Numerical simulation for the fractional diabetes model by fractional variational iteration method

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Abstract: *In this paper, a mathematical study of the size of a population of diabetes mellitus patients is carried out. The study also monitors the number of patients with complications. By appropriate definition of a parameter, the mathematical model may be classified as linear or non-linear. The non-linear case is discussed and the critical values of the population are analysed for stability. The model equations were solved using the Fractional Variational Iteration Method (FVIM). Graphs were generated from the results obtained using Maple software. It was observed that the parameters involved play a crucial role in the size of population of diabetics and the number of diabetics with complications at time t. Numerical methods are developed for solving the model equations and the results of numerical simulations are reported.*

Keywords: *Fractional diabetes model; Fractional variational iteration method; Caputo fractional derivative*

I. Introduction

In past few decades, fractional order calculus has emerged as a potential tool in various domains of science and engineering such as fluid dynamic traffic [1], neurophysiology [2], visco elasticity [3], bioengineering [4], control theory [5], electromagnetic theory [6], potential theory [7],electric technology [8], biology [9], industrial robotics [10], mathematical economy [11], etc. The real–world processes which we require to deal with, are generally of fractional order. Heat diffusion in to a semi–infinite solid where heat flow equals half–derivative of temperature is an example of fractional order system.

Differential equations which govern the systems with memory are fractional differential equations (FDEs). The arbitrariness in their order introduces more degrees of freedom in design and analysis, resulting in more accurate modelling, better robustness in control and greater flexibility in signal processing [12]. By this time, it is established that the electrochemical phenomena like double layer charge distribution or the diffusion process can be better explained with fractional order system. As a result, the modelling of lithium in battery, fuel cells, super capacitors are carried out with FDEs. The characterization of ceramic bodies, fractal structures, viscoelastic materials, the decay rate of fruits and meats, study of corrosion in metal surface are also promising areas of its applications. Fractional order system is also a popular choice to study real time events such as earthquake propagation, volcanic phenomenon, design of phermo-kinetics, modelling of human lungs and skin. Even the characteristics of economic market fluctuation adopts fractional calculus based system modelling. So, F.O. analysis has now reached from inert physical network to living network of biology, ecology, physiology and sociology reminding us Leibnitz's predication in his letter to L'Hopital in 1695 that fractional differential operator is "an apparent paradox from which one day useful consequences will be drawn".

Diabetes Mellitus is simply caused by the failure of the body to produce the right amount of insulin to stabilize the amount of sugar in the body [13]. Most patients who suffer this type of body failure are recommended to take insulin injection. This is called diabetes type I. Diabetes type II is the patient's body rejection to insulin. This type of patient is recommended to undergo a certain health meal program as well as performing exercises tolose weight in addition of oral medication. However, heart diseases are likely to strike these patients in the long run [14]. Mainly, two types of diabetes are studied: Type 1 diabetes, too familiar as insulin dependent diabetes mellitus, attacking people below the age of 40 and representing 10 to 15 percent of the diabetic population. Then, Type II diabetes, previously recognized as non-insulin dependent diabetes mellitus, describing the majority (85 to 90 percent). Although in all age groups with the spreading epidemic of obesity, it is anticipated that in 10 year's time, there will be additional children having Type II rather than having Type I; see [15–22].

Diabetes mellitus is not notifiable to the health authorities in the United Kingdom, where estimates of the prevalence of diabetes are usually based upon samples taken from various parts of the country, or from the small number of diabetes registers which exist, or from population surveys (British Diabetic Association [BDA] [23]). The

1993 Health Survey for England carried out by the Office of Population Censuses and Surveys found that in a sample of adults aged 16 or over, the prevalence of self-reported diabetes was 3% in men and 2% in women (BDA [23]). The recent reports released by the World Health Organisation (WHO) [24] and the International Diabetes Federation (IDF) [25] give rise for concern. In 2003 it was estimated that the prevalence of diabetes in the world population was still 3% (5.1% for those aged 20 to 79), showing that 194 million people worldwide are diabetic, most of whom live in developing countries (IDF[25]).

Gestational Diabetes can occur temporarily during Pregnancy which is due to the hormonal changes and usually begins in the fifth or sixth month of pregnancy (between the 24th and 28th weeks). Gestational Diabetes usually resolves once the baby is born. However, 25-50 % of women with gestational diabetes will eventually develop diabetes later in their life, especially those who require insulin during pregnancy and those who are overweight after their delivery [26].

