2)-1}\right)^2\left(\frac{2\sigma_i^4}{n_i-1}\right).$$

Therefore,

$$Var\left(\hat{\theta}_i\right) = \frac{4\sigma_i^2}{n_i} + \frac{2\sigma_i^4}{n_i-1} + \left[\left(\frac{\exp(\sigma_i^2)}{\exp(\sigma_i^2)-1}\right)^2\left(\frac{2\sigma_i^4}{n_i-1}\right)\right].$$

III. THE APPROACHES OF CONFIDENCE INTERVAL ESTIMATION

1) The generalized confidence interval approach

An important concept of generalized confidence intervals (GCI) is based on the generalized pivotal quantity (GPQ) for a parameter θ, which was introduced by Weerahandi [6]. A generalized pivot $R(X, x, \theta, \nu)$ for interval estimation, where X is a random sample from a distribution which depends on a vector of parameters $\theta = (\theta, \nu)$, ν is a nuisance parameters, x is an observed value of X, as a random variable having the following two properties:

1. $R(X, x, \theta, \nu)$ has a distribution free of the vector of nuisance parameters ν.
2. The observed value of $R(X, x, \theta, \nu)$ is θ.

Let R_α be the 100α-th percentile of R. Then R_α becomes the $100(1-\alpha)\%$ lower bound for θ and .. becomes a $100(1-\alpha)\%$ two-side generalized confidence interval for θ.

Let $\bar{X}_i$ and S_i^2 denote the sample mean and variance for log-transformed data X_{il} for the i-th sample respectively, $\bar{x}_i$ and s_i^2 denote the observed sample mean and variance respectively. So, $\sigma_i^2 = (n_i-1)S_i^2/V_i$ where V_i is χ^2 variates with degree of freedom $n_i - 1$.

The generalized pivotal quantity to estimate σ_i^2 based on the i-th sample can be defined as

$$R_{\sigma_i^2} = \frac{(n_i-1)s_i^2}{V_i} \sim \frac{(n_i-1)s_i^2}{\chi^2_{n_i-1}}. \tag{7}$$

The generalized pivotal quantity to estimate μ_i based on the i-th sample can be defined as

$$R_{\mu_i} = \bar{x}_i - \frac{Z_i}{\sqrt{U_i}}\sqrt{\frac{(n_i-1)s_i^2}{n_i}}, \tag{8}$$

where Z_i and U_i denote standard normal variate and χ^2 variate with degree of freedom $n_i - 1$ respectively.

The generalized pivotal quantity for estimating θ based on the i-th sample is

$$R_{\theta_i} = 2R_{\mu_i} + R_{\sigma_i^2} + \log\left\{\exp(R_{\sigma_i^2}) - 1\right\}. \tag{9}$$

From the i-th sample, the maximum likelihood estimator of θ is

$$\hat{\theta}_i = 2\hat{\mu}_i + \hat{\sigma}_i^2 + \log\left\{\exp(\hat{\sigma}_i^2) - 1\right\}, \text{ where } \hat{\mu}_i = \bar{X}_i,\ \hat{\sigma}_i^2 = S_i^2 \tag{10}$$

The sample variance for $\hat{\theta}_i$ is

$$Var(\hat{\theta}_i) = \frac{4\sigma_i^2}{n_i} + \frac{2\sigma_i^4}{n_i - 1} + \left[\left(\frac{\exp(\sigma_i^2)}{\exp(\sigma_i^2) - 1}\right)^2 \left(\frac{2\sigma_i^4}{n_i - 1}\right)\right], \text{ see Theorem 1.} \tag{11}$$

The generalized pivotal quantity proposed for the common log-variance $\theta = \log\eta$ is a weighted average of the generalized pivot R_{θ_i} based on k individual samples as; Tian and Wu [4].

$$R_\theta = \frac{\sum_{i=1}^{k} R_w R_\theta^{(i)}}{\sum_{i=1}^{k} R_{w_i}}, \tag{12}$$

where

$$R_{w_i} = \frac{1}{R_{Var(\hat{\theta}_i)}}, \tag{13}$$

$$R_{Var(\hat{\theta}_i)} = \frac{4R_{\sigma_i^2}}{n_i} + \frac{2\left(R_{\sigma_i^2}\right)^2}{n_i - 1} + \left[\left(\frac{\exp(R_{\sigma_i^2})}{\exp(R_{\sigma_i^2}) - 1}\right)^2 \left(\frac{2\left(R_{\sigma_i^2}\right)^2}{n_i - 1}\right)\right]. \tag{14}$$

That is, $R_{Var(\hat{\theta}_i)}$ is $Var(\hat{\theta}_i)$ with σ_i^2 replaced by $R_{\sigma_i^2}$.

Therefore, the 100(1- α)% two-sided confidence interval for the common variance of lognormal distributions η based on the GCI approach is

$$CI_{GCI} = (L_{GCI}, U_{GCI}) = \left(\exp\left(R_\theta(\alpha/2)\right), \exp\left(R_\theta(1-\alpha/2)\right)\right), \tag{15}$$

where $R_\theta(\alpha/2)$ and $R_\theta(1-\alpha/2)$ denote the $100(\alpha/2)$-th and $100(1-\alpha/2)$-th percentiles of R_θ, respectively.

Algorithm 1

For a given data set X_{ij} for $i = 1, 2, \ldots, k$, $j = 1, 2, \ldots, n_i$, the generalized confidence intervals for η can be computed by the following steps.

1) Compute $\bar{x}_i$ and s_i^2 for $i = 1, 2, \ldots, k$.
2) Generate $V_i \sim \chi^2_{n_i - 1}$ and then calculate $R_{\sigma_i^2}$ from (1) for $i = 1, 2, \ldots, k$.
3) Generate $Z_i \sim N(0,1)$ and $U_i \sim \chi^2_{n_i - 1}$, then calculate R_{μ_i} from (2) for $i = 1, 2, \ldots, k$.
4) Calculate R_{θ_i} in equation (9) for $i = 1, 2, \ldots, k$.
5) Repeat steps 2-3, calculate R_{w_i} from (7) and (8) for $i = 1, 2, \ldots, k$.
6) Compute R_θ following (6).
7) Repeat step 2- 6 a total m times and obtain an array of R_θ's.
8) Rank this array of R_θ's from small to large.

The $\left(R_\theta(\alpha/2), R_\theta(1-\alpha/2)\right)$ is a two-sided 100(1- α)% confidence interval.

2) The Large Sample Confidence Interval for the Common Variance

The large sample confidence interval is the standard confidence interval. According to Graybill and Deal [20], the large sample estimate of the log-variance of lognormal distribution is a pooled estimated unbiased estimator of the log-variance defined as follows:

$$\hat{\theta} = \sum_{i=1}^{k} \frac{\hat{\theta}_i}{Var(\hat{\theta}_i)} \Bigg/ \sum_{i=1}^{k} \frac{1}{Var(\hat{\theta}_i)}, \tag{16}$$

where $\hat{\theta}_i$ is defined in equation (5) and $Var(\hat{\theta}_i)$ is an estimate of $Var(\hat{\theta}_i)$ in equation (6) with μ_i and σ_i^2 replaced by $\bar{x}_i$ and s_i^2, respectively.

