Approximation of Phillips-operators on new parameters and via Dunkl analogue

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Abstract

The present article is to study the convergence properties of Lebesgue measurable functions based on Dunkl Analogue. We construct a new modified version of Phillips-operators by introducing the new parameters α, β and υ. To obtain the results of uniform convergence in a better way of the Phillips-operators we calculate the qualitative results in a Korovkin and weighted Korovkin spaces.

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1. Introduction and preliminaries

The Bernstein operators[3] preserve a simpler nd a constructive way to prove the First Weierstrass approximation theorem on the interval $[0, 1]$ defined by S.N. Bernstein in 1912. In 1950 O. Szász ([21])was able to gave an extension type positive linear operators on interval $[0, \infty)$ known as Szász operators. In recent years, many results about the generalization of Szász type operators have been obtained by several mathematicians (see [6], [7], [10], [11]). These Dunkl types generalizations is a very recent and crucial work which plays an important role in the field of approximation theory. Furthermore, basic definitions and properties of approximation theory can be found in ([5], [12], [13], [18]) and ([19], [23]). The Dunkl type generalizations is obtained by Sucu [22] by proposed a exponential generalization of the function given by [20]. For the set of all continuous function $f \in C[0, \infty)$ defined for all $x \in [0, \infty)$, $\upsilon \geq 0$ and $n \in \mathbb{N}$ the Dunkl type construction of Szász operators given as:

$$\mathcal{S}_n^*(f; x) := \frac{1}{e_\upsilon(nx)} \sum_{s=0}^{\infty} \frac{(nx)^s}{\gamma_\upsilon(s)} f\left(\frac{s + 2\upsilon\theta_s}{n}\right), \tag{1.1}$$

where

$$e_\upsilon(x) = \sum_{s=0}^{\infty} \frac{x^s}{\gamma_\upsilon(s)}. \tag{1.2}$$

$$\gamma_\upsilon(2s) = \frac{2^{2s}s!\Gamma\left(s+\upsilon+\frac{1}{2}\right)}{\Gamma\left(\upsilon+\frac{1}{2}\right)}, \quad \gamma_\upsilon(2s+1) = \frac{2^{2s+1}s!\Gamma\left(s+\upsilon+\frac{3}{2}\right)}{\Gamma\left(\upsilon+\frac{1}{2}\right)}. \tag{1.3}$$

This type of generalization by exponential function was introduced and is a generalization of in Hermite type polynomials, expressed in form of the confluent hypergeometric function Φ (see [20]). For $s = 0, 1, 2, \ldots$ a recursion for γ_υ

$$\frac{\gamma_\upsilon(s+1)}{(s+1+2\upsilon\theta_{s+1})} = \gamma_\upsilon(s),$$

$$\theta_s = \begin{cases} 0 & \text{if } s = 2n,\ n = 0, 1, 2\cdots \\ 1 & \text{if } s = 2n+1,\ n = 0, 1, 2\cdots. \end{cases} \tag{1.4}$$

Moreover, in the recent years in the field of approximation theory the (p, q)-generalization of Bernstein operators have obtained by Mursaleen et., all [9] and various results on (p, q) approximation have obtained by authors (see [16], [1], [17], [8], [14]).

The main Ideas of our research article to extend the recent work of [15] by obtaining the degrees of approximation of Stancu type generalization of Phillips operators in the form the modulus of continuity, Lipschitz functions, Peetre's K-functional and second order modulus of continuity. The approximation obtained in present article designed by Dunkl type modification on Stancu type operators and which provides a better way to approximate the operators depends upon the parameters $\upsilon \geq 0$ and $0 \leq \alpha \leq \beta$. For any $f_i \in f$, in the recent work defined by [15] for a Phillips operators $\mathcal{P}^*_{n,\upsilon}(\cdot\,;\cdot)$ via Dunkl generalization the moments estimated such as:

$$\mathcal{P}^*_{n,\upsilon}(f_i; x) = \begin{cases} 1 & \text{if } i = 1, \\ x + \frac{1}{n}, & \text{if } i = 2, \\ \frac{2}{n^2} + \frac{2}{n}\Big(2 + \upsilon\chi_{n,\upsilon}(x)\Big)x + x^2, & \text{if } i = 3, \\ \frac{6}{n^3} + \frac{2}{n^2}\Big(9 + 2\upsilon + 8\upsilon\chi_{n,\upsilon}(x)\Big)x + \frac{1}{n}\left(9 - 2\upsilon\chi_{n,\upsilon}(x)\right)x^2 + x^3, & \text{if } i = 4, \\ \frac{24}{n^4} + \frac{2}{n^3}\Big(63 + 26\upsilon^2 + 2\upsilon(29 + 2\upsilon^2)\chi_{n,\upsilon}(x)\Big)x & \\ + \frac{4}{n^2}\Big(18 + \upsilon^2 - 7\upsilon\chi_{n,\upsilon}(x)\Big)x^2 + \frac{4}{n}\Big(4 + \upsilon\chi_{n,\upsilon}(x)\Big)x^3 + x^4 & \text{if } i = 4, \end{cases} \tag{1.5}$$

where $f_i = t^{i-1}$ for $i = 1, 2, 3, 4, 5$ and $\chi_{n,\upsilon}(x) = \frac{e_\upsilon(-nx)}{e_\upsilon(nx)}$.