These rates of change are formalized by the ordinary differential equations (ODEs)

$$\frac{dD(t)}{dt} = I - (\lambda + \mu)D + \gamma C,$$
$$\frac{dC(t)}{dt} = I + \lambda D - (\gamma + \mu + \nu + \delta)C, \tag{1}$$

where D indicates the no. of diabetics with no complications at time t. C denotes the no. of diabetics having complications. I represents the occurrence of diabetes mellitus.

The fractional diabetes model when N(t) = C(t) + D(t) and it indicates the size of diabetes at the time t, then we get,

$$\frac{d^{\alpha}C(t)}{dt^{\alpha}} = -(\lambda + \theta)C + \lambda N, \ \ t > 0$$
$$\frac{d^{\alpha}N(t)}{dt^{\alpha}} = I - (\nu + \delta)C - \mu N, \tag{2}$$

In above Eq. $\theta = \gamma + \mu + \nu + \delta$ with initial conditions $C(0) = C_0, \ N(0) = N_0$.
where μ denotes the rate of natural mortality, λ indicates the probability of a diabetic person spreading a complication, γ shows the rate of healing complications, ν stands for the rate at which diabetic patients having complication converts critically disabled and δ denotes the mortality rate because of complications.

Fractional variational iteration method (FVIM) [27–30] directly attacks the nonlinear FDEs without a need to find certain polynomials for nonlinear terms and gives result in an infinite series, that rapidly converges to analytical solution. This method does not require linearization, discretization, little perturbations or any restrictive assumptions. It lessens mathematical computations significantly.

The aim of this paper is to obtain numerical solution of fractional diabetes model by FVIM. This paper is structured in the following manner. Section 1, is Introductory. In section 2, we give brief review of preliminary description of Caputo fractional derivative and some other results, helpful in the learning of FDEs. In section 3, basic plan of proposed numerical method FVIM is shown by taking the problem under consideration. In section 4, deals with discussion of obtained numerical results and their significance. Figures are drawn using Maple package. In last section 5, we recapitulate our outcomes and draw inferences.

II. Preliminaries

Definition 2.1. Consider a real function h (χ), $\chi > 0$. It is called in space $C_{\zeta}, \zeta \in R$ if $\ni$ a real no. b ($> \zeta$), s.t. h (χ) = χ^{b} $h_1(\chi)$, $h_1 \in C[0, \infty]$. It is clear that $C_{\zeta} \subset C_{\gamma}$ if $\gamma \le \zeta$.

Definition 2.2. Consider a function h (χ), $\chi > 0$. It is called in space $C_{\zeta}^{m}, m \in \mathbb{N} \cup \{0\}$ if $h^{(m)} \in C_{\zeta}$.

Definition 2.3. Left sided Caputo fractional derivative of h, $h \in C_{-1}^{m}, m \in \mathbb{N} \cup \{0\}$,

$$D_t^{\beta} h(t) = \begin{cases} I^{m-\beta} h^{(m)}(t), m-1 < \beta < m, m \in \mathbb{N}, \\ \frac{d^m}{dt^m} h(t), \ \beta = m, \end{cases}$$

a. $I_t^{\zeta} h(x,t) = \frac{1}{\Gamma\zeta} \int_0^t (t-s)^{\zeta-1} h(x,s) ds; \ \zeta, t > 0.$

b. $D_{\tau}^{\nu} V(x,\tau) = I_{\tau}^{m-\nu} \frac{\partial^m V(x,\tau)}{\partial t^m}, m-1 < \nu \le m.$

c. $D_t^{\zeta} I_t^{\zeta} h(t) = h(t), m-1 < \zeta \le m, m \in \mathbb{N}.$

d. $I_t^{\zeta} D_t^{\zeta} h(t) = h(t) - \sum_{k=1}^{m-1} h^k(0^+) \frac{t^k}{k!}, m-1 < \zeta \le m,, m \in \mathbb{N}.$

e. $I^{\nu} t^{\zeta} = \frac{\Gamma(\zeta+1)}{\Gamma(\nu+\zeta+1)} t^{\nu+\zeta}.$

Definition 2.4. Laplace transform of Caputo fractional derivative is

$$L[D^{\alpha} g(t)] = p^{\alpha} F(p) - \sum_{k=0}^{n-1} p^{\alpha-k-1} g^{(k)}(0), n-1 < \alpha \leq n.$$

Lemma[37]**.** If u and its partial derivatives are continuous, the fractional derivative $D_t^{\alpha} u(x,y,z,t)$ is bounded.

