Confidence Intervals for the Ratio of the Means of Two Normal Distributions with One Variance Unknown

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Abstract : *In this paper, we proposes confidence intervals for the ratio of normal means with one variance unknown based on the generalized confidence interval (GCI) approach, the method of variance estimates recovery (MOVER), and a Bootstrap technique (Bootstrap confidence interval (BCI). The coverage probabilities and expected lengths of these confidence intervals were then compared via a Monte Carlo simulation.The simulation results indicated that the MOVER approaches are satisfactory in terms of the coverage probability. The results indicate that the MOVER confidence intervals are better than those constructed via the GCI and BCI. The expected lengths of the MOVER approach are shorter than expected lengths of GCI and BCI. The coverage probabilities of the MOVER confidence intervals are more appropriate than those using the GCI and BCI. Simulation results show that the MOVER approach is satisfactory performances for all sample case which was presented by Thangjai et al. [19]. Our approaches are applied to an analysis of a real data set of drugs or treatments.*

Keywords : *Coverage probability, Expected length, Normal Mean, Unknown coefficient of variation, Generalized confidence interval, Method of variance estimates recovery, Boostrap confidence interval*

I. INTRODUCTION

Statistical problems have been successfully solved using the confidence interval for the ratio of the means of normal populations. Fieller's [2] theorem is a widely used general procedure to construct confidence intervals for the ratio of means to compare the effectiveness of treatments, a scenario that occurs frequently in bioassay and bioequivalence research. Koschat [5] suggested that Fieller's theorem had exact coverage probabilities of the confidence intervals when the variances were equal, while Lee and Lin [7] proposed generalized confidence intervals (GCIs) for the ratio of the means of two normal populations. A recurrent theme in bioassays and bioequivalence studies is the necessity to compare the relative potency of two drugs or treatments, and there is often the need to compare the efficacies of a new drug with a routinely used one. Historical data on the standard drug means that the variance is known whereas that of the new drug is assumed to be unknown because of insufficient data. In practice, the new (unknown) treatment is compared with the standard (known) treatment, and many researchers have studied this problem as one variance known and one unknown that must be estimated. For example, Maity and Sherman [10] proposed a new test statistic using a two-sample t-test with one variance known and the other unknown, an approach that was extended by Liqian and Tiejun [8]. Meanwhile, Niwitpong [12] proposed confidence intervals for the difference between two normal population means with one variance unknown.

In fact, in a few real-life situations, the variance of a normal population is unknown and so needs to be estimated. Data on the population of interest are collected in different settings from various experiments and situations. A variety of approaches have been proposed to construct confidence intervals for the most effective approach to estimate the parameter of interest appropriately and confidence intervals to estimate normal population means have been widely studied. Galeone and Pollastri [3] proposed confidence intervals for the ratio of two means using the distribution of the quotient of two normal populations. Niwitpong [15] proposed confidence intervals for the difference between two normal population means with one variance unknown that gave a shorter expected length than that of the well-known Welch-Satterthewaite confidence interval when the ratio of their population variances is large. In addition, Niwitpong [13] developed a simple confidence interval for the difference between two normal population means with one variance unknown.

In this study, we use the ratio of the means of normal distributions with one variance unknown, for which we construct confidence intervals for the ratio using generalized confidence interval (GCI), the method of variance estimates recovery (MOVER), and Bootstrap approaches (Boostrap confidence intervals : BCI). Confidence intervals based on GCI have been used by several researchers. For example, the idea of GCI was introduced by [20], [4], [6]. Sodanin [16] proposed GCIs for the normal mean with an unknown coefficient of variation.

Meanwhile, Wongkhao [21], [22] studied new confidence intervals for the inversion of a normal mean using GCI and MOVER. Therefore, GCI are significant when comparing the estimations of two means. Niwitpong [14] presented confidence intervals for the bounded parameters of the difference and the ratio of lognormal means using the MOVER approach. Moreover, Sangnawakij and Niwitpong [17] used MOVER to construct confidence intervals for the coefficient of variation of a two-parameter exponential distribution. Li [9] constructed separate confidence intervals for two individual Poisson rates and then combined them into a single confidence interval for the ratio of rates using the MOVER approach. Donner and Zou [1] presented closed-form confidence intervals for functions of the mean and standard deviation for a normal distribution. Zhou and Tu [23] suggested a bootstrap approach for estimating the interval for the ratio of means of log-normally distributed medical costs with zero values.