Therefore, the 100(1- α)% two-sided confidence interval for the common variance of lognormal distributions η based on the large sample approach is

$$CI_{LS} = (L_{LS}, U_{LS}) = \left(\exp\left(\hat{\theta} - z_{1-\alpha/2}\sqrt{1\Big/\sum_{i=1}^{k}\frac{1}{Var(\hat{\theta}_i)}} \right), \exp\left(\hat{\theta} + z_{1-\alpha/2}\sqrt{1\Big/\sum_{i=1}^{k}\frac{1}{Var(\hat{\theta}_i)}} \right) \right), \quad (17)$$

where $z_{1-\alpha/2}$ is the $(1-\alpha/2)$-th quantile of the standard normal distribution.

3) The Adjusted Method of Variance Estimates Recovery Confidence Interval for the Common Variance

The concept of the adjusted method of variance estimates recovery approach (the Adjusted MOVER) is based on the large sample approach and the method of variance estimates recovery approach (the Mover). The Mover approach was proposed by Zou and Donner [14], which considers confidence interval for two parameters $\theta_1 + \theta_2$, the lower limit L and the upper limit U are given by

$$[L,U] = (\hat{\theta}_1 + \hat{\theta}_2) \pm z_{\alpha/2}\sqrt{\hat{\text{var}}(\hat{\theta}_1) + \hat{\text{var}}(\hat{\theta}_2)}, \quad (18)$$

Suppose the $100(1-\alpha)\%$ two-sided confidence interval for θ_i is given by (l_i, u_i), $i = 1,2$.

Thus, the estimates variances near the lower $\hat{\text{var}}(\hat{\theta}_i)$ and upper $\hat{\text{var}}(\hat{\theta}_i)$ limits of θ_i as

$$\hat{\text{var}}(\hat{\theta}_i) = \left(\hat{\theta}_i - l_i\right)^2 \big/ z_{\alpha/2}^2, \quad \hat{\text{var}}(\hat{\theta}_i) = \left(u_i - \hat{\theta}_i\right)^2 \big/ z_{\alpha/2}^2.$$

For $i = 1,2$, two-side $100(1-\alpha)\%$ confidence limits for $\theta_1 + \theta_2$ given as

$$[L,U] = \left((\hat{\theta}_1 + \hat{\theta}_2) - \sqrt{\left(\hat{\theta}_1 - l_1\right)^2 + \left(\hat{\theta}_2 - l_2\right)^2}, (\hat{\theta}_1 + \hat{\theta}_2) + \sqrt{\left(u_1 - \hat{\theta}_1\right)^2 + \left(u_2 - \hat{\theta}_2\right)^2} \right). \quad (19)$$

Let $\theta_1, \theta_2, \ldots, \theta_k$ be k parameters of interest, where the estimates $\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_k$ are independent. The concept of the Mover was used to construct of a $100(1-\alpha)\%$ two-sided confidence interval (L, U) for $\theta_1 + \theta_2 + \ldots + \theta_k$, the lower limit L and upper limit U is defined as follows:

$$L = (\hat{\theta}_1 + \hat{\theta}_2 + \ldots + \hat{\theta}_k) - z_{\alpha/2}\sqrt{V\hat{a}r(\hat{\theta}_1) + V\hat{a}r(\hat{\theta}_2) + \ldots + V\hat{a}r(\hat{\theta}_k)} \quad (20)$$

and

$$U = (\hat{\theta}_1 + \hat{\theta}_2 + \ldots + \hat{\theta}_k) + z_{\alpha/2}\sqrt{V\hat{a}r(\hat{\theta}_1) + V\hat{a}r(\hat{\theta}_2) + \ldots + V\hat{a}r(\hat{\theta}_k)}, \quad (21)$$

Suppose the $100(1-\alpha)\%$ two-sided confidence interval for θ_i is given by (l_i, u_i), where $i = 1,2,..,k$. Thus, the variance estimate for $\hat{\theta}_i$ at $\theta_i = l_i$ and $\theta_i = u_i$ are equal to

$$\hat{\text{var}}(\hat{\theta}_i) = (\hat{\theta}_i - l_i)^2 \big/ z_{\alpha/2}^2, \quad \hat{\text{var}}(\hat{\theta}_i) = (u_i - \hat{\theta}_i)^2 \big/ z_{\alpha/2}^2. \quad (22)$$

Therefore, the lower limit L and upper limit U for $\theta_1 + \theta_2 + \ldots + \theta_k$ is given by

$$L = (\hat{\theta}_1 + \hat{\theta}_2 + \ldots + \hat{\theta}_k) - \sqrt{(\hat{\theta}_1 - l_1)^2 + (\hat{\theta}_2 - l_2)^2 + \ldots + (\hat{\theta}_k - l_k)^2} \quad (23)$$

and

$$U = (\hat{\theta}_1 + \hat{\theta}_2 + \ldots + \hat{\theta}_k) + \sqrt{(u_1 - \hat{\theta}_1)^2 + (u_2 - \hat{\theta}_2)^2 + \ldots + (u_k - \hat{\theta}_k)^2}, \quad (24)$$

where $z_{\alpha/2}$ denotes the $(\alpha/2)$-th quantile of the standard normal distribution.

The Adjusted MOVER approach was motivated based on concepts of the large sample approach in equations (16), (17) and the MOVER approach in equations (18)-(24). According to Graybill and Deal [20], the common log-variance θ is weighted average of the log-variance $\hat{\theta}_i$ based on k individual samples is defined as follows:

$$\hat{\theta} = \sum_{i=1}^{k}\frac{\hat{\theta}_i}{Var(\hat{\theta}_i)} \Big/ \sum_{i=1}^{k}\frac{1}{Var(\hat{\theta}_i)}, \quad (25)$$

where the variance estimate for $\hat{\theta}_i$ at $\theta_i = l_i$ and $\theta_i = u_i$ is the average variance between these two variances and given by

$$Var(\hat{\theta}_i) = \frac{1}{2}\left(\frac{(\hat{\theta}_i - l_i)^2}{z_{\alpha/2}^2} + \frac{(u_i - \hat{\theta}_i)^2}{z_{\alpha/2}^2}\right);\ i = 1, 2, \ldots, k\ , \tag{26}$$

where $z_{\alpha/2}$ denotes the $(\alpha/2)$-th quantile of the standard normal distribution.

Therefore, the lower limit L and upper limit U for the common variance θ are given by

$$L = \hat{\theta} - z_{1-\alpha/2}\sqrt{1\Big/\sum_{i=1}^{k}\frac{z_{\alpha/2}^2}{(\hat{\theta}_i - l_i)^2}}\ , \quad U = \hat{\theta} + z_{1-\alpha/2}\sqrt{1\Big/\sum_{i=1}^{k}\frac{z_{\alpha/2}^2}{(u_i - \hat{\theta}_i)^2}}\ , \tag{27}$$

where $\hat{\theta}_i = 2\bar{X}_i + S_i^2 + \log\{\exp(S_i^2) - 1\}$ and $\hat{\theta}$ is defined in equation (25).