2. Construction of operators and their associated moments

In this section we introduce the Stancu type generalization to the operators defined by [15] for obtaining a better approximation and a flexibility in the results by their Dunkl generalization. Let $C[0, \infty)$ be the set of all continuous functions

on $[0,\infty)$, then for all $x \in [0,\infty)$ and every $f \in C_{\varrho}[0,\infty) = \{f \in C[0,\infty) : f(t) = O(t^{\varrho}),\}$ as $t \to \infty$, and $\varrho > n,\ n \in \mathbb{N}$, we define

$$\mathcal{S}_n^{\alpha,\beta}(f;x) = \frac{n^2}{e_{\upsilon}(nx)} \sum_{s=0}^{\infty} \frac{(nx)^s}{\gamma_{\upsilon}(s)} \int_0^{\infty} \frac{e^{-nt} n^{s+2\upsilon\theta_s-1} t^{s+2\upsilon\theta_s}}{\gamma_{\upsilon}(s)} f\left(\frac{nt+\alpha}{n+\beta}\right) \mathrm{d}t, \tag{2.1}$$

where $0 \le \alpha \le \beta$ and the parameter $\upsilon \ge 0$, and θ_s defined in (1.4). It should be noted that if we take $\alpha = \beta = 0$ in (2.1), then the operators $\mathcal{S}_n^{\alpha,\beta}$ reduce to operators defined by [15]

Lemma 2.1. *Let $f_i = t^{i-1}$ for $i = 1,2,3,4,5$ and $\mathcal{P}_n^*(\,\cdot\,;\,\cdot\,)$ defined by (1.5). Then the operators $\mathcal{S}_n^{\alpha,\beta}(\,\cdot\,;\,\cdot\,)$ satisfying the following estimates:*

$$\begin{aligned}
(1)^* \quad \mathcal{S}_n^{\alpha,\beta}(f_1;x) &= 1,\\
(2)^* \quad \mathcal{S}_n^{\alpha,\beta}(f_2;x) &= \frac{n}{n+\beta}\mathcal{P}_n^*(f_2;x) + \frac{\alpha}{n+\beta},\\
(3)^* \quad \mathcal{S}_n^{\alpha,\beta}(f_3;x) &= \frac{n^2}{(n+\beta)^2}\mathcal{P}_n^*(f_3;x) + \frac{2n\alpha}{(n+\beta)^2}\mathcal{P}_n^*(f_2;x) + \frac{\alpha^2}{(n+\beta)^2},\\
(4)^* \quad \mathcal{S}_n^{\alpha,\beta}(f_4;x) &= \frac{n^3}{(n+\beta)^3}\mathcal{P}_n^*(f_4;x) + \frac{3n^2\alpha}{(n+\beta)^3}\mathcal{P}_n^*(f_3;x) + \frac{3n\alpha^2}{(n+\beta)^3}\mathcal{P}_n^*(f_2;x) + \frac{\alpha^3}{(n+\beta)^3},\\
(5)^* \quad \mathcal{S}_n^{\alpha,\beta}(f_5;x) &= \frac{n^4}{(n+\beta)^4}\mathcal{P}_n^*(f_5;x) + \frac{4n^3\alpha}{(n+\beta)^4}\mathcal{P}_n^*(f_4;x) + \frac{6n^2\alpha^2}{(n+\beta)^4}\mathcal{P}_n^*(f_3;x)\\
&+ \frac{4n\alpha^3}{(n+\beta)^4}\mathcal{P}_n^*(f_2;x) + \frac{\alpha^4}{(n+\beta)^4}.
\end{aligned}$$

Proof. Take $f = f_1$, then, $\mathcal{S}_n^{\alpha,\beta}(f_1;x) = \mathcal{P}_{n,\upsilon}^*(1;x) = 1$. If $f = f_2$, then $\mathcal{S}_n^{\alpha,\beta}(f_2;x) = \frac{n}{n+\beta}\mathcal{P}_{n,\upsilon}^*(t;x) + \frac{\alpha}{n+\beta}$. In case of $f = f_3$, we have $\mathcal{S}_n^{\alpha,\beta}(f_3;x) = \frac{n^2}{(n+\beta)^2}\mathcal{P}_{n,\upsilon}^*(t^2;x) + \frac{2n\alpha}{(n+\beta)^2}\mathcal{P}_{n,\upsilon}^*(t;x) + \frac{\alpha^2}{(n+\beta)^2}$. Other results also easily can be obtained. □

Lemma 2.2. *Suppose $g_i = (t-x)^i$ for $i = 1,2,3,4$. Then, for $x \in [0,\infty)$ the operators $\mathcal{S}_n^{\alpha,\beta}(\,\cdot\,;\,\cdot\,)$ defined by (2.1) satisfying the following identities:*