The aim of this study is to develop the coverage probabilities and expected lengths of new confidence intervals based on the GCI, MOVER, and BCI and then compare them. The procedures for these three methods are significant when comparing the estimation of the means from two normally distributed populations where one variance is unknown. The MOVER method can only be applied to find the confidence interval for the ratio of means using different simulation study settings as it is a practical approach rather than a theoretical one.

This paper is divided into the following sections. The GCI, MOVER, and BCI for the ratio of normal means with one variance unknown are described in Sections 2, 3, and 4, respectively. The simulation study and results of the comparison between the three methods are covered in Section 5. Finally, the comparison of the three confidence intervals is discussed and concluded in Section 6.

II. THE GENERALIZED CONFIDENCE INTERVAL (GCI)

Let $X=\left(X_1,X_2,...,X_n\right)$ be a random sample from normal distribution $N\left(\mu_x,\sigma_x^2\right)$ with mean μ_x and variance σ_x^2, and let $Y=\left(Y_1,Y_2,...,Y_m\right)$ be a random sample from normal distribution $N\left(\mu_y,\sigma_y^2\right)$ with mean μ_y and variance σ_y^2, then

$$\bar{X}=\frac{1}{n}\sum_{i=1}^{n}X_i\,,\qquad \bar{Y}=\frac{1}{m}\sum_{j=1}^{m}Y_j$$

$$S_x^2=\frac{\sum_{i=1}^{n}\left(X_i-\bar{X}\right)}{n},\qquad S_y^2=\frac{\sum_{i=1}^{m}\left(Y_i-\bar{Y}\right)}{m},$$

where $\bar{X}$ and S_X^2 are the sample mean and sample variance for X, respectively; $\bar{Y}$ and S_Y^2 are the sample mean and sample variance for Y, respectively; and $\bar{x}$, $\bar{y}$, s_X^2, and s_Y^2 are the observed samples of $\bar{X}$, $\bar{Y}$, S_X^2, and S_Y^2, respectively.

Lee and Lin [7] used the generalized pivotal to the function of unknown parameters. θ is constructed based on the random quantities $Z_x=\dfrac{\sqrt{n}\left(\bar{X}-\mu_x\right)}{\sigma_x}\sim N\left(0,1\right)$, $Z_y=\dfrac{\sqrt{m}\left(\bar{Y}-\mu_y\right)}{\sigma_y}\sim N\left(0,1\right)$,

$u_x=\dfrac{nS_x^2}{\sigma_X^2}\sim\chi_{n-1}^2$, and $u_y=\dfrac{mS_y^2}{\sigma_y^2}\sim\chi_{m-1}^2$.

The distributions are free of unknown parameters:

$$\theta=\frac{\bar{Y}-Z_y\sigma_y/\sqrt{m}}{\bar{X}-Z_x\sigma_x/\sqrt{n}}=\frac{\bar{Y}-Z_yS_y/\sqrt{u_y}}{\bar{X}-Z_xS_x/\sqrt{u_x}}.$$

Thus, the generalized pivotal can be defined as

$$Q\left(X,Y,x,y,\mu_x,\mu_y,\sigma_x^2,\sigma_y^2\right)=\frac{\mu_y}{\mu_x}=\frac{\bar{Y}-Z_yS_y/\sqrt{u_y}}{\bar{X}-Z_xS_x/\sqrt{u_x}}=\frac{\bar{Y}-T_yS_y/\sqrt{m-1}}{\bar{X}-T_xS_x/\sqrt{n-1}}.$$

Following the idea presented by Lee and Lin [7], we suppose that σ_X^2 is known and σ_Y^2 is unknown, then the respective means for X and Y become

$$\mu_x=\bar{x}-\left(T_1\sigma_x/\sqrt{n-1}\right)\tag{1}$$

and
$$\mu_y = \bar{y} - \left(T_2 S_y / \sqrt{m-1}\right), \tag{2}$$
where T_1 and T_2 are t-distributions with $n-1$ and $m-1$ degrees of freedom, respectively.