Therefore, the adjusted MOVER solution for confidence interval estimation is

$$(\mathrm{L}, \mathrm{U}) = \left(\hat{\theta} - z_{1-\alpha/2}\sqrt{1\Big/\sum_{i=1}^{k}\frac{z_{\alpha/2}^2}{(\hat{\theta}^{(i)} - l_i)^2}}, \hat{\theta} + z_{1-\alpha/2}\sqrt{1\Big/\sum_{i=1}^{k}\frac{z_{\alpha/2}^2}{(u_i - \hat{\theta}^{(i)})^2}}\right). \tag{28}$$

From the i-th sample, where $i = 1, 2, .., k$. The common log-variance θ is

$$\begin{aligned}\theta = \log\eta &= 2\mu + \sigma^2 + \log\{\exp(\sigma^2) - 1\} \\ &= 2\left(\mu + \sigma^2/2\right) + \log\{\exp(\sigma^2) - 1\}\end{aligned} \tag{29}$$

Let $\theta_1 = 2\left(\mu + \sigma^2/2\right)$, $\hat{\theta}_1^{(i)} = 2\left(\hat{\mu}_i + \hat{\sigma}_i^2/2\right)$, where $\hat{\mu}_i = \bar{X}_i$, $\hat{\sigma}_i^2 = S_i^2$.

Following Chami et al. [21], two methods for constructing the 100(1- α)% confidence interval for log-mean $\left(\mu_i + \sigma_i^2/2\right)$ of lognormal are Cox's method and Angus's conservative method.

According to Cox's method, the confidence interval for $\hat{\theta}_1^{(i)}$ is

$$(l_{i1}, u_{i1}) = \left(2\left(\bar{X}_i + \frac{S_i^2}{2} - z_{1-\alpha/2}\sqrt{\frac{S_i^2}{n_i} + \frac{S_i^4}{2(n_i - 1)}}\right), 2\left(\bar{X}_i + \frac{S_i^2}{2} + z_{1-\alpha/2}\sqrt{\frac{S_i^2}{n_i} + \frac{S_i^4}{2(n_i - 1)}}\right)\right). \tag{30}$$

According to Angus's conservative method, the confidence interval for $\hat{\theta}_1^{(i)}$ is

$$(l_{i2}, u_{i2}) = \left(2\left(\bar{X}_i + \frac{S_i^2}{2} - \frac{t_{1-\alpha/2,(n_i-1)}}{\sqrt{n_i}}\sqrt{S_i^2\left(1 + \frac{S_i^2}{2}\right)}\right), 2\left(\bar{X}_i + \frac{S_i^2}{2} + \frac{q_{\alpha/2,(n_i-1)}}{\sqrt{n_i}}\sqrt{S_i^2\left(1 + \frac{S_i^2}{2}\right)}\right)\right) \tag{31}$$

which $t_{1-\alpha/2,(n-1)}$ be the 1- α percentile of a *t*-distribution with n-1 degrees of freedom, and let

$q_{\alpha/2,(n-1)} = \sqrt{\frac{n}{2}\left(\frac{n-1}{\chi^2_{\alpha,(n-1)}} - 1\right)}$ where $\chi^2_{\alpha,(n-1)}$ is the α - percentile of the chi-square distributions with *n*-1 degrees of freedom.

Let $\theta_2 = \log\{\exp(\sigma_i^2) - 1\}$, $\hat{\theta}_2^{(i)} = \log\left(\exp(\hat{\sigma}_i^2) - 1\right)$, where $\hat{\sigma}_i^2 = S_i^2$.

The confidence interval for $\hat{\theta}_2^{(i)}$ is

$$(l_i, u_i) = \left(\log\left[\exp\left(\frac{(n_i - 1)\mathrm{S}_i^2}{V}\right) - 1\right], \log\left[\exp\left(\frac{(n_i - 1)\mathrm{S}_i^2}{U}\right) - 1\right]\right), \tag{32}$$

where V_i denotes the $(1-\alpha/2)$-th quantile of the chi-square distribution with $n_i - 1$ degrees of freedom and U_i denotes the $(\alpha/2)$-th quantile of the chi-square distribution with $n_i - 1$ degrees of freedom.

The common parameter θ_1 and θ_2 is weighted average of $\hat{\theta}_1^{(i)}$ and $\hat{\theta}_2^{(i)}$ based on k individual samples. It is defined in equation (25) and variance estimate for $\hat{\theta}_1^{(i)}$ and $\hat{\theta}_2^{(i)}$ in equation (26)

Consequently, L and U are defined in equation (28). One obtains two groups of confidence intervals $(l_{i1}, \mathrm{u}_{i1})$ in equation (30) and $(l_{i2}, \mathrm{u}_{i2})$ in equation (31) which are defined by am1 and am2.

Hence, the confidence intervals based on cox's method (am1) of common θ_1 is

$$(\mathrm{L}_{am1},\mathrm{U}_{am1})=\left(\hat{\theta}_1-z_{1-\alpha/2}\sqrt{1\Big/\sum_{i=1}^{k}\frac{z_{\alpha/2}^2}{(\hat{\theta}_1^{(i)}-l_{i1})^2}},\hat{\theta}+z_{1-\alpha/2}\sqrt{1\Big/\sum_{i=1}^{k}\frac{z_{\alpha/2}^2}{(u_{i1}-\hat{\theta}_1^{(i)})^2}}\right). \quad (33)$$

The confidence intervals based on Angus's conservative (am2) of common θ_1 is

$$(\mathrm{L}_{am2},\mathrm{U}_{am2})=\left(\hat{\theta}_1-z_{1-\alpha/2}\sqrt{1\Big/\sum_{i=1}^{k}\frac{z_{\alpha/2}^2}{(\hat{\theta}_1^{(i)}-l_{i2})^2}},\hat{\theta}+z_{1-\alpha/2}\sqrt{1\Big/\sum_{i=1}^{k}\frac{z_{\alpha/2}^2}{(u_{i2}-\hat{\theta}_1^{(i)})^2}}\right). \quad (34)$$

Consequently, L and U are defined in equation (28). One gains a group of confidence intervals (l_i,u_i) in equation (33) which is defined by am.

Hence, the confidence intervals (am) of common θ_2 is

$$(\mathrm{L}_{am},\mathrm{U}_{am})=\left(\hat{\theta}_2-z_{1-\alpha/2}\sqrt{1\Big/\sum_{i=1}^{k}\frac{z_{\alpha/2}^2}{(\hat{\theta}_2^{(i)}-l_i)^2}},\hat{\theta}_2+z_{1-\alpha/2}\sqrt{1\Big/\sum_{i=1}^{k}\frac{z_{\alpha/2}^2}{(u_i-\hat{\theta}_2^{(i)})^2}}\right). \quad (35)$$

Therefore, $\theta=2\mu+\sigma^2+\log\{\exp(\sigma^2)-1\}=\theta_1+\theta_2$.