$$
\begin{aligned}
(1)^\circ \quad \mathcal{S}_n^{\alpha,\beta}(g_1;x) &= \left(\frac{n}{n+\beta}-1\right)x+\frac{1}{(n+\beta)}(1+\alpha),\\
(2)^\circ \quad \mathcal{S}_n^{\alpha,\beta}(g_2;x) &= \left(\frac{n^2}{(n+\beta)^2}-\frac{2n}{n+\beta}+1\right)x^2\\
&+ \left[\frac{2n}{(n+\beta)^2}\left(2+\alpha+\upsilon\chi_{n,\upsilon}(x)\right)-\frac{2}{n+\beta}(1+\alpha)\right]x\\
&+ \frac{1}{(n+\beta)^2}\left(2+2\alpha+\alpha^2\right),\\
(3)^\circ \quad \mathcal{S}_n^{\alpha,\beta}(g_4;x) &= \left[\frac{n^4}{(n+\beta)^4}-\frac{4n^3}{(n+\beta)^3}+\frac{6n^2}{(n+\beta)^2}-\frac{4n}{n+\beta}+1\right]x^4\\
&+ \left[\frac{4n^3}{(n+\beta)^4}\left(4+\alpha+4\upsilon\chi_{n,\upsilon}(x)\right)\right.\\
&- \frac{4n^2}{(n+\beta)^3}\left(9+3\alpha-2\upsilon\chi_{n,\upsilon}(x)\right)\\
&+ \left.\frac{12n}{(n+\beta)^2}\left(2+\alpha+2\upsilon\chi_{n,\upsilon}(x)\right)-\frac{4\alpha}{n+\beta}\right]x^3\\
&+ \left[\frac{2n^2}{2(n+\beta)^4}\left(2\alpha(9-2\upsilon\chi_{n,\upsilon}(x))+36\upsilon^2-14\upsilon\chi_{n,\upsilon}(x)+3\alpha^2\right)\right.\\
&- \frac{4n}{(n+\beta)^3}\left(18+4\upsilon+16\upsilon\chi_{n,\upsilon}(x)+6\alpha(2+\upsilon\chi_{n,\upsilon}(x))+3\alpha^2\right)\\
&+ \left.\frac{6}{(n+\beta)^2}(2\alpha+1)\right]x^2\\
&+ \left[\frac{2n}{2(n+\beta)^4}\left(631+26\upsilon^2+2\upsilon(29+2\upsilon^2\chi_{n,\upsilon}(x))\right.\right.\\
&+ \left.\left.4\alpha(9+2\upsilon+8\upsilon\chi_{n,\upsilon}(x))+2\alpha^3+6\alpha^2(2+\upsilon\chi_{n,\upsilon}(x))\right)\right.\\
&- \left.\frac{4}{(n+\beta)^3}\left(6+6\alpha+3\alpha^2+\alpha^3\right)\right]x\\
&+ \frac{}{(n+\beta)^4}\left(24+24\alpha+12\alpha^2+4\alpha^3+\alpha^4\right).
\end{aligned}
$$

3. Convergence in Korovkin and weighted Korovkin space

The Korovkin and weighted Korovkin spaces is a purely emphasize on some newly results based on logical concepts and mathematical experiences, which plays a crucial role in the approximation theory. In the last quarter of 20th century and in 21th century the Korovkin's type approximation theorem has many useful connections between the classical approximation theory and other branches of mathematics.

In the present article, we suppose $C_B[0,\infty)$ defined such as the set of all bounded and continuous functions on the interval $[0,\infty)$ and a linear normed

space with the supremum norm

$$\| f \|_{C_{B[0,\infty)}} = \sup_{x\in[0,\infty)} | f(x) | .$$

Let

$$\mathfrak{E}_f = \{f(x) : x \in [0,\infty)\},$$

for which the function $\frac{f(x)}{1+x^2}$ is uniformly convergent as approaches to ∞.

Theorem 3.1. *Let the function $f \in C[0,\infty) \cap \mathfrak{E}_f$ and the operators $\mathcal{S}_n^{\alpha,\beta}(\,\cdot\,;\,\cdot\,)$ defined by (2.1). Then*

$$\lim_{n\to\infty} \mathcal{S}_n^{\alpha,\beta}(f;x) = f(x)$$

is uniformly on each compact subset of $[0,\infty)$.

Proof. The well-known Korovkin's theorem is very useful to approximates the operators $\mathcal{S}_n^{\alpha,\beta}$ uniformly to x^{j-1} for $j = 1,2,3$ as $n \to \infty$. Therefore, for $j = 1,2,3$ $\lim_{n\to\infty} \mathcal{S}_n^{\alpha,\beta}(f_j;x) \to x^{j-1}$ is uniformly convergent on $[0,\infty)$. Clearly, $\lim_{n\to\infty} \mathcal{S}_n^{\alpha,\beta}(f_1;x) = 1$, and $n \to \infty$, $\frac{1}{n} \to 0$, thus obvious we conclude that

$$\lim_{n\to\infty} \mathcal{S}_n^{\alpha,\beta}(f_2;x) = x, \ \lim_{n\to\infty} \mathcal{S}_n^{\alpha,\beta}(f_3;x) = x^2.$$

Which complete the proof. □

To obtain the best approximation to the operators we take the weight function $\varrho(x) = 1 + x^2$ is and for all $x \in [0,\infty)$, $f \in C[0,\infty)$ we suppose $\mathfrak{A}_\varrho(x)$ and $\mathfrak{B}_\varrho(x)$ defined in weighted spaces such as

$$\begin{aligned}
\mathfrak{A}_\varrho(x) &= \{f :| f(x) |\leq \mathfrak{C}_f \varrho(x)\}, \\
\mathfrak{B}_\varrho(x) &= \{f : f \in \mathfrak{A}_\varrho(x) \cap C[0,\infty)\}, \\
\mathfrak{B}_\varrho^\eta(x)\big|_{x\in[0,\infty)} &= \left\{f : f \in \mathfrak{B}_\varrho(x) \text{ and } \lim_{x\to\infty} \frac{f(x)}{\varrho(x)} = \eta\right\},
\end{aligned}$$

where $\mathfrak{C}_f$ is constant depends on f. It should be noted that, $\mathfrak{B}_\varrho(x)$ is a normed space defined with the norm of $\| f \|_\varrho = \sup_{x\in[0,\infty)} \frac{|f(x)|}{\varrho(x)}$.