The generalized pivotal quantity for the ratio of the means is given by
$$Q\left(X,Y,x,y,\mu_x,\mu_y,\sigma_x^2,s_Y^2\right) = \frac{\mu_y}{\mu_x} = \frac{\bar{y}-\left(T_2\, s_y / \sqrt{n-1}\right)}{\bar{x}-\left(T_1\, \sigma_x / \sqrt{m-1}\right)}. \tag{3}$$

Therefore, the $100(1-\alpha)\%$ two-sided confidence interval for the ratio of the means with one variance unknown based on the GCI approach is obtained as

$$CI_{GCI} = \left(Q(\alpha/2), Q(1-\alpha/2)\right),, \tag{4}$$

where $Q(\alpha/2)$ and $Q(1-\alpha/2)$ are the $100(\alpha/2)$-th and $100(1-\alpha/2)$-th percentiles of Q, respectively.

III. THE METHOD OF VARIANCE ESTIMATES RECOVERY CONFIDENCE INTERVAL (MOVER)

Let (l_1,u_1) be the confidence interval for the mean of Y, where l_1 and u_1 become
$$l_1 = \bar{Y} - Z_{1-\alpha/2}\frac{S_Y}{\sqrt{m}} \tag{5}$$
and
$$u_1 = \bar{Y} + Z_{1-\alpha/2}\frac{S_Y}{\sqrt{m}}. \tag{6}$$
Thus,
$$CI_1 = (l_1,u_1) = \left[\bar{Y} - Z_{1-\alpha/2}\frac{S_y}{\sqrt{m}}, \bar{Y} + Z_{1-\alpha/2}\frac{S_y}{\sqrt{m}}\right]. \tag{7}$$

In addition, let $(l_2 u_2)$ be the confidence interval for the mean of X, then l_2 and u_2 become
$$l_2 = \bar{X} - Z_{1-\alpha/2}\frac{\sigma_x}{\sqrt{n}} \tag{8}$$
and
$$u_2 = \bar{X} + Z_{1-\alpha/2}\frac{\sigma_x}{\sqrt{n}}. \tag{9}$$
Thus,
$$CI_2 = (l_2,u_2) = \left[\bar{X} - Z_{1-\alpha/2}\frac{S_x}{\sqrt{n}}, \bar{X} + Z_{1-\alpha/2}\frac{S_x}{\sqrt{n}}\right]. \tag{10}$$

Following Donner and Zou [1], the lower and upper limits for the ratio of two means are respectively expressed as

$$L_{MOVER} = \frac{\hat{\mu}_Y\hat{\mu}_X - \sqrt{\left(\hat{\mu}_Y\hat{\mu}_X\right)^2 - l_1u_2\left(2\hat{\mu}_Y - l_1\right)\left(2\hat{\mu}_X - u_2\right)}}{u_2\left(2\hat{\mu}_X - u_2\right)} \tag{11}$$

and
$$U_{MOVER} = \frac{\hat{\mu}_Y\hat{\mu}_X + \sqrt{\left(\hat{\mu}_Y\hat{\mu}_X\right)^2 - u_1l_2\left(2\hat{\mu}_Y - u_1\right)\left(2\hat{\mu}_X - l_2\right)}}{l_2\left(2\hat{\mu}_X - l_2\right)}, \tag{12}$$

where l_1, u_1, l_2, and u_2 are defined as in Equations (5)–(10), respectively.

Therefore, the $100(1-\alpha)\%$ two-sided confidence interval for the ratio of means with one variance unknown based on the MOVER approach is obtained as $CI_{MOVER} = CI_M = \left(L_{MOVER}, U_{MOVER}\right) = \left(L_M, U_M\right)$,

$CI_M = \left(L_M, U_M\right)$,

$$CI_M = \left(\frac{\hat{\mu}_y\hat{\mu}_x - \sqrt{\left(\hat{\mu}_y\hat{\mu}_x\right)^2 - l_1u_2\left(2\hat{\mu}_y - l_1\right)\left(2\hat{\mu}_x - u_2\right)}}{u_2\left(2\hat{\mu}_x - u_2\right)}, \frac{\hat{\mu}_y\hat{\mu}_x + \sqrt{\left(\hat{\mu}_y\hat{\mu}_x\right)^2 - u_1l_2\left(2\hat{\mu}_y - u_1\right)\left(2\hat{\mu}_x - l_2\right)}}{l_2\left(2\hat{\mu}_x - l_2\right)} \right), \tag{13}$$

where L_{MOVER} and U_{MOVER} are defined as in Equations (11)–(13).

