Lower bounds for the length of non-associative algebras

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Abstract: *We obtain a sharp lower bound for the length of arbitrary non- associative algebra and present an example demonstrating the sharpness of our bound. To show this we introduce a new method of characteristic sequences based on linear algebra technique. This method provides an efficient tool for computing the length function in non-associative case. Then we apply anothermethod to obtain an lower bound for the length of an arbitrary locally complex algebra. We also show that the obtained bound is sharp. In the last case the length is bounded in terms of Fibonacci sequence.*

Keywords: *Finite subset, locally complex, linear span, finite dimensional.*

I. Introduction

In the present paper$\mathcal{A}$is a unital finite dimensional not necessarily associative algebra over a field $\mathbb{F}$.Let $\mathcal{S}$=$\{a_{1,}a_2,..a_k\}$be a finite subset of elements of the algebra $\mathcal{A}$. We define the length function of$\mathcal{S}$as follows.

Any product of a finite number of elements of $\mathcal{S}$ is a word in letters from $\mathcal{S}$,or simply a word in $\mathcal{S}$. The length of the word equals to the number of letters in the corresponding product. We consider 1 as a word in $\mathcal{S}$ of the length 0.

It is worth noting that different choices of brackets provide different words of the same length due to the non-associativity of $\mathcal{A}$.

The set of all words in $\mathcal{S}$ with lengths less than or equal to i is denoted by S_i, $i \geq 0$.

Note that similar to the associative case, m <n implies that $S_m \subseteq S_{n.}$

The set $\mathcal{L}_i(\mathcal{S})$ = $\langle S_i \rangle$is the linear span of the setS_i (the set of all finite linear combinations with coefficients belonging to $\mathbb{F}$). We write $\mathcal{L}_i instead\ of\ \mathcal{L}_i\,(\mathcal{S})$ if $\mathcal{S}$ is determined from the context. It should be noted that $\mathcal{L}_0(\mathcal{S})$=$\langle 1 \rangle$=$\mathbb{F}$ for any $\mathcal{S}$.the set $\mathcal{L}(\mathcal{S})$ stands for $\bigcup_{i=0}^{\infty} \mathcal{L}_i(\mathcal{S})$.

II. Preliminaries

2.1 Remark:$\mathcal{S}$ is a generating set of $\mathcal{A}$ if and only if$\mathcal{A}$=$\mathcal{L}$ ($\mathcal{S}$).

2.2 Definition: The length of a generating set of a finite-dimensional algebra $\mathcal{A}$ is defined as follows$l(\mathcal{S})$=min $\{k \in \mathbb{Z}_+ : \mathcal{L}_k(\mathcal{S}) = \mathcal{A}\}$.

2.3 Definition: The length of an algebra $\mathcal{A}$ is $l(\mathcal{A})$ =max $\{l(\mathcal{S}) : \mathcal{L}(\mathcal{S}) = \mathcal{A}\}$.

2.4 Theorem: Let $\mathbb{F}$be an arbitrary field. Then

$$l(M_n(\mathbb{F})) \leq \left[\frac{n^2+2}{3}\right],$$

Where [.] denotes the least integer function.

2.5 Theorem: Let $\mathbb{F}$ be an arbitrary field.$\mathcal{R}$ be an associative $\mathbb{F}$-algebra, and let

f(d, m) =m$\sqrt{\frac{2d}{m-1}+\frac{1}{4}}+\frac{m}{2}-2$.

Then $l(\mathcal{R})$ <f (dim$\mathcal{R}$, m ($\mathcal{R}$)).

Over an infinite field.

In this paper we obtain a lower bound for the length of arbitrary non-associative algebra and provide an example demonstrating that our bound on the length is sharp. let $\mathcal{A}$ be an $\mathbb{F}$-algebra dim$\mathcal{A}$=n<2.We show that $l(\mathcal{A})\geq 2^{n-2}$ and provide an example of an n-dimensional algebra of the length exactly 2^{n-2}.