Theorem 3.2. *For all $f \in \mathfrak{B}_\varrho(x)$ and $x \in [0,\infty)$ the operators $\mathcal{S}_n^{\alpha,\beta}$ satisfying*

$$\lim_{n\to\infty} \| \mathcal{S}_n^{\alpha,\beta}(f;x) - f \|_\varrho = 0.$$

Proof. Suppose for all $x \in [0,\infty)$, $f \in \mathfrak{B}_\varrho(x)$. We take $f = f_\tau = t^{\tau-1}$. Then by the Korovkin's theorem if it satisfies $\mathcal{S}_n^{\alpha,\beta}(f_\tau;x) \to x^{\tau-1}$, for $\tau = 1,2,3$ uniformly, whenever $n \to \infty$. Then from the Lemma 2.1 for $\tau = 1$, we get

$$\left|\left|\mathcal{S}_n^{\alpha,\beta}(f_1;x) - 1\right|\right|_\varrho = 0. \tag{3.1}$$

For $\tau = 2$, we have

$$\begin{aligned}
&\left|\left|\mathcal{S}_n^{\alpha,\beta}\left(f_2;x\right)-x\right|\right|_{\varrho} \\
&= \sup_{x\geqq 0}\frac{\left|\mathcal{S}_n^{\alpha,\beta}(f_2;x)-x\right|}{1+x^2} \\
&= \sup_{x\geqq 0}\frac{\left|\frac{n}{n+\beta}\mathcal{P}^*_{n,\upsilon}(f_2;x)-x+\frac{\alpha}{n+\beta}\right|}{1+x^2} \\
&\leq \left(\frac{n}{n+\beta}-1\right)\sup_{x\geqq 0}\frac{x}{1+x^2}+\frac{1+\alpha}{n+\beta}\sup_{x\geqq 0}\frac{1}{1+x^2}.
\end{aligned}$$

As, $n \to \infty$, then

$$\left|\left|\mathcal{S}_n^{\alpha,\beta}\left(f_2;x\right)-x\right|\right|_{\varrho} = 0. \tag{3.2}$$

In similar way if take $\tau = 3$ we get

$$\begin{aligned}
&\left|\left|\mathcal{S}_n^{\alpha,\beta}\left(f_3;x\right)-x^2\right|\right|_{\varrho} \\
&= \sup_{x\geqq 0}\frac{\left|\mathcal{S}_n^{\alpha,\beta}(f_3;x)-x^2\right|}{1+x^2} \\
&= \sup_{x\geqq 0}\frac{\left|\frac{n^2}{(n+\beta)^2}\mathcal{P}^*_{n,\upsilon}(f_3;x)+\frac{2\alpha n}{(n+\beta)^2}\mathcal{P}^*_{n,\upsilon}(f_2;x)+\frac{\alpha^2}{(n+\beta)^2}-x^2\right|}{1+x^2} \\
&\leqq \left(\frac{n^2}{(n+\beta)^2}-1\right)\sup_{x\geqq 0}\frac{x^2}{1+x^2}+\frac{2n}{(n+\beta)^2}\left(\alpha+2+\upsilon\chi_{n,\upsilon}(x)\right)\sup_{x\geqq 0}\frac{x}{1+x^2} \\
&+\frac{1}{(n+\beta)^2}\left(2+2\alpha+\alpha^2\right)\sup_{x\geqq 0}\frac{1}{1+x^2},
\end{aligned}$$

whenever, $n \to \infty$,

$$\left|\left|\mathcal{S}_n^{\alpha,\beta}\left(f_3;x\right)-x^2\right|\right|_{\varrho} = 0. \tag{3.3}$$

Which completes the proof. □

4. Order of approximation and rate of convergence $\mathcal{S}_n^{\alpha,\beta}$

In the present section we obtain the order of approximation with the modulus of continuity of the function $f \in C^*[0,\infty)$ which are enabled to give a maximum oscillation of f, where $C^*[0,\infty)$ is the set of all uniformly continuous functions on $[0,\infty)$. Here also to obtain the rate of convergence of the operators we use the usual class of Lipschitz functions. Therefore let for $\mathfrak{H}_f = \{f \mid f \in C^*[0,\infty)$, and for every $f \in C^*[0,\infty)$, suppose $\omega^*(f;\delta^*)$ denotes the modulus of continuity which gives the maximum oscillation of f on $\delta^* > 0$. For $x_1, x_2 \in [0,\infty)$ one given

$$\omega^*(f;\delta^*) = \sup_{|x_1-x_2|\leq\delta^*} \mid f(x_1)-f(x_2) \mid . \tag{4.1}$$

Note that, for $f \in C^*[0,\infty)$ and $\delta^* > 0$ one has $\lim_{\delta^*\to 0^+} \omega^*(f;\delta^*) = 0$,

$$| f(x_1) - f(x_2) | \leq \left(1 + \frac{| x_1 - x_2 |}{\delta^*}\right) \omega^*(f;\delta^*). \tag{4.2}$$