IV. THE BOOTSTRAP CONFIDENCE INTERVAL (BCI)

Let $x = (x_1, x_2, \ldots, x_n)$ be the observed values of $X = (X_1, X, \ldots, X_n)$. Suppose that $\hat{F}$ is the empirical distribution with the probability $\frac{\mu_y}{\mu_x}$ on each of the observed values x_i, where $i = 1, 2, \ldots, n.$

Let $X^* = (X_1^*, X_2^*, \ldots, X_n^*)$ be the random sample drawn from $\hat{F}$ (called the bootstrap sample) and let $x^* = (x_1^*, x_2^*, \ldots, x_n^*)$ be the observed values in the bootstrap sample, i.e. $x^* = (x_1^*, x_2^*, \ldots, x_n^*)$ is the result of resampling $x = (x_1, x_2, \ldots, x_n)$.

The Bootstrap confidence interval can be constructed based on several methods .We constructed this one based on the bootstrap percentiles of the bootstrap distribution. The variance estimator of normal distribution based on the bootstrap sample is defined as

$$\hat{\theta}^* = \frac{\overline{Y} - Z_y\sigma_y / \sqrt{m}}{\overline{X} - Z_x\sigma_x / \sqrt{n}} = \frac{\overline{Y} - Z_yS_y / \sqrt{u_y}}{\overline{X} - Z_xS_x / \sqrt{u_x}}. \tag{14}$$

Therefore, the Bootstrap confidence interval for the variance of normal distribution is defined as

$$CI_{PB} = [L_{PB}, U_{PB}] = [\hat{\theta}^*(\alpha/2), \hat{\theta}^*(1-\alpha/2), \tag{15}$$

where $\hat{\theta}^*(\alpha/2)$ and $\hat{\theta}^*(1-\alpha/2)$ are the $100(\alpha/2)$-th and $100(1-\alpha/2)$-th percentiles of $\hat{\theta}^*$, respectively.

V. SIMULATION STUDIES

A Monte Carlo simulation study was performed to compare the coverage probabilities (CPs) and expected lengths of the confidence intervals for the ratio of means of normal distributions with one variance unknown based on the generalized confidence interval (GCI), the method of variance estimates recovery $(MOVER)$, and the Bootstrap confidence interval (BCI) (CI_{GCI}, CI_{MOVER}, and CI_{BCI}, respectively). In this simulation study, the sample sizes were varied as $n = 20$, 30, 50, 100, 200, 250, 500 and $m = 20$, 30, 50, 100, 200, 250, 500; the population means of normal data for each sample were set as $\mu_x = \mu_y = 1$; and the population standard deviations were varied as $\sigma_1 =$ 0.1, 0.3, 0.5, 0.7, 0.9, 1.0, 1.1, 1.3, 1.5, 1.7, 2.0 and $\sigma_2 = 1.0$. For each set of parameters, the number of simulation runs was M = 5,000, from which R_θ was obtained for N = 2,500 random samples.

Table 1 reports the coverage probabilities and expected lengths for CI_{GCI}, CI_{MOVER}, and CI_{BCI}, from which it can be seen that the coverage probabilities for CI_{MOVER} were closer to the nominal 0.95 confidence level than the others for almost all cases in the study. For small sample sizes ($n < 30$, $m < 30$), the CPs for GCI (CI_{GCI}) were larger than those for CI_{MOVER}. The coverage probabilities for CI_{GCI} for large sample sizes ($n \geq 30$, $m \geq 30$) were slightly better than for small sample sizes, thereby inferring a slight improvement in performance with increasing sample size. Moreover, the expected lengths of CI_{MOVER} and CI_{GCI} decreased with increasing sample size. Nevertheless, the performances of CI_{GCI} and CI_{MOVER} were still good for moderate-to-large sample sizes ($n \geq 50$, $m \geq 50$). In all cases, CI_{BCI} attained larger CPs than CI_{GCI} and CI_{MOVER} that were closer to the nominal 0.95 confidence interval and its expected lengths were longer than the others. The expected lengths of

CI_{GCI} and CI_{MOVER} were slightly different in each case, but those of CI_{MOVER} were always shorter. Thus, the results show that CI_{MOVER} performed better in most situations.