III. Basic plan of FVIM for fractional diabetes model

Consider the mathematical model described by Eq. (2) as

$$\frac{d^{\alpha} C(t)}{dt^{\alpha}} = -(\lambda+\theta)C + \lambda N, \quad t > 0$$

$$\frac{d^{\alpha} N(t)}{dt^{\alpha}} = I - (\nu+\delta)C - \mu N, \tag{3}$$

with initial condition $C(0) = C_0, \ N(0) = N_0$

A correction functional is built for Eq. (3) as,

$$C_{n+1}(t) = C_n(t) + \int_0^t \lambda \left(\frac{d^{\alpha} C(t)}{d\xi^{\alpha}} + (\lambda+\theta)C - \lambda N\right)(d\xi)^{\alpha},$$

$$N_{n+1}(t) = N_n(t) + \int_0^t \lambda \left(\frac{d^{\alpha} N(t)}{d\xi^{\alpha}} - I + (\nu+\delta)C + \mu N\right)(d\xi)^{\alpha} \tag{4}$$

where λ is Lagrange's multiplier. By variational theory, λ must satisfy

$$\frac{d^{\alpha}\lambda}{d\xi^{\alpha}}\Big|_{\xi=x} = 0 \qquad \text{and,} \qquad 1 + \lambda|_{\xi=x} = 0.$$

We quickly get, $\lambda = -1$. Then, using it in Eq. (4), we get

$$C_{n+1}(t) = C_n(t) - \int_0^t \left(\frac{d^{\alpha} C(t)}{d\xi^{\alpha}} + (\lambda+\theta)C - \lambda N\right)(d\xi)^{\alpha},$$

$$N_{n+1}(t) = N_n(t) - \int_0^t \left(\frac{d^{\alpha} N(t)}{d\xi^{\alpha}} - I + (\nu+\delta)C + \mu N\right)(d\xi)^{\alpha}, \tag{5}$$

Consecutive approximations $C_n(t), N_n(t)$ $n \geq 0$ can be built henceforth. C_n *and* N_n is restricted variation i.e. $\delta\tilde{C}_n = 0$ and $\delta\tilde{N}_n = 0$. Finally, we obtain sequences $C_{n+1}(t), N_{n+1}(t)$, $n \geq 0$ of solution. Consequently, exact solution is gained as $C(t) = \lim_{n\to\infty} C_n(t)$ *and* $N(t) = \lim_{n\to\infty} N_n(t)$.

IV. Numerical Implementation of FVIM.

By using conditions, we may initialize with $C(0) = C_0$, $N(0) = N_0$ and using the application of FVIM to Eqs. (2), we get

$$C_1(t) = C_0 - \int_0^t \left(\frac{d^{\alpha} C_0}{d\xi^{\alpha}} + (\lambda+\theta)C_0 - \lambda N_0\right)(d\xi)^{\alpha},$$

$$= C_0 - \frac{(-(\lambda+\theta)C_0 + \lambda N_0)\, t^{\alpha}}{\Gamma(1+\alpha)}, \tag{6}$$

$$N_1(t) = N_0 - \int_0^t \left(\frac{d^{\alpha} N_0}{d\xi^{\alpha}} - I + (\nu+\delta)C_0 + \mu N_0\right)(d\xi)^{\alpha},$$

$$= N_0 - \frac{(I - (\nu+\delta)\, C_0 - \mu\, N_0)\, t^{\alpha}}{\Gamma(1+\alpha)}, \tag{7}$$

$$C_2(t) = C_1 - \int_0^t \left(\frac{d^{\alpha} C_1}{d\xi^{\alpha}} + (\lambda+\theta)C_1 - \lambda N_1\right)(d\xi)^{\alpha},$$

$$= C_0 + \frac{((\lambda+\theta)C_0 - \lambda\, N_0)\, t^{\alpha}}{\Gamma(1+\alpha)} + \frac{((-\delta\lambda + (\theta+\lambda)^2 - \lambda\nu)C_0 + \lambda\,(I - (\theta+\lambda+\mu))N_0)\, t^{2\alpha}}{\Gamma(1+2\alpha)}, \tag{8}$$