Theorem 4.1. *Let $f \in \mathfrak{H}_f$, then for all $x \in [0,\infty)$ and the operators $\mathcal{S}_n^{\alpha,\beta}(\cdot;\cdot)$ defined by (2.1) satisfying*

$$| \mathcal{S}_n^{\alpha,\beta}(f;x) - f(x) | \leq 2\omega^* \left(f; \sqrt{\delta_n^*(x)}\right).$$

Proof. In the view of the results(4.1), (4.2) and well-known Cauchy-Schwarz inequality, we conclude that
$| \mathcal{S}_n^{\alpha,\beta}(f;x) - f(x) |$

$$\begin{aligned}
&\leq \frac{n^2}{e_\upsilon(nx)} \sum_{s=0}^{\infty} \frac{(nx)^s}{\gamma_\upsilon(s)} \int_0^\infty \frac{e^{-nt} n^{s+2\upsilon\theta_s-1} t^{s+2\upsilon\theta_s}}{\gamma_\upsilon(s)} | f(t) - f(x) | \, \mathrm{d}t, \\
&\leq \frac{n^2}{e_\upsilon(nx)} \sum_{s=0}^{\infty} \frac{(nx)^s}{\gamma_\upsilon(s)} \int_0^\infty \frac{e^{-nt} n^{s+2\upsilon\theta_s-1} t^{s+2\upsilon\theta_s}}{\gamma_\upsilon(s)} \left(1 + \frac{1}{\delta^*} | t - x |\right) \omega^*(f;\delta^*) \, \mathrm{d}t \\
&= \left\{1 + \frac{1}{\tilde{\delta}} \left(\frac{n^2}{e_\upsilon(nx)} \sum_{s=0}^{\infty} \frac{(nx)^s}{\gamma_\upsilon(s)} \int_0^\infty \frac{e^{-nt} n^{s+2\upsilon\theta_s-1} t^{s+2\upsilon\theta_s}}{\gamma_\upsilon(s)} | t - x | \, \mathrm{d}t\right)\right\} \omega^*(f;\delta^*) \\
&\leq \left\{1 + \frac{1}{\delta^*} \left(\frac{n^2}{e_\upsilon(nx)} \sum_{s=0}^{\infty} \frac{(nx)^s}{\gamma_\upsilon(s)} \int_0^\infty \frac{e^{-nt} n^{s+2\upsilon\theta_s-1} t^{s+2\upsilon\theta_s}}{\gamma_\upsilon(s)} (t-x)^2 \mathrm{d}t\right)^{\frac{1}{2}}\right\} \omega^*(f;\delta^*) \\
&= \omega^*(f;\delta^*) + \frac{1}{\delta^*} \left(\mathcal{S}_n^{\alpha,\beta}(g_2;x)\right)^{\frac{1}{2}} \omega^*(f;\delta^*).
\end{aligned}$$

Choose $\delta^* = \sqrt{\mathcal{S}_n^{\alpha,\beta}(g_2;x)} = \sqrt{\delta_n^*(x)}$, then we get our result. □

For the class of Lipschitz functions to obtain the rate of convergence of the operators $\mathcal{S}_n^{\alpha,\beta}(f;x)$, we let $Lip_{\mathcal{M},\zeta}(f)$ be the class of all Lipschitz functions in which $f \in C[0,\infty), \mathcal{M} > 0$ and $0 < \zeta \leq 1$ given as

$$Lip_{\mathcal{M},\zeta}(f) = \left\{f :| f(\tau_1) - f(\tau_2) | \leq \mathcal{M} | \varsigma_1 - \tau_2 |^\zeta; \ (\tau_1, \tau_2 \in [0,\infty))\right\} \tag{4.3}$$

Theorem 4.2. *Let $\mathcal{M} > 0$ and $0 < \zeta \leq 1$, then for every $f \in Lip_{\mathcal{M},\zeta}$, we have*

$$| \mathcal{S}_n^{\alpha,\beta}(f;x) - f(x) | \leq \mathcal{M} \left(\delta_n^*(x)\right)^{\frac{\zeta}{2}}.$$

Proof. In the light of the well-known Hölder inequality and (4.3), we get

$$\begin{aligned}
| \mathcal{S}_n^{\alpha,\beta}(f;x) - f(x) | &\leq | \mathcal{S}_n^{\alpha,\beta}(f(t) - f(x);x) | \\
&\leq \mathcal{S}_n^{\alpha,\beta}(| f(t) - f(x) |;x) \\
&\leq \mathcal{M}\mathcal{S}_n^{\alpha,\beta}\left(| t - x |^\zeta;x\right).
\end{aligned}$$