In the overall image, the performance of CI_{MOVER} was better than CI_{GCI} and CI_{BCI} , and it attained the nominal confidence interval on most occasions. Thus, the CPs for CI_{MOVER} show that it is more appropriate than CI_{GCI} and CI_{BCI} for all sample sizes and situations tested. Note that these simulation results are similar to those of Wongkhao [21],[22] who studied the confidence intervals for the ratio of two independent coefficients of variation of normal distributions, and Niwitpong [15], who studied the confidence intervals for the difference and the ratio of coefficients of variation of normal distributions with a known ratio of variances.

TABLE I

COVERAGE PROBABILITIES AND EXPECTED LENGTHS OF APPROXIMATELY 95% OF THE PROPOSED CONFIDENCE INTERVALS FOR THE RATIO OF TWO NORMAL MEANS WITH ONE VARIANCE UNKNOWN

n	m	$\frac{\sigma_2}{\sigma_1}$	CI_{GCI} (GCI)		CI_{MOVER} (MOVER)		CI_{BCI} (Bootstrap)	
			CP	Ex.Length	CP	Ex.Length	CP	Ex.Length
20	30	0.1	0.9808	2.0007	0.9486	2.1164	0.9970	3.7516
		0.3	0.9826	1.9938	0.9514	2.4454	0.9966	3.8146
		0.5	0.9790	2.1829	0.9510	1.9392	0.9982	4.0434
		0.7	0.9776	2.1656	0.9504	1.7034	0.9972	4.0097
		0.9	0.9798	2.1925	0.9540	2.3548	0.9986	4.1045
		1.0	0.9774	2.1871	0.9518	1.9576	0.9976	4.0729
		1.1	0.9710	2.2637	0.9458	1.8007	0.9968	4.1488
		1.3	0.9738	2.3001	0.9484	2.2435	0.9978	4.1980
		1.5	0.9704	2.3334	0.9446	2.8061	0.9970	4.2181
		1.9	0.9704	2.4604	0.9466	-3.5173	0.9986	4.4138
		2.0	0.9620	2.8004	0.9460	4.4694	0.9980	4.8622
30	20	0.1	0.9670	1.1239	0.9538	1.0647	0.9966	1.9191
		0.3	0.9618	1.2658	0.9450	1.1760	0.9970	2.1510
		0.5	0.9614	1.3310	0.9472	1.2196	0.9958	2.1930
		0.7	0.9674	1.4423	0.9492	1.3172	0.9966	2.3589
		0.9	0.9672	1.5334	0.9508	1.3786	0.9964	2.4915
		1.0	0.9620	1.5732	0.9438	1.4308	0.9968	2.5296
		1.1	0.9630	1.6164	0.9416	1.3322	0.9952	2.6172
		1.3	0.9598	1.6819	0.9432	1.4904	0.9934	2.6764
		1.5	0.9632	1.7279	0.9484	1.5786	0.9938	2.7360
		1.9	0.9670	1.8774	0.9476	1.8625	0.9950	2.9308
		2.0	0.9586	1.9116	0.9410	1.7272	0.9910	2.9881
30	30	0.1	0.9646	1.1257	0.9514	1.0298	0.9976	1.9549
		0.3	0.9602	1.2308	0.9480	1.1157	0.9968	2.0969
		0.5	0.9620	1.2576	0.9486	1.1677	0.9974	2.1155
		0.7	0.9562	1.3106	0.9450	1.1775	0.9956	2.1789
		0.9	0.9590	1.3731	0.9466	1.1957	0.9954	2.2729
		1.0	0.9630	1.3878	0.9512	1.2444	0.9962	2.2672
		1.1	0.9632	1.4336	0.9524	1.3366	0.9982	2.3697
		1.3	0.9630	1.4666	0.9488	1.3504	0.9962	2.3990
		1.5	0.9600	1.5357	0.9458	1.4173	0.9952	2.5066
		1.9	0.9626	1.6381	0.9504	1.6016	0.9954	2.6449
		2.0	0.9596	1.6682	0.9468	1.5438	0.9950	2.6922
30	50	0.1	0.9626	1.1191	0.9478	1.0110	0.9974	1.9206
		0.3	0.9670	1.1312	0.9538	1.0543	0.9980	1.9307
		0.5	0.9608	1.1833	0.9492	1.0854	0.9964	2.0007
		0.7	0.9602	1.2256	0.9466	1.1626	0.9958	2.0894
		0.9	0.9586	1.2931	0.9464	1.1857	0.9970	2.2202
		1.0	0.9672	1.3168	0.9572	1.1753	0.9976	2.2103
		1.1	0.9622	1.3001	0.9508	1.3756	0.9974	2.2006