$$N_2(t) = N_1 - \int_0^t \left(\frac{d^{\alpha} N_0}{d\xi^{\alpha}} - I + (\nu+\delta)C_1 + \mu N_1\right)(d\xi)^{\alpha},$$

$$= N_0 + \frac{(-I + (\nu+\delta)\, C_0 + \mu\, N_0)\, t^{\alpha}}{\Gamma(1+\alpha)} + \frac{(-I\,\mu + (\theta+\lambda+\mu)(\nu+\delta)C_0 + \left(\mu^2 - \lambda\,(\nu+\delta)\right)N_0)\, t^{2\alpha}}{\Gamma(1+2\alpha)}. \tag{9}$$

Proceeding in this way, the next iteration components can be found with the help of Maple package. At last, we can get solution as

$$C(t) = \lim_{n\to\infty} C_n(t),$$

$$N(t) = \lim_{n\to\infty} N_n(t). \tag{10}$$

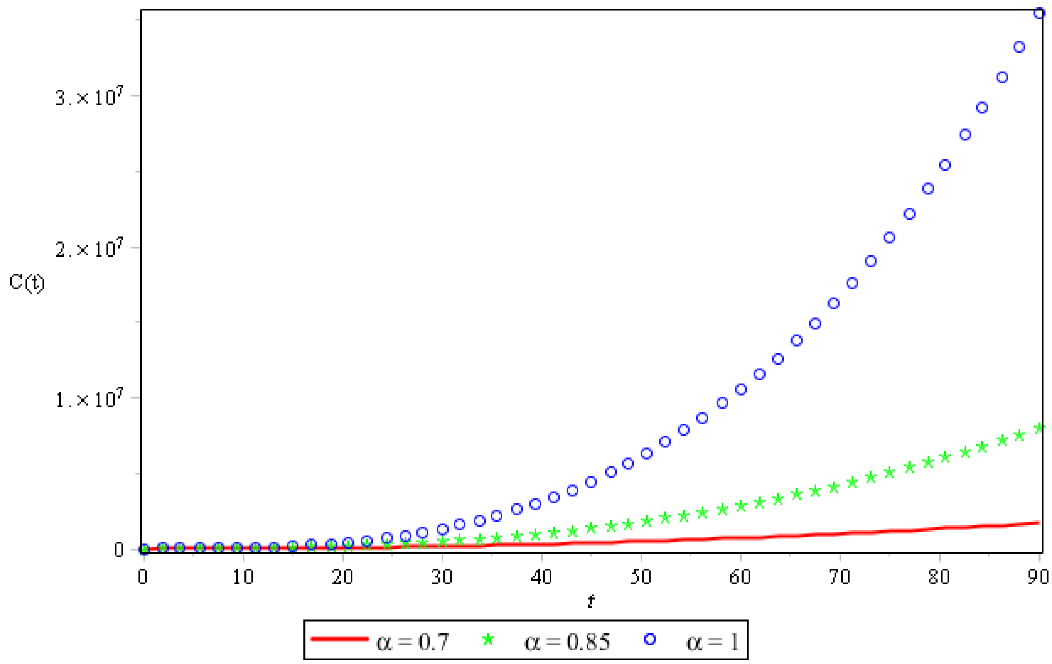

Fig. 1. Behavior of C(t) w.r.t. time t for different value of α

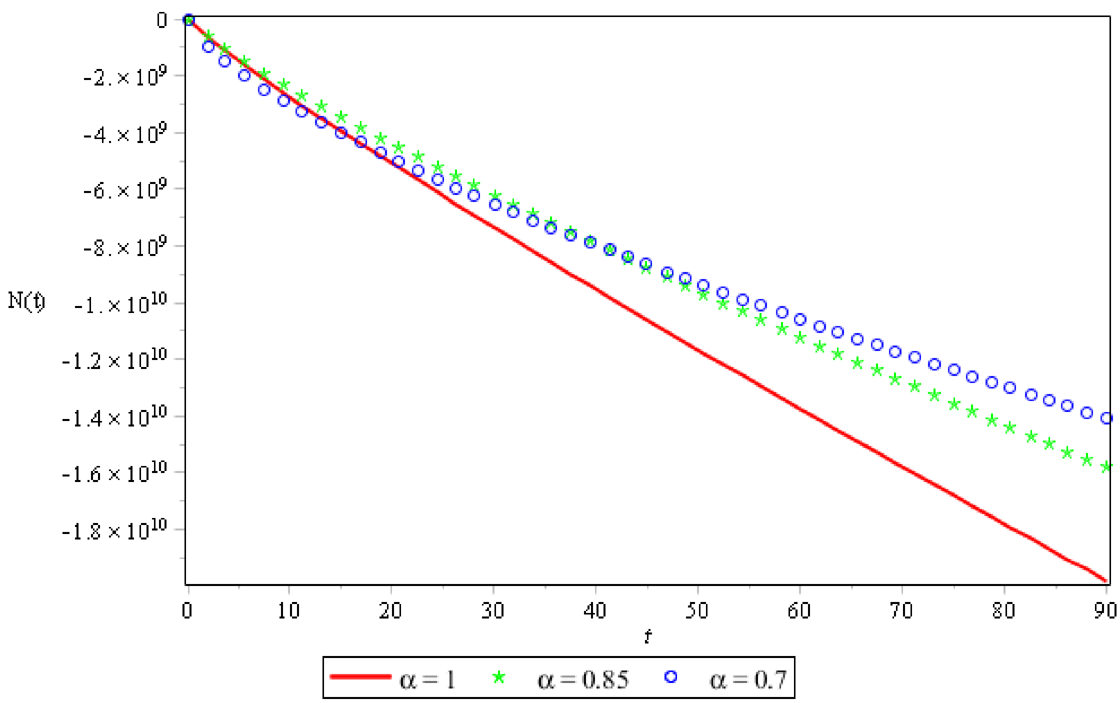

Fig. 2. Behavior of N(t) w.r.t. time t for different value of α

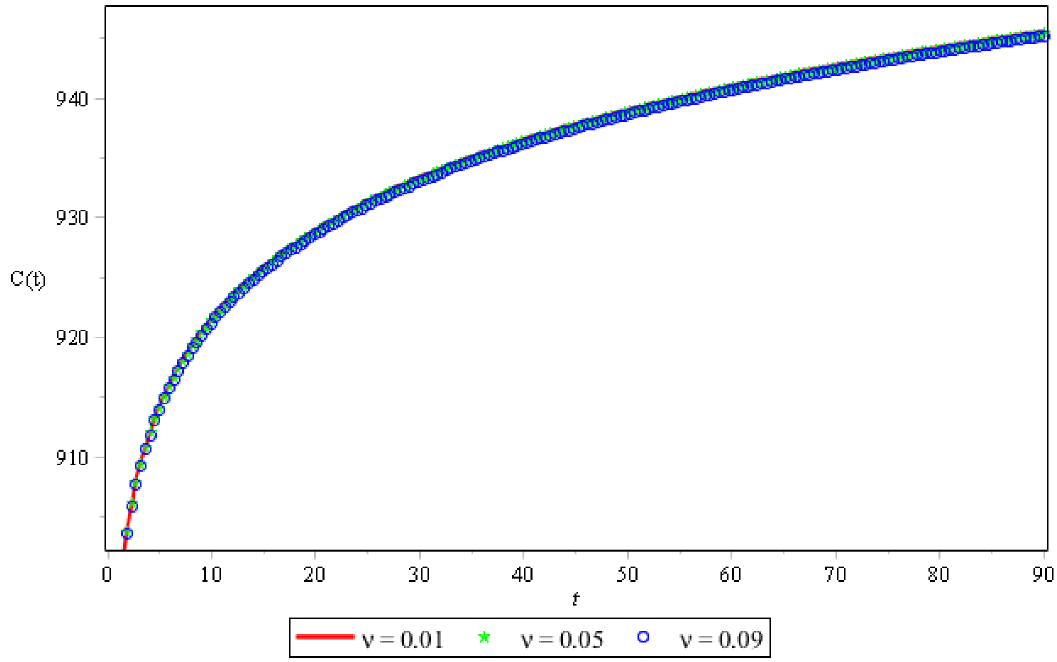

Fig. 3. Behavior of C(t) w.r.t. time t for different value of ν