Therefore,

$$
\begin{aligned}
&\mid \mathcal{S}_n^{\alpha,\beta}(f;x) - f(x) \mid \\
\leq\ & \mathcal{M}\frac{n^2}{e_v(nx)}\sum_{s=0}^{\infty}\frac{(nx)^s}{\gamma_v(s)}\int_0^{\infty}\frac{e^{-nt}n^{s+2v\theta_s-1}t^{s+2v\theta_s}}{\gamma_v(s)} \mid t-x \mid \mathrm{d}t \\
\leq\ & \mathcal{M}\frac{n^2}{e_v(nx)}\sum_{s=0}^{\infty}\left(\frac{(nx)^s}{\gamma_v(s)}\right)^{\frac{2-\zeta}{2}}\left(\frac{(nx)^s}{\gamma_v(s)}\right)^{\frac{\zeta}{2}}\int_0^{\infty}\frac{e^{-nt}n^{s+2v\theta_s-1}t^{s+2v\theta_s}}{\gamma_v(s)} \mid t-x \mid \mathrm{d}t \\
\leq\ & \mathcal{M}\left(\frac{n^2}{e_v(nx)}\sum_{s=0}^{\infty}\frac{(nx)^s}{\gamma_v(s)}\int_0^{\infty}\frac{e^{-nt}n^{s+2v\theta_s-1}t^{s+2v\theta_s}}{\gamma_v(s)}\mathrm{d}t\right)^{\frac{2-\zeta}{2}} \\
\times\ & \left(\frac{n^2}{e_v(nx)}\sum_{s=0}^{\infty}\frac{(nx)^s}{\gamma_v(s)}\int_0^{\infty}\frac{e^{-nt}n^{s+2v\theta_s-1}t^{s+2v\theta_s}}{\gamma_v(s)} \mid t-x \mid^2 \mathrm{d}t\right)^{\frac{\zeta}{2}} \\
=\ & \mathcal{M}\left(\mathcal{S}_n^{\alpha,\beta}(t-x)^2;x\right)^{\frac{\zeta}{2}} = \mathcal{M}\left(\mathcal{S}_n^{\alpha,\beta}(g_2;x)\right)^{\frac{\zeta}{2}}.
\end{aligned}
$$

Which completes the proof. □

In order to get the convergence of the operators $\mathcal{S}_n^{\alpha,\beta}$ for every $\varphi \in C_B^2[0,\infty)$, one has

$$C_B^2[0,\infty) = \{\varphi \in C_B[0,\infty) : \varphi', \varphi'' \in C_B[0,\infty)\}, \tag{4.4}$$

where $C_B[0,\infty)$ is the space of all bounded and continuous functions on the interval $[0,\infty)$ and the normed defined on $C_B^2[0,\infty)$ such as

$$\| \varphi \|_{C_B^2[0,\infty)} = \| \varphi'' \|_{C_B[0,\infty)} + \| \varphi' \|_{C_B[0,\infty)} + \| \varphi \|_{C_B[0,\infty)}, \tag{4.5}$$

where the norm defined on $C_B[0,\infty)$,

$$\| \varphi \|_{C_B[0,\infty)} = \sup_{x\in[0,\infty)} \mid \varphi(x) \mid . \tag{4.6}$$

Theorem 4.3. *For all $x \in [0,\infty)$ and every $\varphi \in C_B^2[0,\infty)$ the operators $\mathcal{S}_n^{\alpha,\beta}(\,\cdot\,;\,\cdot\,)$ satisfying*

$$\mid \mathcal{S}_n^{\alpha,\beta}(\varphi;x) - \varphi(x) \mid \leq \left(\mathcal{S}_n^{\alpha,\beta}(g_1;x) + \frac{\mathcal{S}_n^{\alpha,\beta}(g_2;x)}{2}\right) \| \varphi \|_{C_B^2[0,\infty)},$$

where $\mathcal{S}_n^{\alpha,\beta}(g_1;x)$ and $\mathcal{S}_n^{\alpha,\beta}(g_2;x)$ are defined by Lemma 2.2.

Proof. Let $\varphi \in C_B^2[0,\infty)$. From expansion of the Taylor series generalized mean value theorem we have

$$\varphi(t) = \varphi(x) + (t-x)\varphi'(x) + \frac{(t-x)^2}{2}\varphi''(\psi)\frac{(t-x)^2}{2},\ \psi \in (x,t).$$

A small calculation leads to linearity on $\mathcal{S}_n^{\alpha,\beta}$, we have

$$\mathcal{S}_n^{\alpha,\beta}(\varphi;x) - \varphi(x) = \varphi'(x)\mathcal{S}_n^{\alpha,\beta}\left((t-x);x\right) + \frac{\varphi''(\psi)}{2}\mathcal{S}_n^{\alpha,\beta}\left((t-x)^2;x\right),$$

which imply that

$$| \mathcal{S}_n^{\alpha,\beta}(\varphi;x) - \varphi(x) | \leq \left(\mathcal{S}_n^{\alpha,\beta}(g_1;x)\right) \| \varphi' \|_{C_B[0,\infty)} + \left(\mathcal{S}_n^{\alpha,\beta}(g_2;x)\right) \frac{\| \varphi'' \|_{C_B[0,\infty)}}{2}.$$

From (4.5) it is obvious that $\| \varphi' \|_{C_B[0,\infty)} \leq \| \varphi \|_{C_B^2[0,\infty)}$, hence
$| \mathcal{S}_n^{\alpha,\beta}(\varphi;x) - \varphi(x) |$

$$| \mathcal{S}_n^{\alpha,\beta}(\varphi;x) - \varphi(x) | \leq \left(\mathcal{S}_n^{\alpha,\beta}(g_1;x)\right) \| \varphi \|_{C_B^2[0,\infty)} + \left(\mathcal{S}_n^{\alpha,\beta}(g_2;x)\right) \frac{\| \varphi \|_{C_B^2[0,\infty)}}{2}.$$

□

5. Some Direct theorems of $\mathcal{S}_n^{\alpha,\beta}$

A potential influences work to obtain a well-known functional known as Peetres K-functional, given by J. Peetre in 1968. Their conflict of interest for K-functional to investigate the interpolation spaces between two Banach spaces and an interactions to the real interpolation is based on K-functional.