		1.3	0.9618	1.3548	0.9504	1.2334	0.9968	2.2488
		1.5	0.9560	1.3722	0.9452	1.2713	0.9966	2.2863
		1.9	0.9576	1.4296	0.9466	1.2077	0.9966	2.3769
		2.0	0.9602	1.4308	0.9488	1.3259	0.9972	2.3649

TABLE 1 CONTINUED

n	m	$\frac{\sigma_2}{\sigma_1}$	CI_{GCI} (GCI)		CI_{MOVER} (MOVER)		CI_{BCI} (Bootstrap)	
			CP	Ex.Length	CP	Ex.Length	CP	Ex.Length
50	30	0.1	0.9560	0.7321	0.9462	0.6998	0.9966	1.1188
		0.3	0.9584	0.8061	0.9522	0.7688	0.9966	1.2144
		0.5	0.9582	0.8959	0.9492	0.8530	0.9934	1.3416
		0.7	0.9552	0.9714	0.9448	0.9225	0.9936	1.4464
		0.9	0.9542	1.0284	0.9438	0.9761	0.9944	1.5200
		1.0	0.9574	1.0668	0.9484	1.0116	0.9928	1.5734
		1.1	0.9556	1.0928	0.9446	1.0355	0.9912	1.6077
		1.3	0.9608	1.1574	0.9514	1.0964	0.9944	1.6978
		1.5	0.9556	1.2087	0.9444	1.1438	0.9920	1.7692
		1.9	0.9598	1.3118	0.9474	1.2404	0.9938	1.9102
		2.0	0.9540	1.3332	0.9444	1.2609	0.9936	1.9410
50	50	0.1	0.9578	0.7054	0.9498	0.6763	0.9944	1.0832
		0.3	0.9582	0.7566	0.9522	0.7261	0.9944	1.1519
		0.5	0.9558	0.8080	0.9492	0.7755	0.9944	1.2258
		0.7	0.9566	0.8569	0.9498	0.8224	0.9962	1.2910
		0.9	0.9550	0.8954	0.9492	0.8602	0.9924	1.3485
		1.0	0.9538	0.9176	0.9460	0.8820	0.9948	1.3798
		1.1	0.9518	0.9429	0.9454	0.9060	0.9932	1.4178
		1.3	0.9564	0.9835	0.9490	0.9453	0.9934	1.4751
		1.5	0.9536	1.0148	0.9442	0.9755	0.9936	1.5128
		1.9	0.9598	1.0866	0.9514	1.0449	0.9952	1.6146
		2.0	0.9496	1.1125	0.9422	1.0698	0.9946	1.6554
50	100	0.1	0.9602	0.6962	0.9512	0.6675	0.9956	1.0716
		0.3	0.9548	0.7202	0.9486	0.6912	0.9942	1.1053
		0.5	0.9536	0.7459	0.9454	0.7176	0.9952	1.1444
		0.7	0.9526	0.7726	0.9472	0.7447	0.9934	1.1851
		0.9	0.9600	0.7959	0.9546	0.7675	0.9950	1.2127
		1.0	0.9532	0.8004	0.9472	0.7727	0.9940	1.2158
		1.1	0.9604	0.8155	0.9546	0.7872	0.9942	1.2387
		1.3	0.9524	0.8416	0.9476	0.8134	0.9946	1.2802
		1.5	0.9608	0.8540	0.9556	0.8257	0.9962	1.2934
		1.9	0.9532	0.9061	0.9482	0.8772	0.9940	1.3692
		2.0	0.9486	0.9162	0.9430	0.8870	0.9932	1.3859
100	50	0.1	0.9510	0.4698	0.9466	0.4599	0.9940	0.6820
		0.3	0.9538	0.5408	0.9502	0.5281	0.9928	0.7803
		0.5	0.9560	0.6010	0.9500	0.5855	0.9956	0.8631
		0.7	0.9536	0.6542	0.9468	0.6364	0.9938	0.9360
		0.9	0.9484	0.7062	0.9428	0.6864	0.9950	1.0083
		1.0	0.9494	0.7332	0.9452	0.7123	0.9938	1.0457
		1.1	0.9514	0.7563	0.9460	0.7345	0.9942	1.0780
		1.3	0.9508	0.7992	0.9452	0.7756	0.9932	1.1377
		1.5	0.9534	0.8424	0.9464	0.8167	0.9952	1.1975
		1.9	0.9484	0.9232	0.9424	0.8948	0.9920	1.3111
		2.0	0.9570	0.9418	0.9512	0.9128	0.9932	1.3368

