New exact solution for a generalized variable coefficient KdV equation with power law nonlinearity in terms of Weierstrass elliptic functions

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Abstract *- In this work, a new exact solitary wave solution expressible in terms of Weierstrass elliptic function of the time-dependent coefficient KdV equation with power-law nonlinearity is obtained. I also obtained a hyperbolic solution as a limiting case.*

Keywords — *Nonlinear evolution equation, exact solution, Weierstrass elliptic function, power-law nonlinearity.*

I. Introduction

Nonlinear evolution equation is a significant research subject in the field of physics and mathematics so far its theoretical value and applicability have been considered. Many powerful methods [1, 2] have been employed by the researchers through decades for seeking the exact solutions to constant coefficient nonlinear evolution equations. But as the constant coefficients are highly idealised assumptions, I therefore aimed at studying the integrability and symmetry of variable coefficient nonlinear evolution equation [3, 4, 5, 6, 7].

A Korteweg-de-Vries(KdV) equation with power law nonlinearity and time-dependent coefficients has been studied and a solitary wave solutions has been obtained by Biswas [8, 9]. In this work I have verified to find exact solutions for a KdV equation with power law nonlinearity and time-dependent coefficients in terms of Weierstrass elliptic function [10].

II. Mathematical analysis

The main goal in this paper is to obtain wave solutions in terms of Weierstrass elliptic function.The KdV equation with power law nonlinearity having time-dependent coefficients is given by [1]

$$u_t(x,t) + \sigma\,(t)u^p(x, t)u_x(x, t) + \delta(t)u_{xxx}(x, t) = 0 \qquad (1)$$

where σ and δ are time-dependent variables. Here the first term is the evolution term, while the second term represents the power law nonlinearity term, with p as the index of power law and the third term is the dispersion term. The special case where p = 1, is the generalized KdV equation with variable coefficients and where p = 2, is known as the generalized modified KdV equation with variable coefficients. Equation (1) arises in various areas of Mathematical physics, plasma physics, fluid dynamics and lattice vibrations of crystal at low temperatures. All these applications start from a more or less general physical model and end up in the generalized KdV equation with variable coefficients by considering a specific limit of the physical problem.

III. New traveling solitary wave solutions in terms of Weierstrass elliptic function

First, we assume that (1) has solutions of the form

$$u(x,t) = v(\xi); \tag{2}$$

$$\xi = x + \lambda t; \tag{3}$$

λ being the speed of the wave. Substituting (2) and (3) into (1), and after simplification we have-

$$\lambda v'(\xi) + \sigma(t)(v(\xi))^p v'(\xi) + \delta(t)v'''(\xi) = 0 \tag{4}$$

Upon integrating, (4) is converted to

$$\lambda v(\xi) + \sigma(t)/(p+1)v^{p+1}(\xi) + \delta(t)v''(\xi) = C, \tag{5}$$

where C is the integrating constant. For the sake of simplicity, we shall take C = 0. With the change of variable

$$v(\xi) = V^{1/p}(\xi) \tag{6}$$

(5) reduces to

$$\lambda V^2(\xi) + \sigma(t)/(p+1)V^3(\xi) + \delta(t)(1-p)/p^2\,(V'(\xi))^2 + \delta(t)/p\,V(\xi)V''(\xi) = 0 \tag{7}$$

Using the idea of the tanh-coth method [11] I choose to search for the solution to (7) in the form of the following expansion

$$V(\xi) = \sum_0^M a_i(t)G^i(\xi, g_1, g_2) + \sum_{M+1}^{2M} a_i(t)G^{M-i}(\xi, g_1, g_2), \tag{8}$$

where M is a positive integer to be determined later, $a_i(t)$, i = 1, 2,......,2M, are arbitrary functions of the variable t and $G(\xi)$ is a Weierstrass elliptic function which satisfies the following equation

$$[G'(\xi : g_2, g_3)]^2 = 4G^3(\xi ; g_2, g_3) - g_2G(\xi ; g_2, g_3) - g_3 \tag{9}$$

$$G''(\xi; g_2, g_3) = 6G^2(\xi; g_2, g_3) - g_2/2 \tag{10}$$

Substituting (8) into (7) and balancing V^3 with $V(\xi)V''(\xi)$ I have 3M = 2M + 1 =>
M = 1 Therefore, (8) takes the form

$$V(\xi) = a_0(t) + a_1(t)G(\xi; g_2, g_3) + a_2(t)G^{-1}(\xi ; g_2, g_3) \tag{11}$$

Substituting of (11) into (7) leads to a system of algebraic equation in the unknowns $a_0(t)$, $a_1(t)$, $a_2(t)$, λ, g_2, g_3, which gives

$$\lambda = -6\sigma(t)a_0(t)/(p+1)(p+2) \tag{12}$$

$$a_1(t) = -2\delta(t)(p+1)(p+2)/\sigma(t)p^2 \tag{13}$$

$$a_2(t) = 0 \tag{14}$$

$$g_2 = \lambda^2p^4/12\delta^2(t) \tag{15}$$

$$g_3 = \lambda^3p^6/216\delta^3(t) \tag{16}$$

The exact solutions to (1)are obtained using (11) is

$$u(x, t) = [a_0(t)-2\delta (t)(p +1)(p+2)/\sigma(t)p^2G(\xi; g_2, g_3)]^{1/p} \quad (17)$$

In limiting case ($e_2 \longrightarrow e_1$) solution (17) reduces to

$$u(x,t)=[a_0(t)-2\delta (t)(p+1)(p+2)/\sigma(t)p^2e_3+(e_1-e_3)\coth^2(e_1-e_3\xi)^{1/2}]^{1/p} \quad (18)$$

Where e_1,e_2,e_3 are the roots of $4y^3-g_2y-g_3=0$ and $\xi=x+\lambda t$ and λ is given by (12).

IV. Summary

It has been observed that the most important key to deriving the exact solutions for a generalization variable coefficient Korteweg-de-Vries equation with power law nonlinearity have been obtained by using generalized Weierstrass elliptic function expansion method for constructing more general exact solutions of nonlinear evolution equations. As a consequence, taking the values p = 1 and p = 2, exact solutions to standard KdV and to mKdV equations with variable coefficient $a_0(t)$is an arbitrary functions in the variable t; therefore, with variables of these coefficients I can obtain abundant solutions to our model. I also obtain coth solution as a limiting case of Weierstrass-eliptic function solution.The advantage of the method is that it can be used to obtain more general exact solutions by the known Weierstrass elliptic function.

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