This well-known functional property defined by Peetre's, which is known as K-functional defined as

$$\mathcal{K}_2(f;\breve{\delta}) = \inf_{C_B^2[0,\infty)} \left\{ \left(\| f - \varphi \|_{C_B([0,\infty))} + \breve{\delta} \| \varphi \|_{C_B^2[0,\infty)} \right) : \varphi \in C_B^2[0,\infty) \right\}, \tag{5.1}$$

For any $\breve{\delta} > 0$, a positive constant $\mathcal{D} > 0$ exits such that $\mathcal{K}_2(f;\breve{\delta}) \leq \mathcal{D}\omega_2(f;\breve{\delta}^{\frac{1}{2}})$, where the second order modulus of continuity $\omega_2(f;\cdot)$ is given by

$$\omega_2(f;\breve{\delta}^{\frac{1}{2}}) = \sup_{0<h<\breve{\delta}^{\frac{1}{2}}} \sup_{t\in[0,\infty)} | f(t+2h) - 2f(t+h) + f(t) | . \tag{5.2}$$

Theorem 5.1. *Suppose the operators $\mathcal{S}_n^{\alpha,\beta}(\cdot;\cdot)$ satisfying the properties (5.1) and (5.2), then for every $f \in C_B^2[0,\infty)$ we have*

$$| \mathcal{S}_n^{\alpha,\beta}(f;x) - f(x) | \leq 2\mathcal{B}\left\{ \omega_2\left(f;\sqrt{\Phi_n(x)}\right) + \min\left(1,\Phi_n(x)\right) \| f \|_{C_B[0,\infty)} \right\},$$

where $\Phi_n(x) = \frac{2\mathcal{S}_n^{\alpha,\beta}(g_1;x)+\mathcal{S}_n^{\alpha,\beta}(g_2;x)}{4}$, $\omega_2(f;\breve{\delta})$ defined in (5.2) and $\mathcal{B}$ is a non-negative constant.

Proof. We use the results obtained in Theorem (4.3) and get

$$\begin{aligned} | \mathcal{S}_n^{\alpha,\beta}(f;x) - f(x) | &\leq | \mathcal{S}_n^{\alpha,\beta}(f-\varphi;x) | + | \mathcal{S}_n^{\alpha,\beta}(\varphi;x) - \varphi(x) | + | f(x) - \varphi(x) | \\ &\leq 2 \| f - \varphi \|_{C_B[0,\infty)} + \left(\mathcal{S}_n^{\alpha,\beta}(g_1;x) + \frac{\mathcal{S}_n^{\alpha,\beta}(g_2;x)}{2} \right) \| \varphi \|_{C_B^2[0,\infty)} \\ &= 2\left(\| f - \varphi \|_{C_B[0,\infty)} + \frac{\mathcal{S}_n^{\alpha,\beta}(g_1;x) + \frac{\mathcal{S}_n^{\alpha,\beta}(g_2;x)}{2}}{2} \| \varphi \|_{C_B^2[0,\infty)} \right). \end{aligned}$$

By taking infimum over all $\varphi \in C_B^2[0,\infty)$ and use the results obtained by (5.1),

$$| \mathcal{S}_n^{\alpha,\beta}(f;x) - f(x) | = 2K_2\left(f; \frac{2\mathcal{S}_n^{\alpha,\beta}(g_1;x) + \mathcal{S}_n^{\alpha,\beta}(g_2;x)}{4}\right)$$

Now from the article [4] an absolute constant $\mathcal{B} > 0$ exits, so we using here

$$\mathcal{K}_2(f;\breve{\delta}) \leq \mathcal{B}\left\{\min(1,\breve{\delta}) \parallel f \parallel_{C_B[0,\infty)} + \omega_2(f;\sqrt{\breve{\delta}})\right\}.$$

This completes the proof. □

Atakut and Ispir [2] introduced the weighted modulus of continuity and defined as, for an arbitrary $f \in Q_\varrho^m(x)$

$$\bar{\Omega}(f;\hat{\delta}) = \sup_{|h|\leq\hat{\delta},\ x\in[0,\infty)} \frac{\mid f(x+h) - f(x) \mid}{(1+x^2)(1+h^2)}, \tag{5.3}$$

with the properties defined as

$$\lim_{\hat{\delta}\to 0} \bar{\Omega}(f;\hat{\delta}) = 0, \tag{5.4}$$

$$\mid f(t) - f(x) \mid \leq 2\left(\frac{\mid t-x \mid}{\hat{\delta}} + 1\right)(1+\hat{\delta}^2)(1+x^2)\left((t-x)^2+1\right)\bar{\Omega}(f;\hat{\delta}), \tag{5.5}$$

where $f \in \mathfrak{B}_\varrho^\eta(x)$ and $t, x \in [0,\infty)$.

Theorem 5.2. *Let $f \in \mathfrak{B}_\varrho^\eta(x)$, $x \in [0,\infty)$ then for the operators $\mathcal{S}_n^{\alpha,\beta}(\,\cdot\,;\,\cdot\,)$ defined by (2.1), we have*

$$\sup_{x\in[0,\mathfrak{M}_\upsilon(n)]} \frac{\mid \mathcal{S}_n^{\alpha,\beta}(f;x) - f(x) \mid}{1+x^2} \leq 2\left(2+\mathcal{U}+\sqrt{\mathcal{V}}\right)\left(1+\mathfrak{M}_\upsilon(n)\right)\bar{\Omega}\left(f;\sqrt{\mathfrak{M}_\upsilon(n)}\right),$$

where the constants $\mathcal{U}$ and $\mathcal{V}$ do not depend on n, and $\mathfrak{M}_\upsilon(n) = \max\left\{u_n, v_n, w_n\right\}$, and $u_n = \left\{\frac{n^2}{(n+\beta)^2} - \frac{2n}{n+\beta} + 1\right\}$, $v_n = \left\{\frac{2n}{(n+\beta)^2}\left(2+\alpha+\upsilon\chi_{n,\upsilon}(x)\right) - \frac{2}{n+\beta}(1+\alpha)\right\}$, $w_n = \left\{\frac{1}{(n+\beta)^2}\left(2+2\alpha+\alpha^2\right)\right\}$.