2.6 Conjecture: Suppose m, n ∈ ℕ are given such that m< n. Then the following statements are equivalent:

1. $\mathcal{L}_n(\mathcal{S})$ =$\mathcal{L}_m(\mathcal{S})$,
2. $dim\mathcal{L}_n(\mathcal{S})$ = dim $\mathcal{L}_m(\mathcal{S})$.

2.7 Conjecture: Let $\mathcal{A}$ be an algebra and$\mathcal{S}_0$ and $\mathcal{S}_1$ be its finite subsets such that

$\mathcal{L}_1(\mathcal{S}_0)\subseteq\mathcal{L}_1(\mathcal{S}_1)$. Then$\mathcal{L}_k(\mathcal{S}_0)\subseteq\mathcal{L}_k(\mathcal{S}_1)$ for every positive integer k.

2.8 Conjecture: If $\mathcal{A}$ is an algebra,$\mathcal{S}_0$ and $\mathcal{S}_1$ are generating sets such that $\mathcal{L}_1(\mathcal{S}_0)\subset\mathcal{L}_1(\mathcal{S}_1)$, then $l(\mathcal{S}_0)\geq l(\mathcal{S}_1)$.

2.9 Conjecture: If $\mathcal{A}$ is an algebra and $\mathcal{S}_0$ and $\mathcal{S}_1$ are its finite subsets such that $\mathcal{L}_1(\mathcal{S}_0)=\mathcal{L}_1(\mathcal{S}_1)$, then $\mathcal{L}_k(\mathcal{S}_0)$=$\mathcal{L}_k(\mathcal{S}_1)$ for every natural k.

2.10 Conjecture: If$\mathcal{A}$ is an algebra,$\mathcal{S}_0$ is its generating set and $\mathcal{S}_1$ is a finite subset such that$\mathcal{L}_1(\mathcal{S}_0)=\mathcal{L}_1(\mathcal{S}_1)$, then $\mathcal{S}_1$generates $\mathcal{A}$ and $l(\mathcal{S}_0)$= $l(\mathcal{S}_1)$.

2.11 Conjecture: Let us consider a finite subset$\mathcal{S}$of $\mathcal{A}$ and integer n ≥ 1.If dim $\mathcal{L}_n(\mathcal{S})=dim\ \mathcal{L}_{n+1}(\mathcal{S})$ =…dim $\mathcal{L}_{2n}(\mathcal{S})$,

Then for all t∈ ℕ it holds that dim $\mathcal{L}_n(\mathcal{S})=\ dim\ \mathcal{L}_{n+t}(\mathcal{S})$.

2.12 Conjecture :A word ω of the length n from generating set $\mathcal{S}$of algebra $\mathcal{A}$ is a fresh word, if for all integer m, 0 ≤ m < n, it holds that $\omega\notin\mathcal{L}_m(\mathcal{S})$.

2.13 Conjecture:A fresh word of the length greater than 1 isa product of two fresh words of non-zero lengths.

III. Main Result

3.1 Definition: Consider a unital $\mathbb{F}$-algebra $\mathcal{A}$ of the dimension dim $\mathcal{A}$=n, and its generating set $\mathcal{S}$.By the characteristic sequence of$\mathcal{S}$ in$\mathcal{A}$we understand a monotonically non-decreasing sequence of natural numbers (m_0,m_1,…,m_N), constructed by the following rules:

1. m_0=0.

2. Denoting s_1= dim $\mathcal{L}_1(\mathcal{S})$ -1,we define m_1 = …=m_{s1} =1.

3.If $m_{0,\ldots\ldots,}m_r$ are already constructed and the sets $\mathcal{L}_1(\mathcal{S})$,…,$\mathcal{L}_{k-1}(\mathcal{S})$ are considered, then we inductively continue the process in the following way. Denote s_k = dim $\mathcal{L}_k(\mathcal{S})$ −dim $\mathcal{L}_{k-1}(\mathcal{S})$.Then $m_{r+1}=\cdots=m_{r+s_k}$ =k.

3.2 Theorem:

Let $\mathcal{A}$ be a$\mathbb{F}$-algebra, dim$\mathcal{A}$=n<2, and $\mathcal{S}$ be a generating set for $\mathcal{A}$.Then

(i)Positiveinteger k appears in the characteristic sequences as many times as many there are linearly independent fresh words of thelength k.