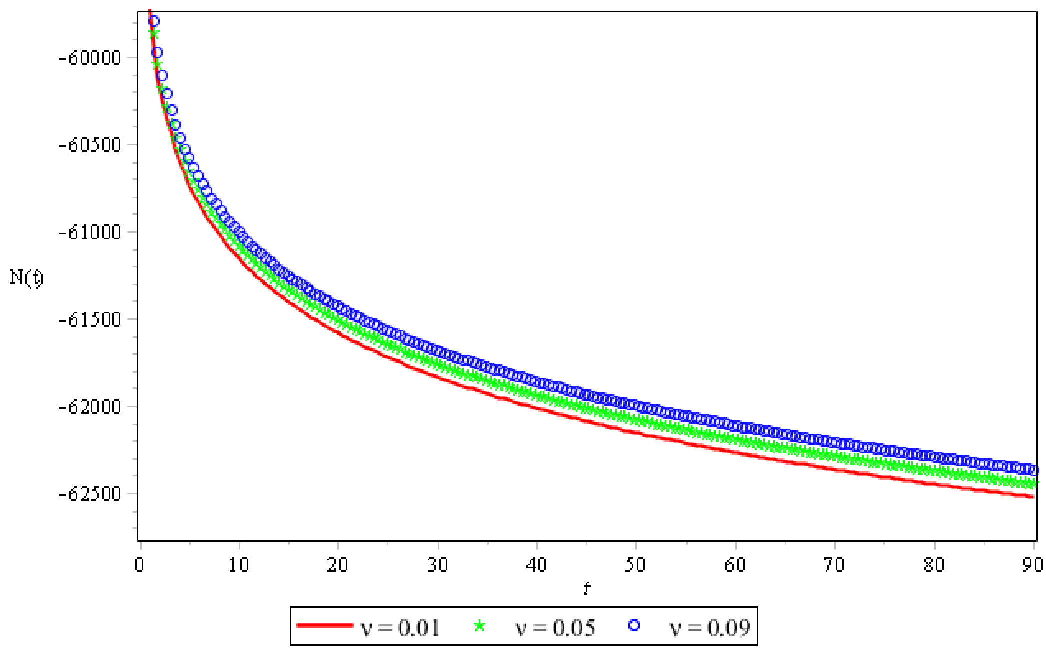

Fig. 4. Behavior of N(t) w.r.t. time t for different value of ν

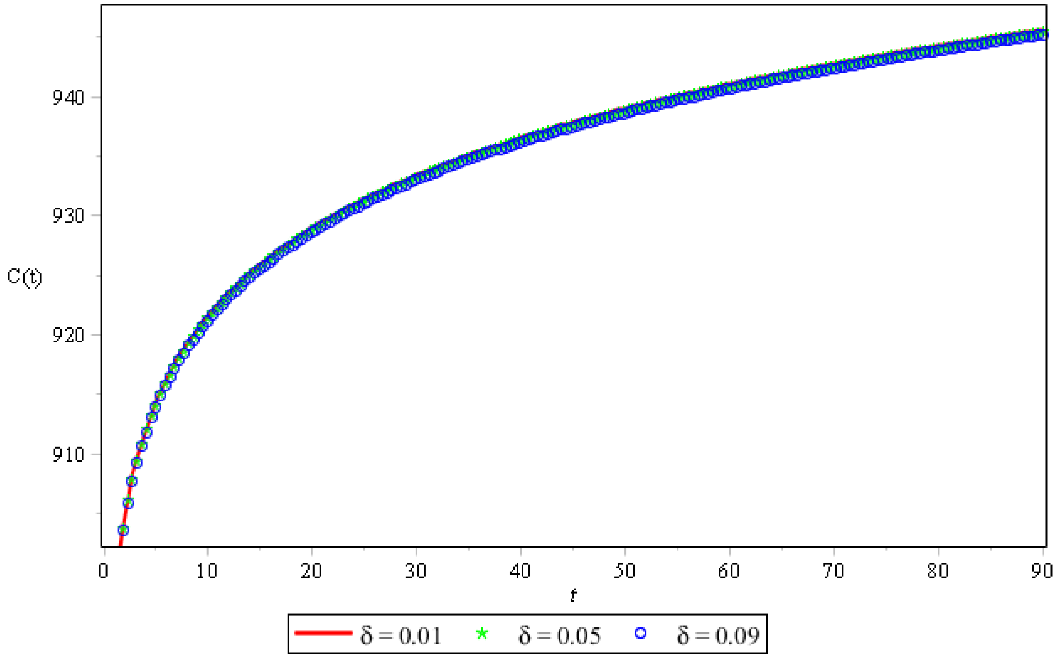

Fig. 5. Behavior of C(t) w.r.t. time t for different value of δ

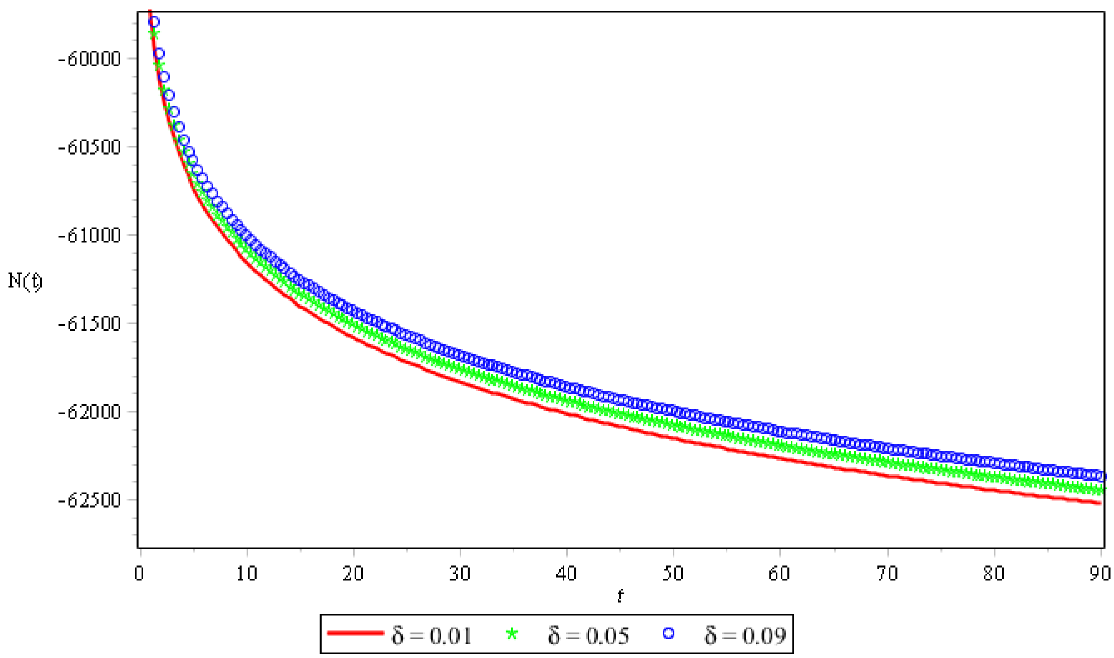

Fig. 6. Behavior of N(t) w.r.t. time t for different value of δ

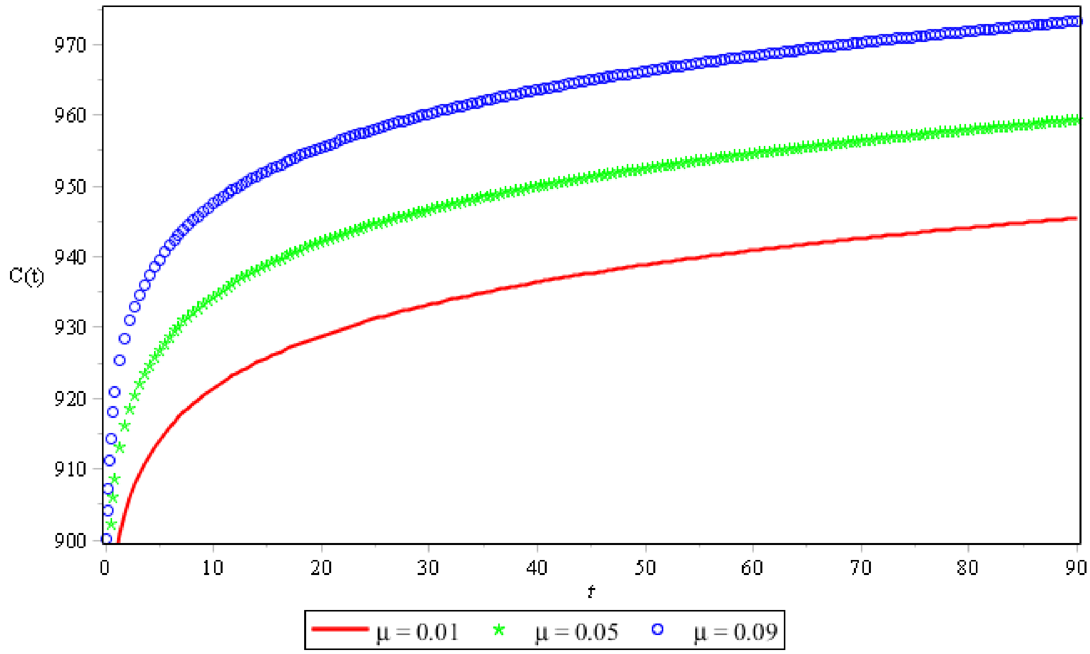

Fig. 7. Behavior of C(t) w.r.t. time t for different value of μ