Hence, the adjusted MOVER approach based on cox's method (AM-Cox) for confidence interval estimation of common variance η is

$$\mathrm{CL}_{AM-Cox}=(\mathrm{L}_{AM-Cox},\mathrm{U}_{AM-Cox})=\left(\exp(\mathrm{L}_{am1}+\mathrm{L}_{am}),\ \exp(U_{am1}+U_{am})\right). \quad (36)$$

The adjusted MOVER approach based on Angus's conservative method (AM-Angus) for confidence interval estimation of common variance η is

$$\mathrm{CL}_{AM-Angus}=(\mathrm{L}_{AM-Angus},\mathrm{U}_{AM-Angus})=\left(\exp(\mathrm{L}_{am2}+\mathrm{L}_{am}),\ \exp(U_{am2}+U_{am})\right). \quad (37)$$

IV. SIMULATION STUDIES AND RESULTS

The performance of the proposed confidence intervals for common variance η of lognormal is investigated in term of coverage probability and the average length through simulation with the R statistical program. We choose a confident interval, which has coverage probability greater than or close to the nominal coverage level at 0.95, and a shortest length interval.

In simulation, the data are generated from lognormal distributions for various combinations of the number of samples k = 2, 4, 6 and 10, as estimated from 5000 randomly generated samples selected from a normal distribution having sizes $n_1=n_2\ldots=n_k=n=$ 10, 30 and 50. The population variances used were $\sigma_1^2=\sigma_2^2=\sigma_k^2=\sigma^2=$ 0.10, 0.30, 0.50, 0.80 and 1.00 and the common $\theta=\log\eta$ take -1, 0 and 1. For the GCI approach, 2500 R_θ's were obtained for each of the random samples. The results of the 95% confidence intervals of common variance η are show in Table 1- Table 4, respectively.

The following algorithm is used to estimate the coverage probability and average length:

Algorithm 2

Step 1 For h =1 to M

Generate data set $x_{i1},x_{i2},\ldots,x_{in_i}$ from $N(\mu,\sigma_i^2)$, $i=1,\ldots,k$

Compute $\bar{x}_i$ and s_i^2

Step 2 Use Algorithm 1 to construct $(L_{GCI(h)},U_{GCI(h)})$

Use equation (18) to construct $(L_{LS(h)},U_{LS(h)})$

Use equation (37) to construct $(L_{AM-Cox(h)},U_{AM-Cox(h)})$

Use equation (38) to construct $(L_{AM-Angus(h)},U_{AM-Angus(h)})$

If $(L_{(h)}\le\theta\le U_{(h)})$, then set $p_{(h)}=1$; else set $p_{(h)}=0$

Compute $U_{(h)}-L_{(h)}$

(end h loop)

Step 3 Compute coverage probabilities and average lengths of the confidence intervals

Table 1-Table 4 present the coverage probabilities and average lengths for 2, 4, 6 and 10 sample cases, respectively. In all sample cases, the GCI approach tends to underestimate the coverage probabilities as the number of samples goes up and depends on k and σ^2 **,** when the k and σ^2 increases. It is shown that the proposed the GCI interval estimates tend to drop from the nominal level 0.95. The coverage probabilities of the LS approach were less than the nominal coverage level for all of the scenarios. The AM-Cox approach provide coverage probabilities close to the nominal confidence level at 0.95, but the coverage probabilities depends on k, n and σ^2 , when n are small and k and σ^2 increases. It was also shown that the AM-Cox approach tends to drop from the confidence level at 0.95. The AM-Angus approach tends to have high coverage probabilities compared with the nominal level 0.95 for 2 and 4 sample cases. However, for $k > 4$, the coverage probabilities depend on n and σ^2 similar to the AM-Cox approach.

Overall, the AM-Cox approach was found to have the coverage probabilities close to the nominal level better than the AM-Angus approach, GCI approach and the LS approach respectively. Additionally, performance of all the approaches in terms of average lengths increased when σ^2 increased. The average lengths of the LS approach was shorter than the GCI approach, AM-Cox approach and the AM-Angus approach, respectively. However, this study selected the approaches with the values of coverage probabilities close to the nominal confidence level at 0.95 and shortest average lengths. Finally, it was also discovered that the AM-Cox approach performed well for all cases and the average lengths were wide compared with another approaches.

V. TABLE I

The coverage probability (CP) and average length (AL) of 95% two-sided confidence intervals for the common variance of lognormal distributions : 2 sample cases

n	σ_i^2	θ	CI_{GCI}		CI_{LS}		CI_{AM-Cox}		$CI_{AM-Angus}$	
			CP	**AL**	**CP**	**AL**				**CP**
10	0.10	-1	0.9466	1.6672	0.8884	1.4816	0.9746	2.0120	0.9886	2.2472
		0	0.9500	1.6723	0.8994	1.4839	0.9808	2.0173	0.9728	2.2541
		1	0.9514	1.6695	0.8996	1.4825	0.9772	2.0144	0.9860	2.2503
	0.30	-1	0.9436	2.2817	0.8592	1.8091	0.9722	2.6676	0.9886	3.0904
		0	0.9406	2.2820	0.8572	1.8087	0.9698	2.6661	0.9838	3.0882
		1	0.9396	2.2827	0.8470	1.8094	0.9682	2.6663	0.9852	3.0882
	0.50	-1	0.9360	2.8754	0.8284	2.1205	0.9584	3.2376	0.9788	3.8001
		0	0.9400	2.8811	0.8252	2.1240	0.9600	3.2443	0.9820	3.8085
		1	0.9420	2.8773	0.8258	2.1213	0.9634	3.2398	0.9830	3.8029
	0.80	-1	0.9424	3.7267	0.8084	2.5730	0.9530	4.0552	0.9780	4.7981
		0	0.9438	3.7484	0.8064	2.5812	0.9538	4.0690	0.9788	4.8146
		1	0.9394	3.7407	0.8042	2.5794	0.9508	4.0667	0.9780	4.8118
	1.00	-1	0.9396	4.3103	0.7954	2.8792	0.9466	4.6069	0.9758	5.4616
		0	0.9336	4.3019	0.7928	2.8760	0.9404	4.6026	0.9720	5.4565
		1	0.9374	4.2960	0.7818	2.8709	0.9462	4.5935	0.9768	5.4459
30	0.10	-1	0.9480	0.9082	0.9186	0.8317	0.9816	1.1040	0.9902	1.2352
		0	0.9516	0.9083	0.9224	0.8316	0.9806	1.1038	0.9898	1.2350
		1	0.9462	0.9078	0.9174	0.8314	0.9792	1.1035	0.9902	1.2345
	0.30	-1	0.9460	1.2184	0.8888	1.0269	0.9780	1.4534	0.9932	1.6899
		0	0.9522	1.2219	0.8982	1.0288	0.9768	1.4564	0.9908	1.6937
		1	0.9422	1.2200	0.8860	1.0279	0.9764	1.4548	0.9904	1.6917
	0.50	-1	0.9486	1.5159	0.8746	1.2163	0.9732	1.7529	0.9904	2.0708
		0	0.9438	1.5121	0.8706	1.2147	0.9706	1.7499	0.9898	2.0669
		1	0.9472	1.5122	0.8726	1.2139	0.9712	1.7491	0.9906	2.0660
	0.80	-1	0.9482	1.9433	0.8504	1.4906	0.9720	2.1700	0.9914	2.5928
		0	0.9448	1.9408	0.8600	1.4899	0.9648	2.1690	0.9872	2.5916
		1	0.9464	1.9392	0.8588	1.4893	0.9678	2.1681	0.9880	2.5906
	1.00	-1	0.9436	2.2209	0.8466	1.6710	0.9644	2.4417	0.9884	2.9293
		0	0.9434	2.2235	0.8448	1.6716	0.9620	2.4424	0.9870	2.9300
		1	0.9368	2.2195	0.8494	1.6686	0.9582	2.4380	0.9884	2.9247
50	0.10	-1	0.9516	0.6961	0.9242	0.6408	0.9834	0.8475	0.9910	0.9631
		0	0.9528	0.6956	0.9288	0.6405	0.9822	0.8471	0.9916	0.9625
		1	0.9524	0.6958	0.9282	0.6406	0.9838	0.8473	0.9912	0.9628
	0.30	-1	0.9488	0.9330	0.9046	0.7939	0.9808	1.1170	0.9942	1.3263
		0	0.9482	0.9321	0.9004	0.7934	0.9788	1.1162	0.9932	1.3252
		1	0.9504	0.9310	0.9066	0.7926	0.9800	1.1151	0.9932	1.3237
	0.50	-1	0.9518	1.1525	0.8854	0.9393	0.9782	1.3415	0.9920	1.6217
		0	0.9482	1.1525	0.8784	0.9393	0.9748	1.3415	0.9914	1.6217
		1	0.9438	1.1527	0.8816	0.9392	0.9734	1.3414	0.9930	1.6216
	0.80	-1	0.9474	1.4726	0.8706	1.1538	0.9698	1.6591	0.9912	2.0332
		0	0.9460	1.4758	0.8670	1.1554	0.9678	1.6614	0.9900	2.0361
		1	0.9478	1.4738	0.8658	1.1545	0.9708	1.6601	0.9908	2.0344
	1.00	-1	0.9448	1.6855	0.8608	1.2968	0.9662	1.8680	0.9868	2.3006
		0	0.9458	1.6860	0.8528	1.2967	0.9676	1.8679	0.9898	2.3006
		1	0.9448	1.6852	0.8576	1.2968	0.9654	1.8681	0.9898	2.3008