(ii)For any term m_h of the characteristic sequence of $\mathcal{S}$ there is a fresh word in $\mathcal{L}(\mathcal{S})$ of the length m_h.

(iii) If there is a fresh word in $\mathcal{L}$ $(\mathcal{S})$ of the length k,then k is included into the characteristic sequence of $\mathcal{S}$.

Proof:

- **i)** Fresh words of lengths greater than or equal to k form a basis of $\mathcal{L}_k(\mathcal{S})$, therefore the number of fresh words of the length exactly k is equal to dim $\mathcal{L}_{k-1}(\mathcal{S}) -$ dim $\mathcal{L}_k(\mathcal{S})$.
- **ii)** If follows directly from (i).
- **iii)** It is also follows from the proof of (i).

3.3 Lemma: Let $\mathcal{A}$ be a$\mathbb{F}$-algebra, dim $\mathcal{A}$ =n<2 and $\mathcal{S}$ be the generating set for $\mathcal{A}$.Then the characteristic sequence of $\mathcal{S}$ contains n terms i.e. N= n −1.Moreover , $m_N = l(\mathcal{S})$.

Proof:

By the definition for each k=1, 2,…,.$l(\mathcal{S})$ on kth step we also add (dim$\mathcal{L}_{k-1}(\mathcal{S}) - dim\ \mathcal{L}_k(\mathcal{S})$) in terms to the characteristic sequence. Hence, the total Number of terms is

1 +(1 −dim $\mathcal{L}_1(\mathcal{S})$)+(dim $\mathcal{L}_1(\mathcal{S})$)− $dim\mathcal{L}_2(\mathcal{S})$)……+(dim $\mathcal{L}_{k-1}(\mathcal{S}) -$ dim $\mathcal{L}_k(\mathcal{S})$+……+ ……+(dim $\mathcal{L}_{l(s)-1}(\mathcal{S}) -$ dim $\mathcal{L}_{l(s)}(\mathcal{S})$) = n.

Since $\mathcal{S}$ is a generating set .Also, the maximal of k such that dim$\mathcal{L}_{k-1}(\mathcal{S}) - \mathcal{L}_k(\mathcal{S})$< 0 is $l(\mathcal{S})$ also by definition 3.1 we obtain $m_N = (\mathcal{S})$.

3.4 Theorem:

Let $\mathcal{A}$be a$\mathbb{F}$-algebra dim$\mathcal{A}$= n< 2. Assume,$\mathcal{S}$ is generating set of $\mathcal{A}$,and(m_0 $m_1, m_{2,}$,….m_{n-1}) is a characteristic sequence of $\mathcal{S}$ .Then for any integer k≥ 0it holds that dim $\mathcal{L}_k(\mathcal{S})$=min{t|$m_t$ ≥ k}+1.

Proof:

We use the induction on K. for k=0 the statement is trivial. let us assume that the theorem is true for K=P then for all K=P+1 one has dim$\mathcal{L}_{q+1}(\mathcal{S})$ = dim $\mathcal{L}_q(\mathcal{S})$+(dim$\mathcal{L}_{q+1}(\mathcal{S}) -$ dim $\mathcal{L}_q(\mathcal{S})$).By definition 3.1 the summand is equals to the number N_0 of terms (P+1) in the characteristic sequence. By the induction hypothesis. N_1 =dim $\mathcal{L}_q(\mathcal{S})$.min {t|$m_t$ ≥ P}+1.i.e it is increased by 1 index of the last position in which m_t ≥ P.By definition 3.1 the sum $N_0 + N_1$ equal to the increased by 1 index of the last position in which $m_t \geq P + 1$ or min{t| m_t ≥ P+1}+1.

3.5 Theorem:

Let $\mathcal{A}$ be an$\mathbb{F}$-Algebra dim $\mathcal{A}$= $n < 2$ Assume,$\mathcal{S}$ is a generating set for $\mathcal{A}$ and (m_0 $m_1, m_{2,}$,….m_{n-1})is the characteristic sequence of $\mathcal{S}$.then for each h satisfying $m_h \leq 2$.it holds that there are indices 0 >$t_1 \geq t_2$ > h such that $m_h = m_{t_1} + m_{t_2}$.