TABLE 1 CONTINUED

n	m	σ_2 / σ_1	CI_{GCI} (GCI)		CI_{MOVER} (MOVER)		CI_{BCI} (Bootstrap)	
			CP	Ex.Length	CP	Ex.Length	CP	Ex.Length
100	100	0.1	0.9556	0.4492	0.9512	0.4407	0.9958	0.6556
		0.3	0.9540	0.4868	0.9508	0.4779	0.9954	0.7083
		0.5	0.9550	0.5171	0.9536	0.5078	0.9946	0.7498
		0.7	0.9534	0.5499	0.9502	0.5399	0.9944	0.7959
		0.9	0.9524	0.5839	0.9484	0.5734	0.9932	0.8439
		1.0	0.9524	0.5968	0.9502	0.5861	0.9930	0.8628
		1.1	0.9486	0.6091	0.9464	0.5986	0.9928	0.8806
		1.3	0.9536	0.6361	0.9496	0.6248	0.9952	0.9172
		1.5	0.9526	0.6615	0.9482	0.6503	0.9948	0.9535
		1.9	0.9514	0.7120	0.9468	0.6994	0.9936	1.0243
		2.0	0.9556	0.7261	0.9506	0.7133	0.9944	1.0449
200	200	0.1	0.9564	0.3027	0.9546	0.2999	0.9942	0.4341
		0.3	0.9504	0.3303	0.9502	0.3273	0.9942	0.4727
		0.5	0.9528	0.3545	0.9534	0.3512	0.9922	0.5070
		0.7	0.9518	0.3760	0.9504	0.3725	0.9950	0.5368
		0.9	0.9546	0.3967	0.9526	0.3933	0.9950	0.5666
		1.0	0.9488	0.4073	0.9482	0.4038	0.9950	0.5814
		1.1	0.9502	0.4152	0.9488	0.4116	0.9936	0.5926
		1.3	0.9500	0.4349	0.9492	0.4310	0.9938	0.6203
		1.5	0.9468	0.4513	0.9456	0.4473	0.9938	0.6433
		1.9	0.9478	0.4870	0.9472	0.4827	0.9930	0.6941
		2.0	0.9540	0.4962	0.9518	0.4917	0.9946	0.7067
250	250	0.1	0.9534	0.2696	0.9526	0.2676	0.9948	0.3855
		0.3	0.9528	0.2921	0.9510	0.2900	0.9944	0.4173
		0.5	0.9502	0.3124	0.9496	0.3102	0.9930	0.4460
		0.7	0.9484	0.3324	0.9492	0.3302	0.9958	0.4740
		0.9	0.9518	0.3511	0.9482	0.3486	0.9958	0.5002
		1.0	0.9514	0.3608	0.9494	0.3584	0.9946	0.5142
		1.1	0.9458	0.3689	0.9444	0.3661	0.9938	0.5254
		1.3	0.9550	0.3868	0.9538	0.3843	0.9952	0.5510
		1.5	0.9556	0.4022	0.9540	0.3995	0.9958	0.5726
		1.9	0.9520	0.4326	0.9494	0.4296	0.9950	0.6154
		2.0	0.9516	0.4398	0.9494	0.4367	0.9924	0.6257
500	500	0.1	0.9492	0.1871	0.9496	0.1864	0.9934	0.2658
		0.3	0.9522	0.2025	0.9528	0.2017	0.9944	0.2878
		0.5	0.9442	0.2181	0.9438	0.2173	0.9938	0.3098
		0.7	0.9482	0.2316	0.9490	0.2307	0.9926	0.3288
		0.9	0.9538	0.2454	0.9544	0.2445	0.9970	0.3483
		1.0	0.9496	0.2513	0.9484	0.2503	0.9934	0.3565
		1.1	0.9470	0.2580	0.9454	0.2570	0.9912	0.3660
		1.3	0.9522	0.2695	0.9532	0.2686	0.9952	0.3825
		1.5	0.9528	0.2808	0.9522	0.2797	0.9960	0.3982
		1.9	0.9502	0.3021	0.9490	0.3009	0.9952	0.4282
		2.0	0.9500	0.3075	0.9508	0.3066	0.9948	0.4366