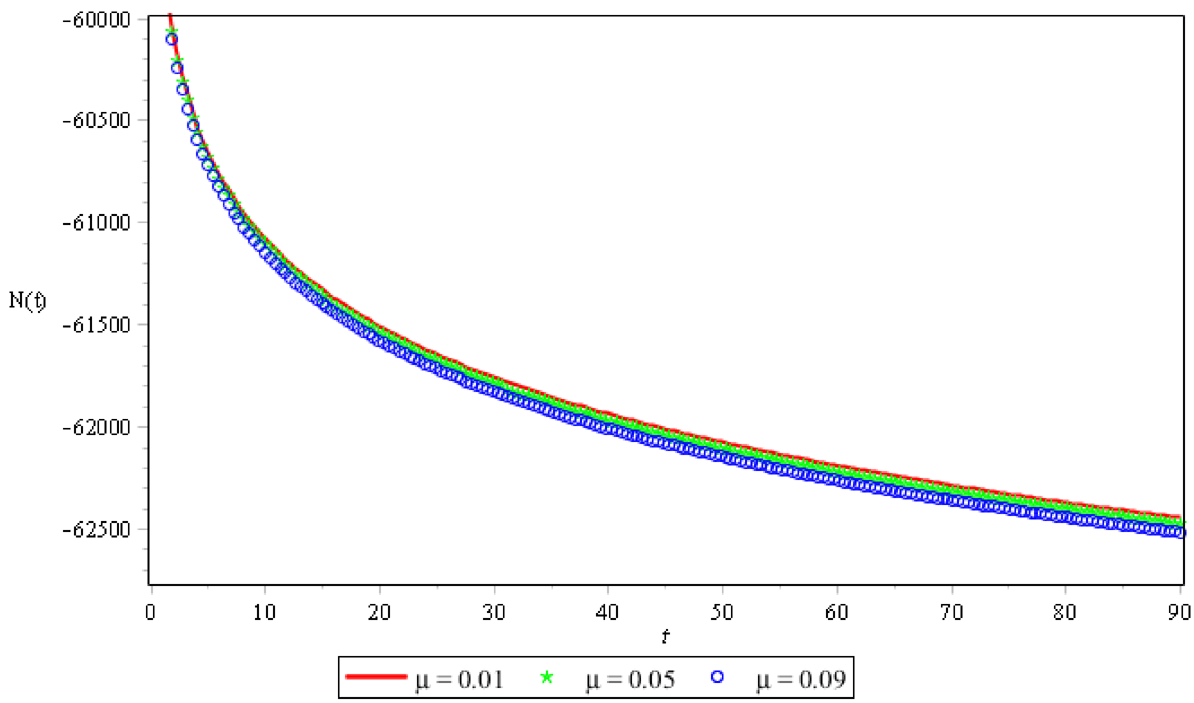

Fig. 8. Behavior of C(t) w.r.t. time t for different value of δ

V. Conclusion

In this study, we have applied Fractional variational iteration method (FVIM) on diabetes model. FVIMM is a pioneering and strong computational scheme to solve noninteger order differential equations. Some numerical results are analyzed to describe the effect of the arbitrary order. The effects of various parameters on the number of diabetics having complications and the size of diabetics with respect to time are shown graphically. The results of this study are very helpful for medical practitioners dealing with diabetes and related issues. Thus, we have concluded that the Caputo derivative is useful in the description of physical, chemical, biological, medical, social, and engineering processes.

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