TABLE II

The coverage probability (CP) and average length (AL) of 95% two-sided confidence intervals for the common variance of lognormal distributions : 4 sample cases

n	σ_i^2	θ	CI_{GCI}		CI_{LS}		CI_{AM-Cox}		$CI_{AM-Angus}$	
			CP	AL	CP	AL	CP	AL	CP	AL
10	0.10	-1	0.9398	1.1447	0.8738	1.0477	0.9766	1.4159	0.9866	1.5792
		0	0.9338	1.1443	0.8664	1.0478	0.9664	1.4156	0.9824	1.5786
		1	0.9332	1.1429	0.8668	1.0470	0.9708	1.4135	0.9836	1.5759
	0.30	-1	0.9086	1.5301	0.8030	1.2727	0.9570	1.8623	0.9780	2.1529
		0	0.9140	1.5261	0.8108	1.2705	0.9568	1.8573	0.9824	2.1465
		1	0.9192	1.5311	0.8142	1.2741	0.9630	1.8645	0.9848	2.1557
	0.50	-1	0.9032	1.8980	0.7726	1.4814	0.9490	2.2433	0.9758	2.6282
		0	0.8982	1.9069	0.7720	1.4854	0.9480	2.2502	0.9788	2.6366
		1	0.8954	1.8996	0.7636	1.4833	0.9430	2.2467	0.9738	2.6323
	0.80	-1	0.8822	2.4418	0.7182	1.7848	0.9274	2.7878	0.9666	3.2943
		0	0.8734	2.4374	0.7120	1.7823	0.9226	2.7833	0.9644	3.2889
		1	0.8808	2.4444	0.7186	1.7869	0.9246	2.7922	0.9650	3.2997
	1.00	-1	0.8746	2.7993	0.7084	1.9828	0.9166	3.1427	0.9638	3.7229
		0	0.8732	2.7918	0.7020	1.9802	0.9104	3.1387	0.9550	3.7183
		1	0.8748	2.8042	0.6956	1.9875	0.9138	3.1519	0.9610	3.7340
30	0.10	-1	0.9462	0.6356	0.9158	0.5877	0.9804	0.7785	0.9906	0.8705
		0	0.9434	0.6355	0.9114	0.5879	0.9784	0.7788	0.9890	0.8708
		1	0.9468	0.6357	0.9170	0.5877	0.9798	0.7786	0.9904	0.8706
	0.30	-1	0.9396	0.8475	0.8794	0.7252	0.9750	1.0246	0.9898	1.1907
		0	0.9344	0.8476	0.8740	0.7249	0.9746	1.0241	0.9922	1.1900
		1	0.9394	0.8473	0.8808	0.7248	0.9782	1.0241	0.9926	1.1900
	0.50	-1	0.9274	1.0489	0.8458	0.8559	0.9688	1.2314	0.9916	1.4536
		0	0.9274	1.0484	0.8522	0.8555	0.9674	1.2307	0.9892	1.4528
		1	0.9374	1.0468	0.8524	0.8547	0.9732	1.2296	0.9932	1.4514
	0.80	-1	0.9274	1.3421	0.8252	1.0471	0.9598	1.5226	0.9888	1.8183
		0	0.9206	1.3406	0.8246	1.0458	0.9580	1.5205	0.9876	1.8157
		1	0.9222	1.3424	0.8242	1.0470	0.9548	1.5224	0.9870	1.8180
	1.00	-1	0.9202	1.5344	0.8140	1.1714	0.9548	1.7098	0.9880	2.0502
		0	0.9222	1.5352	0.8150	1.1723	0.9562	1.7111	0.9898	2.0518
		1	0.9164	1.5339	0.8030	1.1717	0.9516	1.7103	0.9860	2.0508
50	0.10	-1	0.9482	0.4891	0.9174	0.4529	0.9820	0.5984	0.9910	0.6797
		0	0.9454	0.4893	0.9164	0.4529	0.9786	0.5984	0.9902	0.6798
		1	0.9498	0.4886	0.9188	0.4527	0.9804	0.5979	0.9912	0.6791
	0.30	-1	0.9388	0.6527	0.8852	0.5603	0.9792	0.7874	0.9948	0.9344
		0	0.9452	0.6527	0.8918	0.5603	0.9780	0.7874	0.9936	0.9343
		1	0.9452	0.6527	0.8960	0.5604	0.9778	0.7876	0.9940	0.9346
	0.50	-1	0.9384	0.8051	0.8668	0.6626	0.9738	0.9455	0.9916	1.1425
		0	0.9366	0.8050	0.8576	0.6626	0.9722	0.9455	0.9916	1.1425
		1	0.9444	0.8058	0.8738	0.6630	0.9766	0.9462	0.9942	1.1435
	0.80	-1	0.9268	1.0283	0.8470	0.8132	0.9628	1.1688	0.9916	1.4318
		0	0.9336	1.0281	0.8412	0.8130	0.9654	1.1684	0.9904	1.4313
		1	0.9338	1.0292	0.8470	0.8135	0.9652	1.1691	0.9926	1.4322
	1.00	-1	0.9306	1.1762	0.8396	0.9133	0.9608	1.3151	0.9892	1.6192
		0	0.9266	1.1763	0.8388	0.9132	0.9558	1.3149	0.9906	1.6190
		1	0.9310	1.1779	0.8474	0.9141	0.9608	1.3163	0.9898	1.6207