Proof:

By theorem 3.2 item(i), each term m_h of the characteristic sequence correspondsto afresh word ofthe Length$m_h$denote it byω_{m_h}.Each fresh word of length $m_h \leq 2$ can be represented as a product of two fresh words, possibly equal, of lesser lengths.Thus,$w_{m_h} = w_{k_1}$ .w_{k_2}.for some fresh wordsw_{k_1},$w_{k_2}$of the lengthsk_1, k_2,correspondingly. Assume $k_1 \leq k_2$,then by theorem 3.2 (iii) there are indices 0 >$t_1 \geq t_2$ > h such that$m_{t_1} = k_1$,$m_{t_2} = k_2$.

Assume $k_1 > k_2$.then by theorem 3.2 (iii) condition there are indices 0 >$t_1 \geq t_2$ > h such that$m_{t_1} = k_2$&$m_{t_2} = k_1$. In both cases the additivity of word length concludes the proof.

3.6 Theorem

Let $\mathcal{A}$be an $\mathbb{F}$-algebra of the dimension dim $\mathcal{A}= n$,n< 2, $\mathcal{S}$ be a generating set for $\mathcal{A}$,$(m_0\ m_1, m_{2,},\ldots.m_{n-1}$) $be\ a$ characteristic sequence of $\mathcal{S}$then for each positive integer h $\geq n-1$ it holds that $m_h \geq 2^{1-h}$.

Proof:

We have to prove this by using induction on h.

If h=1 ,$m_1 \geq 2^{1-1}$

$$m_1 \geq 2^0$$

Let us assume that for all positive integers K such that h$\geq k > n-1$the statement holds.

Now we have to prove it for h= $k+1 \geq n-1$by theorem 3.5 we have $m_{k+1} = m_{t_1} + m_{t_2}$where 0 >$t_1 \geq t_2$ > k+1

According to the induction hypothesis

$$m_{t_1} + m_{t_2} \geq 2^{t_1-1} + 2^{t_2-1} \geq 2^{k-1} + 2^{k-1}$$

$\geq 2(2^{k-1})$

$\geq 2(2^{k})$

This concludes the proof.

3.7 Proposition:

Let $\mathcal{A}$ be an $\mathbb{F}$-Algebra dim$\mathcal{A}= n < 2$ then $l(\mathcal{A}) \geq 2^{n-2}$.

Proof:

Let $\mathcal{S}$ be an arbitrary generating set of $\mathcal{A}$ by lemma 3.3 the length l(s) is equal to the last element of the characteristic sequence of $\mathcal{S}$. The index of this element is dim $\mathcal{A}$-1= $n-1$.henceby theorem 3.6 we get

l(S) $\geq (2^{(n-1)-1} \geq 2^{n-2}$.

Hence proved.

3.8 Example:

Let as consider an arbitrary field $\mathbb{F}$ and non-associative $\mathbb{F}$-Algebra $\mathcal{A}$ of the dimension n < 2 with the basis $\{e_0 = 1, e_1, e_{2,} \ldots\ldots, e_{n-1}\}$ and the following multiplication rules. for every K such that $1 \geq k \ \geq n-2$ we take $e_{k^2} = \ e_{k+1},\ e_{n-1^2}$=0, for all p, q, p$\neq q$, 1$\geq p$ and q$\geq n-1, e_p e_q = 0$.

Since each new fresh word except the first one, is the square of the previous one thus l(S) =2^{n-2}In general l(A)$\geq l(S)$we get l(A)= l(S) = 2^{n-2}.

3.9 Theorem:

Let $\mathcal{A}$ be a locally complex algebra of the dimension, dim $\mathcal{A}$=n, n< 2 Assume that $\mathcal{S}$ is a generating set of $\mathcal{A}$and($m_1, m_{2,},\ldots m_{n-1)}$ is the characteristic sequence of $\mathcal{S}$. Then for each positiveinteger h $\geq n-1$ it holds that$m_h \geq F_h$.