VI. SIMULATION STUDIES

Example 1 : In this section, two datasets are used to compare the three confidence interval approaches. The first dataset was obtained from Montgomery and Runger [11] concerning a polymer is manufactured in a batch chemical process. Viscosity measurements are normally made on each batch, and long experience with the process has indicated that the variability in the process is fairly stable with $\sigma_1 = 20$; 15 batch viscosity measurements are available:

724, 718, 776, 760, 745, 759, 795, 756, 742, 740, 761, 749, 739, 747, 742.

A process change is made which involves switching the type of catalyst used in the process. Following the process change, eight batch viscosity measurements are available:

735, 775, 729, 755, 783, 760, 738, 780.

Assuming that process variability is unaffected by the catalyst change, the 90% confidence interval for the difference in mean batch viscosity resulting from the process change can be calculated. The sample mean (sample variance) of the normal data were 750.2000 (19.1281) and 756.8750 (21.2834) for the standard and new processes, respectively.

The confidence intervals for the ratio of normal means based on the GCI, MOVER, and Bootstrap approaches for the lower and upper bound were (0.9802, 1.0395) with a length of interval of 0.0594, (0.9852, 1.0330) with a length of interval of 0.0478, and (0.9736, 1.0434) with a length of interval of 0.0699, respectively.

Example 2 : The second dataset from Montgomery and Runger [11] consists of the filled volumes of bottles from two machines (Table 2). The two machines are used to fill plastic bottles with a net volume of 16.0 ounces. The fill volume can be assumed as a normal distribution with standard deviation $\sigma_1 = 0.020$ ounces. A member of the quality engineering staff suspected that both machines fill to the same mean net volume (whether or not this volume is 16.0 ounces) and so a random sample of 10 bottles was taken from the output of each machine. The sample mean (sample variance) of the normal data were 16.015 (0.0303) and 16.005 (0.0255) for machines 1 and 2, respectively.

The confidence intervals for the ratio of normal means based on the GCI, MOVER, and Bootstrap approaches for the lower and upper bound were (0.9979, 1.0009) with a length of interval of 0.0031, (0.9981, 1.0006) with a length of interval of 0.0025, and (0.9973, 1.0015) with a length of interval of 0.0042. The results from these two examples support the simulation results.

TABLE II

THE 20 OBSERVATIONS OF THE FILLED PLASTIC BOTTLES FROM THE TWO MACHINES

Machine 1		Machine 2	
16.03	16.01	16.02	16.03
16.04	15.96	15.97	16.04
16.05	15.98	15.96	16.02
16.05	16.02	16.01	16.01
16.02	15.99	15.99	16.00

VII. SIMULATION STUDIES

In this paper, we proposes confidence intervals for the ratio of normal means with one variance unknown based on the generalized confidence interval (GCI) approach, the method of variance estimates recovery (MOVER), and a Bootstrap technique (Bootstrap confidence interval (BCI). Three new confidence intervals for the ratio of means of normal distributions with one variance unknown: CI_{GCI} , CI_{MOVER}, and CI_{BCI} are proposed. Through a simulation study, the CPs of CI_{MOVER} showed that it was close to the nominal 0.95 confidence interval when the sample size n > 30 and performed as well as or better than CI_{GCI} and CI_{BCI} in all cases. The expected lengths of CI_{MOVER} and CI_{GCI} were not much different, but those of CI_{MOVER} were shorter than those of CI_{GCI} and CI_{BT} in all cases. The results indicate CI_{GCI} and CI_{MOVER} were closer to the nominal level of 0.95 than CI_{BCI} . Overall, CI_{MOVER} performed better than CI_{GCI} and CI_{BCI} . In conclusion, the main finding of this study is that CI_{MOVER} is more appropriate than CI_{GCI} and CI_{BCI} for constructing confidence intervals for the ratio of normal means with one variance unknown. In the future, our approaches are applied to an analysis of a real data set of drugs or treatments.

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