TABLE III

The coverage probability (CP) and average length (AL) of 95% two-sided confidence intervals for the common variance of lognormal distributions : 6 sample cases

n	σ_i^2	θ	CI_{GCI}		CI_{LS}		CI_{AM-Cox}		$CI_{AM-Angus}$	
			CP	AL	CP	AL	CP	AL	CP	AL
10	0.10	-1	0.9302	0.9247	0.8452	0.8548	0.9688	1.1514	0.9840	1.2829
		0	0.9252	0.9239	0.8450	0.8546	0.9692	1.1510	0.9834	1.2823
		1	0.9238	0.9247	0.8396	0.8549	0.9668	1.1515	0.9806	1.2829
	0.30	-1	0.8788	1.2253	0.7662	1.0365	0.9490	1.5110	0.9782	1.7450
		0	0.8808	1.2241	0.7690	1.0358	0.9460	1.5100	0.9764	1.7438
		1	0.8866	1.2258	0.7628	1.0367	0.9456	1.5116	0.9760	1.7458
	0.50	-1	0.8566	1.5187	0.7106	1.2060	0.9318	1.8203	0.9686	2.1311
		0	0.8520	1.5178	0.7050	1.2055	0.9248	1.8194	0.9666	2.1299
		1	0.8556	1.5200	0.7142	1.2069	0.9250	1.8223	0.9676	2.1336
	0.80	-1	0.8292	1.9453	0.6636	1.4499	0.8976	2.2569	0.9510	2.6654
		0	0.8208	1.9447	0.6462	1.4492	0.9026	2.2562	0.9558	2.6647
		1	0.8226	1.9494	0.6552	1.4515	0.8942	2.2602	0.9558	2.6696
	1.00	-1	0.7988	2.2259	0.6198	1.6078	0.8736	2.5396	0.9412	3.0072
		0	0.7980	2.2227	0.6090	1.6062	0.8756	2.5363	0.9370	3.0031
		1	0.8010	2.2239	0.6152	1.6056	0.8750	2.5352	0.9404	3.0018

TABLE IV (CONTINUED)

The coverage probability (CP) and average length (AL) of 95% two-sided confidence intervals for the common variance of lognormal distributions : 6 sample cases

n	σ_i^2	θ	CI_{GCI}		CI_{LS}		CI_{AM-Cox}		$CI_{AM-Angus}$	
			CP	**AL**	**CP**	**AL**	**CP**	**AL**	**CP**	**AL**
30	0.10	-1	0.9470	0.5171	0.9068	0.4798	0.9816	0.6351	0.9928	0.7100
		0	0.9448	0.5173	0.9056	0.4799	0.9832	0.6355	0.9912	0.7105
		1	0.9366	0.5177	0.9014	0.4800	0.9784	0.6355	0.9898	0.7105
	0.30	-1	0.9322	0.6882	0.8606	0.5915	0.9722	0.8351	0.9930	0.9701
		0	0.9344	0.6885	0.8662	0.5917	0.9788	0.8354	0.9944	0.9706
		1	0.9224	0.6876	0.8604	0.5912	0.9694	0.8346	0.9914	0.9695
	0.50	-1	0.9170	0.8500	0.8264	0.6973	0.9654	1.0026	0.9922	1.1831
		0	0.9092	0.8502	0.8262	0.6979	0.9602	1.0034	0.9898	1.1842
		1	0.9134	0.8503	0.8280	0.6977	0.9664	1.0033	0.9918	1.1841
	0.80	-1	0.8942	1.0875	0.7976	0.8532	0.9480	1.2402	0.9830	1.4807
		0	0.9018	1.0870	0.8032	0.8530	0.9508	1.2400	0.9856	1.4805
		1	0.9070	1.0877	0.8046	0.8531	0.9566	1.2400	0.9884	1.4806
	1.00	-1	0.8908	1.2444	0.774	0.9551	0.9418	1.3937	0.9840	1.6709
		0	0.8994	1.2453	0.7920	0.9561	0.9470	1.3952	0.9846	1.6728
		1	0.8952	1.2438	0.7752	0.9547	0.9442	1.3930	0.9818	1.6700
50	0.10	-1	0.9408	0.3081	0.9020	0.2863	0.9780	0.3781	0.9930	0.4294
		0	0.9442	0.3081	0.9042	0.2863	0.9824	0.3780	0.9948	0.4293
		1	0.9458	0.3084	0.9080	0.2864	0.9784	0.3781	0.9920	0.4294
	0.30	-1	0.9258	0.4102	0.8674	0.3541	0.9724	0.4973	0.9924	0.5900
		0	0.9286	0.4100	0.8684	0.3540	0.9710	0.4972	0.9944	0.5898
		1	0.9322	0.4104	0.8656	0.3541	0.9748	0.4974	0.9932	0.5901
	0.50	-1	0.9094	0.5061	0.8312	0.4187	0.9634	0.5972	0.9906	0.7215
		0	0.9122	0.5059	0.8270	0.4186	0.9668	0.5970	0.9932	0.7212
		1	0.9112	0.5062	0.8368	0.4189	0.9666	0.5974	0.9928	0.7218
	0.80	-1	0.8818	0.6449	0.7824	0.5128	0.9504	0.7368	0.9916	0.9024
		0	0.8904	0.6456	0.7992	0.5134	0.9526	0.7376	0.9926	0.9034
		1	0.8940	0.6457	0.7954	0.5133	0.9558	0.7375	0.9930	0.9032
	1.00	-1	0.8780	0.7378	0.7750	0.5757	0.9434	0.8287	0.9910	1.0201
		0	0.8876	0.7379	0.7898	0.5759	0.9476	0.8291	0.9900	1.0207
		1	0.8826	0.7376	0.7818	0.5756	0.9454	0.8286	0.9902	1.0199