Proof. We prove it by using (5.3), (5.5) and Cauchy-Schwarz inequality.
$| \mathcal{S}_n^{\alpha,\beta}(f;x) - f(x) |$

$$\begin{aligned}
&\leq \frac{n^2}{e_\upsilon(nx)} \sum_{s=0}^{\infty} \frac{(nx)^s}{\gamma_\upsilon(s)} \int_0^\infty \frac{e^{-nt} n^{s+2\upsilon\theta_s-1} t^{s+2\upsilon\theta_s}}{\gamma_\upsilon(s)} \mid f(t) - f(x) \mid \mathrm{d}t \\
&\leq 2(1+\hat{\delta}^2)(1+x^2)\Omega(f;\hat{\delta}) \frac{n^2}{e_\upsilon(nx)} \sum_{s=0}^{\infty} \frac{(nx)^s}{\gamma_\upsilon(s)} \\
&\times \int_0^\infty \frac{e^{-nt} n^{s+2\upsilon\theta_s-1} t^{s+2\upsilon\theta_s}}{\gamma_\upsilon(s)} \left(1 + \frac{1}{\hat{\delta}} \mid t-x \mid\right) \left(1 + (t-x)^2\right) d_q(t) \\
&= 2(1+\hat{\delta}^2)(1+x^2)\bar{\Omega}(f;\hat{\delta}) \frac{n^2}{e_\upsilon(nx)} \sum_{s=0}^{\infty} \frac{(nx)^s}{\gamma_\upsilon(s)} \int_0^\infty \frac{e^{-nt} n^{s+2\upsilon\theta_s-1} t^{s+2\upsilon\theta_s}}{\gamma_\upsilon(s)} \mid f(t) - f(x) \mid \mathrm{d}t \\
&\times \left(1 + (t-x)^2 + \frac{1}{\hat{\delta}} \mid t-x \mid + \frac{1}{\hat{\delta}} \mid t-x \mid (t-x)^2\right) \\
&\leq 2(1+\hat{\delta}^2)(1+x^2)\bar{\Omega}(f;\hat{\delta}) \\
&\times \left(1 + \mathcal{S}_n^{\alpha,\beta}(g_2;x) + \frac{1}{\hat{\delta}}\sqrt{\mathcal{S}_n^{\alpha,\beta}(g_2;x)} + \frac{1}{\hat{\delta}}\sqrt{\mathcal{S}_n^{\alpha,\beta}(g_2;x)\mathcal{S}_n^{\alpha,\beta}(g_4;x)}\right),
\end{aligned}$$

From Lemma 2.2, we easily see that,

$$\mathcal{S}_n^{\alpha,\beta}(g_2;x) \leq \mathfrak{W}_\upsilon(n)(1+x+x^2),$$

where$\mathfrak{M}_\upsilon(n) = \max\left\{u_n, v_n, w_n\right\}$ and since $\frac{1}{n+\beta} \to 0$ as $n \to \infty$, then there exits a constant $\mathcal{U} > 0$ such that

$$\mathcal{S}_n^{\alpha,\beta}(g_2;x) \leq \mathcal{U}(1+x+x^2).$$

In similar way there exits a constant $\mathcal{V} > 0$, such that

$$\mathcal{S}_n^{\alpha,\beta}(g_4;x) \leq \mathcal{V}\big(1+x+x^2+x^3+x^4\big).$$

Therefore,

$$\begin{aligned}
| \mathcal{S}_n^{\alpha,\beta}(f;x) - f(x) | &\leq 2(1+\hat{\delta}^2)(1+x^2)\bar{\Omega}(f;\hat{\delta}) \\
&\times \left(1 + \mathcal{U}(1+x+x^2) + \frac{1}{\hat{\delta}}\sqrt{\mathfrak{W}_\upsilon(n)(1+x+x^2)}\right. \\
&+ \left.\frac{1}{\hat{\delta}}\sqrt{\mathcal{V}\mathfrak{W}_\upsilon(n)(1+x+x^2)(1+x+x^2+x^3+x^4)}\right).
\end{aligned}$$

If we taking the supremum for all $x \in [0, \mathfrak{M}_\upsilon(n)$ and choose $\hat{\delta} = \sqrt{\mathfrak{M}_\upsilon(n)}$, then easily led to the result asserted by Theorem 5.2. □

6. Conclusion

This research manuscript has an appropriate uniform approximation properties and the flexibility in the error estimation with their generalization rather than the article [15]. We establish a generalized version of the classic Phillips Operators [15] via Dunkl type generalization. The point should be noted that in case of $\alpha = \beta = 0$, the operators (2.1) reduce to the Phillips Operators given by [15]. The approximation obtained by these operators designed by Dunkl type provide a better generalization and a educational platform to the researcher to obtain the error estimations of the uniform convergence depending on υ.

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