Proof:

We prove this by induction on h

If h=1, then m_1 =1≥ F_1

Assume that for all positive integers h= 1,.k the statement holds we have to prove it now for h=k+1 $\geq n+1$ Also we know that $m_{k+1} = m_{t_1}+ m_{t_2}$where 0 >$t_1 \geq t_2$ > k +1 According to the induction hypothesis

$m_{t_1} + m_{t_2} \geq F_{t_1} + F_{t_2} \geq F_k + F_{k-1}= F_{k+1}$

Thisconcludes the proof.

3.10 Theorem:

Let $\mathcal{A}$ be a locally complex algebra of the dimension, dim $\mathcal{A}$= n<2.then the length of $\mathcal{A}$ is greater than or equal to the $(n-1)^{th}$Fibonacci numberF_{n-1}.

Proof:

Let $\mathcal{S}$ be an arbitrary generating set of $\mathcal{A}$, and($m_1, m_{2,},\ldots m_{n-1)}$be its characteristic sequence. By theorem 3.9 l(S)=$m_{n-1} \geq F_{n-1}$ $then\ len$gth of $\mathcal{A}$ is greater than or equal to F_{n-1}.

3.11 Theorem:

If $\mathcal{A}$ is a locally complex algebra of the dimension dim $\mathcal{A}$=n, $\mathcal{S}$ be the generating set containing Klinearly independent modulo R elements then l(S) $\leq F_{n-k+1.}$

Proof:

Let($m_1, m_{2,},\ldots m_{n-1}$) be the characteristic sequence of $\mathcal{S}$. It should be noted that $m_1 = m_{2=\ldots}m_k = 1$.Since dim $\mathcal{L}_0(S)$ − dim $\mathcal{L}_{1\,(S)} = k$ we use the induction to prove that$m_{k+h} \geq F_{n+2}$for all integer h, -1 ≤ $h \geq n-k-$ 1. for h= -1 and h=0 one has$m_{k-1} = m_k = 1 = F_1 = F_{2.}$let us assume that for h=d+1≥ $n-k-1$.we know that m_{k+d+1}=$m_{t_1} + m_{t_2}$ where 0 >$t_1 > t_2$ > k +d+1 According to the induction hypothesis .

$m_{t_1} + m_{t_2} \geq F_{t_{1-k+2}}$+$F_{t_{2-k+2}} \geq F_{d+1}$+$F_{d+2}$= F_{d+3}

This concludes the proof.

3.12 Example:

Let us consider locally -complex algebra $\mathcal{A}$ over real number with basis $\{e_0 = 1, \ldots . e_{n-1}\}$ andfollowing multiplication rule for every k, such that 1≥ $k \geq n-3$

$e_k e_{k+1}$=e_{k+2}

$e_{k+1} e_k$= $-e_{k+2}$

For every m, such that 1 ≥m≥n-1

$e_m e_m$= -1and for other combination of p, q: 1≥p, q≥n-1

$e_p e_q$=0

The set $\mathcal{S}$={$e_1 e_2$} generates$\mathcal{A}$,and its characteristic sequence is exactly (0,1,1,2,…F_{n-1}),since every fresh word is obtained as a product of two previous fresh words. We getF_{n-1} =l(S)≥ $l(A) \geq F_{n-1}$,which means l(A)=$F_{n-1.}$

IV. Conclusion

In the present paper, A lower bound for the length of non-associative algebra in term of a function of two invariant of the algebra, the dimension and the maximal degree of the minimal polynomial for the element of the algebra, is obtained. As a conjecture, a formula for the length of the algebra of diagonal matrices over an arbitrary field obtained.

References

[1] M.Bresar, P.Semrl, S.Spenko, On locally-complex algebras and low-dimensional Cayley-Dickson algebras, Journal of Alegbra,327(2011) 107-125

[2] A.E.Guterman, D.K.Kudryavtsev, General method to compute the lengths of non-associative algebras, Preprint.

[3] A.E.Guterman, O.V.Markova, M.A.Khrystik, On the lengths of group algebras of finite Abelian groups in the semi-simple case.

[4] A.E.Guterman,O.V.Markova, The length of group algebras of small order groups, Journal of Mathematical Sciences(New York), (2019) to appear.