TABLE V

The coverage probability (CP) and average length (AL) of 95% two-sided confidence intervals for the common variance of lognormal distributions : 10 sample cases

n	σ_i^2	θ	CI_{GCI}		CI_{LS}		CI_{AM-Cox}		$CI_{AM-Angus}$	
			CP	**AL**	**CP**	**AL**	**CP**	**AL**	**CP**	**AL**
10	0.10	-1	0.9060	0.7104	0.7920	0.6617	0.9516	0.8892	0.9762	0.9900
		0	0.9146	0.7101	0.8012	0.6617	0.9574	0.8893	0.9810	0.9901
		1	0.9110	0.7107	0.7952	0.6620	0.9534	0.8897	0.9732	0.9906
	0.30	-1	0.8218	0.9355	0.6846	0.8011	0.9190	1.1646	0.9656	1.3439
		0	0.8176	0.9360	0.6790	0.8013	0.9132	1.1650	0.9602	1.3444
		1	0.8258	0.9357	0.6798	0.8011	0.9222	1.1650	0.9682	1.3446
	0.50	-1	0.7632	1.1551	0.6114	0.9314	0.8862	1.4019	0.9540	1.6401
		0	0.7732	1.1570	0.6122	0.9319	0.8910	1.4032	0.9546	1.6418
		1	0.7640	1.1552	0.6048	0.9313	0.8884	1.4017	0.9534	1.6400
	0.80	-1	0.6984	1.4747	0.5272	1.1165	0.8404	1.7324	0.9268	2.0449
		0	0.6986	1.4757	0.5252	1.1171	0.8354	1.7338	0.9286	2.0467
		1	0.7124	1.4781	0.5390	1.1183	0.8372	1.7365	0.9264	2.0501
	1.00	-1	0.6628	1.6815	0.4838	1.2350	0.8100	1.9431	0.9130	2.2998
		0	0.6570	1.6789	0.4790	1.2340	0.8016	1.9417	0.9058	2.2982
		1	0.6622	1.6847	0.4972	1.2368	0.8038	1.9475	0.9120	2.3053
30	0.10	-1	0.9398	0.3996	0.8900	0.3716	0.9768	0.4918	0.9924	0.5497
		0	0.9388	0.3996	0.8894	0.3717	0.9750	0.4918	0.9906	0.5498
		1	0.9376	0.3996	0.8860	0.3716	0.9772	0.4917	0.9914	0.5496
	0.30	-1	0.9012	0.5306	0.8238	0.4578	0.9652	0.6459	0.9910	0.7502
		0	0.9068	0.5303	0.8306	0.4576	0.9682	0.6457	0.9922	0.7500
		1	0.9140	0.5306	0.8384	0.4578	0.9680	0.6459	0.9924	0.7501
	0.50	-1	0.8862	0.6555	0.7906	0.5402	0.9520	0.7764	0.9856	0.9162
		0	0.8920	0.6557	0.7936	0.5403	0.9568	0.7766	0.9906	0.9164
		1	0.8850	0.6551	0.7966	0.5398	0.9536	0.7758	0.9870	0.9154
	0.80	-1	0.8636	0.8383	0.7448	0.6603	0.9446	0.9594	0.9872	1.1453
		0	0.8598	0.8365	0.7468	0.6591	0.9368	0.9576	0.9836	1.1431
		1	0.8592	0.8367	0.7448	0.6593	0.9398	0.9580	0.9854	1.1436
	1.00	-1	0.8392	0.9567	0.7168	0.7378	0.9230	1.0762	0.9808	1.2901
		0	0.8456	0.9589	0.7348	0.7390	0.9294	1.0781	0.9796	1.2924
		1	0.8414	0.9580	0.7298	0.7386	0.9174	1.0775	0.9782	1.2917

TABLE VI (CONTINUED)

The coverage probability (CP) and average length (AL) of 95% two-sided confidence intervals for the common variance of lognormal distributions : 10 sample cases

n	σ_i^2	θ	CI_{GCI}		CI_{LS}		CI_{AM-Cox}		$CI_{AM-Angus}$	
			CP	**AL**	**CP**	**AL**	**CP**	**AL**	**CP**	**AL**
50	0.10	-1	0.9408	0.3081	0.9020	0.2863	0.9780	0.3781	0.9930	0.4294
		0	0.9442	0.3081	0.9042	0.2863	0.9824	0.3780	0.9948	0.4293
		1	0.9458	0.3084	0.9080	0.2864	0.9784	0.3781	0.9920	0.4294
	0.30	-1	0.9258	0.4102	0.8674	0.3541	0.9724	0.4973	0.9924	0.5900
		0	0.9286	0.4100	0.8684	0.3540	0.9710	0.4972	0.9944	0.5898
		1	0.9322	0.4104	0.8656	0.3541	0.9748	0.4974	0.9932	0.5901
	0.50	-1	0.9094	0.5061	0.8312	0.4187	0.9634	0.5972	0.9906	0.7215
		0	0.9122	0.5059	0.8270	0.4186	0.9668	0.5970	0.9932	0.7212
		1	0.9112	0.5062	0.8368	0.4189	0.9666	0.5974	0.9928	0.7218
	0.80	-1	0.8818	0.6449	0.7824	0.5128	0.9504	0.7368	0.9916	0.9024
		0	0.8904	0.6456	0.7992	0.5134	0.9526	0.7376	0.9926	0.9034
		1	0.8940	0.6457	0.7954	0.5133	0.9558	0.7375	0.9930	0.9032
	1.00	-1	0.8780	0.7378	0.7750	0.5757	0.9434	0.8287	0.9910	1.0201
		0	0.8876	0.7379	0.7898	0.5759	0.9476	0.8291	0.9900	1.0207
		1	0.8826	0.7376	0.7818	0.5756	0.9454	0.8286	0.9902	1.0199

TABLE VII

The coverage probability (CP) and average length (AL) of 95% two-sided confidence intervals for the common variance of lognormal distributions : 30 sample cases

n	σ_i^2	θ	CI_{GCI}		CI_{LS}		CI_{AM-Cox}		$CI_{AM-Angus}$	
			CP	**AL**	**CP**	**AL**				**CP**
10	0.10	-1	0.8334	0.4074	0.5806	0.3821	0.8772	0.5125	0.9374	0.5703
		0	0.8308	0.4073	0.5672	0.3820	0.8718	0.5124	0.9310	0.5701
		1	0.8330	0.4073	0.5664	0.3820	0.8748	0.5122	0.9326	0.5699
	0.30	-1	0.5476	0.5334	0.3714	0.4621	0.7448	0.6699	0.8708	0.7726
		0	0.5624	0.5337	0.3802	0.4623	0.7488	0.6705	0.8774	0.7732
		1	0.5610	0.5338	0.3722	0.4623	0.7560	0.6704	0.8790	0.7732
	0.50	-1	0.3772	0.6560	0.2628	0.5362	0.6332	0.8046	0.8092	0.9407
		0	0.3902	0.6572	0.2734	0.5368	0.6362	0.8058	0.8032	0.9422
		1	0.3956	0.6565	0.2654	0.5365	0.6334	0.8051	0.8090	0.9413
	0.80	-1	0.2390	0.8341	0.1800	0.6419	0.4964	0.9925	0.7016	1.1709
		0	0.2318	0.8345	0.1670	0.6419	0.4898	0.9926	0.6992	1.1710
		1	0.2364	0.8334	0.1782	0.6416	0.4860	0.9920	0.6956	1.1702
	1.00	-1	0.1690	0.9493	0.1332	0.7087	0.4074	1.1113	0.6218	1.3147
		0	0.1780	0.9522	0.1374	0.7105	0.4324	1.1149	0.6438	1.3192
		1	0.1770	0.9503	0.1346	0.7095	0.4206	1.1128	0.6360	1.3165
30	0.10	-1	0.9158	0.2302	0.8022	0.2145	0.9528	0.2837	0.9838	0.3170
		0	0.9142	0.2300	0.8080	0.2145	0.9562	0.2837	0.9842	0.3170
		1	0.9156	0.2302	0.8088	0.2145	0.9556	0.2837	0.9852	0.3171
	0.30	-1	0.8324	0.3053	0.7046	0.2643	0.9338	0.3727	0.9822	0.4329
		0	0.8184	0.3049	0.6874	0.2641	0.9262	0.3724	0.9814	0.4325
		1	0.8216	0.3050	0.6940	0.2642	0.9338	0.3726	0.9836	0.4327
	0.50	-1	0.7286	0.3763	0.6180	0.3114	0.8880	0.4473	0.9728	0.5277
		0	0.7330	0.3761	0.6112	0.3113	0.8906	0.4472	0.9736	0.5276
		1	0.7340	0.3764	0.6246	0.3114	0.8920	0.4474	0.9756	0.5278
	0.80	-1	0.6270	0.4803	0.5292	0.3801	0.8288	0.5521	0.9552	0.6589
		0	0.6304	0.4807	0.5372	0.3802	0.8334	0.5522	0.9546	0.6591
		1	0.6316	0.4807	0.5318	0.3802	0.8468	0.5523	0.9602	0.6592
	1.00	-1	0.5692	0.5495	0.4864	0.4254	0.8046	0.6203	0.9458	0.7435
		0	0.5624	0.5492	0.4874	0.4253	0.7962	0.6201	0.9460	0.7433
		1	0.5784	0.5494	0.4946	0.4253	0.8026	0.6202	0.9388	0.7434
50	0.10	-1	0.9306	0.1776	0.8630	0.1653	0.9642	0.2182	0.9902	0.2478
		0	0.9282	0.1777	0.8604	0.1653	0.9682	0.2182	0.9926	0.2477
		1	0.9304	0.1776	0.8596	0.1653	0.9684	0.2182	0.9934	0.2478
	0.30	-1	0.8704	0.2362	0.7812	0.2043	0.9520	0.2869	0.9924	0.3403
		0	0.8738	0.2362	0.7784	0.2043	0.9512	0.2868	0.9926	0.3402
		1	0.8790	0.2363	0.7828	0.2043	0.9512	0.2869	0.9924	0.3404
	0.50	-1	0.8270	0.2914	0.7312	0.2416	0.9352	0.3446	0.9914	0.4163
		0	0.8126	0.2913	0.7206	0.2415	0.9302	0.3444	0.9900	0.4160
		1	0.8242	0.2914	0.7258	0.2416	0.9316	0.3446	0.9878	0.4163
	0.80	-1	0.7514	0.3714	0.6546	0.2960	0.8986	0.4252	0.9880	0.5208
		0	0.7500	0.3714	0.6492	0.2960	0.9006	0.4252	0.9838	0.5207
		1	0.7620	0.3716	0.6656	0.2961	0.9046	0.4254	0.9854	0.5209
	1.00	-1	0.7028	0.4242	0.6142	0.3318	0.8790	0.4776	0.9798	0.5878
		0	0.7076	0.4242	0.6194	0.3318	0.8770	0.4776	0.9816	0.5878
		1	0.7150	0.4242	0.6318	0.3320	0.8808	0.4779	0.9840	0.5882

VI. AN EMPIRICAL APPLICATION

In this section, two real data examples are used to illustrate the given approaches in section 2. Both examples also used by Lin and Wang (2013) to illustrate the approaches to calculate the 95% confidence intervals for common mean of lognormal distributions. The first example (A) was the medical charge data discussed by McDonald et al. [22], Zhou et al. [23], and Tian and Wu [4]. The data set (A) was divided into two groups, 119 of them were an American group and 106 of them were the white group. The second example (B) was the pharmacokinetics data from alcohol interaction in men which was studied by Bradstreet and Liss [24]. The data set (B) was equally divided into three groups, 22 of them were group 1, group 2 and group 3. Both examples also used by Lin and Wang [5] to illustrate the approaches to calculate the 95% confidence intervals for common mean of lognormal distributions. The sample mean and the sample variance of the log-transformed data for the data set (A) were (9.067, 1.825) and (8.693, 2.693), respectively and the sample mean and the sample variance of the log-transformed data for the data set (B) were (2.601, 0.24), (2.596, 0.20) and (2.599, 0.17), respectively.

Using the purposed approaches to construct 95% confidence intervals for overall variance of lognormal distributions, the results showed that the GCI approach CI_{GCI} =(21.0999, 23.0355) with an interval length of 1.9356, the LS approach CI_{LS} =(21.2582, 22.6841) with an interval length of 1.4259, the AM-Cox approach CI_{AM-Cox} = (21.0265, 23.0577) with an interval length of 2.0312 and the AM-Angus approach $CI_{AM-Angus}$ = (21.0228, 23.7104) with an interval length of 2.6876. For data set (B), results showed that the GCI approach was CI_{GCI} = (3.4638, 4.5040) with the length of interval 1.0402. In comparison, the confidence interval by the large sample approach was CI_{LS} = (3.4420, 4.3380) with the length of interval 0.8960. In comparison, the confidence interval by the Adjusted MOVER based on cox's method approach was CI_{AM-Cox} = (3.3376, 4.5941) with the length of interval 1.2564 and the confidence interval by the adjusted MOVER based on Angus's conservative method approach was $CI_{AM-Angus}$ = (3.3243, 4.7579) with the length of interval 1.4336.

In summary, the results from above two examples show that the Adjusted MOVER based on cox's method approach is much closer to sample variances and length of interval is shorter than the other approaches, which support the simulation results in the previous section. In simulation, the large sample confidence interval has the shortest average lengths because the coverage probabilities provide less than the nominal confidence level of 0.95. Moreover, the large sample confidence interval uses the concept of the central limit theorem. Hence, the coverage probability of the large sample confidence interval is close to nominal confidence level of $1-\alpha$ when the sample size is large. Therefore, the large sample approach is not recommended to construct the confidence interval for the common variance of lognormal distributions when the sample size is small. Clearly, results from the two real data examples support the results of the simulation.

VII. CONCLUSIONS

This article has presented a simple approach to construct confidence intervals for common variance of lognormal distributions. The four approaches were the generalized confidence interval approach (GCI), the large sample approach (LS) and the adjusted MOVER approach based on cox's method (AM-Cox) and Angus's conservative method (AM-Angus). Simulation results showed that the LS approach gave unsatisfactory coverage probabilities. The coverage probabilities of the GCI approach was close to the nominal confidence level at 0.95. The AM-Cox approach overestimated the coverage probabilities same as the AM-Angus approach, but the AM-Angus approach gave an excellent result in almost all cases. Additionally, the coverage probabilities of all approaches depend on k, n and σ^2, when n were small and when k and σ^2 increased. It has been shown that the interval estimates tend to drop from the nominal level at 0.95.

Overview, the average lengths increased when the value of σ^2 increased for all approaches. The results indicated that the confidence interval for the common variance of lognormal distributions based on the AM-Angus approach provides stable coverage probabilities and average lengths. In conclusion, the AM-Angus approach can be successfully used to estimate the common variance of lognormal distributions.

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