Different Types of Orthogonalities In Functional Analysis

B.Kokila [#1], N.Mythili[#2], S.Sravanthi[# 3]

[# 1, 2, 3] *PG and Research Department of Mathematics, Arcot Sri Mahalakshmi Women's College, Villapakkam- 632521 Ranipet Dist, Tamilnadu state ,india*

ABSTRACT:

Orthogonality is the generalization of the notion of perpendicularity to the linear algebra of bilinear forms. Two elements of x and y of a vector space with bilinear form B are Orthogonal when B<x, y> =0. Depending on the bilinear form B are the vector space may contain non- zero self- Orthogonal vectors.

KEYWORDS:

Banach space , Orthogonal ,Orthonormal, Orthogonal sequence, self adjoint Operator, bounded operator.

INTRODUCTION

Orthonormal sets are not especially significant on their own. However they display certain features that make them fundamental in exploring the notation of diagonalizablity of certain operators on vector spaces. many definitions of Orthogonality in Banach spaces were introduced over the years. Roberts (1934),defined Orthogonality relation for pairs of elements in Banach space as: two elements x and y of X are said to be Orthogonal in the sence of Roberts if and only if $\|x+ky\| = \|x-ky\|$, for all k ∈R Brikhoff (1935)suggested author definition as : two elements x and y are said to be Orthogonal in the sense of brikhoff if and only if $\|x\| \leq \|x+\lambda y\|$, for all $\lambda \in R$

In 1945 James introduced Pythagorean and isosceles Orthogonality two elements x and y of X are said to be Orthogonal in Pythagorean sense if and only if $\|x-y\|^2 = \|x\|^2 + \|y\|^2$. Saidi (2002) improved the definition of Khalil and presented a new types of Orthogonality for sequence in Banach space and studied this Orthogonality in specific spaces namely l^p spaces. In this paper we discuss about new types of Orthogonality in Banach spaces with examples

Definition: Banach space

A complete normed linear space is said to be Banach space.

Definition: Orthonormal

A set $\{w_1,w_2,\ldots\ldots w_n\}$ is said to be Orthonormal if the following properties will be satisfied,

a. $(w_i, w_j) = 0$ for $i \neq j$
b. $\|w_i\| = 1$ for every i

Definition: Orthogonal

Two vectors v_1 & v_2 in a Hilbert space H is said to be Orthogonal if $(v_1, v_2) = 0$ and it's denoted by $v_1 \perp v_2$

Definition: Orthonormal sequence

If a finite or infinite sequence of vectors forms an Orthonormal system then the sequence is said to be an Orthonormal sequence

Example:

Let $\{x_1, x_2, \ldots. x_n\}$ be a finite Orthonormal set in Hilbert space H. prove that for any $x \in H$ the vector $x - \sum_{k=1}^{n} (x, x_k)\, x_k$ is Orthogonal to x_k for every k = 1, 2,3…..n

Solution:

$$\left(x - \sum_{k=1}^{n} (x, x_k)\, x_k, x_k\right) = (x, x_k) - \sum_{k=1}^{n} (x, x_k)\,(x_k, x_l)$$
$$= (x, x_k) - (x, x_k)$$

If k = l then $(x_k, x_l) = 1$

$$\left(x - \sum_{k=1}^{n} (x, x_k)\, x_k, x_k\right) = 0$$

$\Rightarrow$ $x - \sum_{k=1}^{n} (x, x_k)\, x_k$ is Orthogonal to x_k.

i.e.) $x - \sum_{k=1}^{n} (x, x_k) x_k \perp x_k$

Definition: Self-adjoint operator

An operator S is said to be a self-adjoint if $S=S^*$ i.e.) $(Sa, b) = (a, Sb)$ $\forall$ a, b $\in$ H

Definition: Bounded operator

An operator T is said to be bounded and T: Y → Y, if there exists a number k such that

$$\|Ty\| \leq k\,\|y\| \qquad \text{for every } y \in Y$$

Example:

If T be a self adjoint operator and T $\neq$ 0 then show that $T^* \neq 0$ for all n $\in$ N

Proof:

Given that T is a self adjoint i.e.) $T = T^*$

by hypothesis, T $\neq$ 0 then $T^* \neq 0$

Example:

If T is a self adjoint operator then show that

$\|Tx + ix\|^2 = \|Tx\|^2 + \|x\|^2$

Proof:

Given that T is self adjoint i.e.) $T = T^*$

$$\begin{aligned}\|Tx + ix\|^2 &= (Tx + ix, Tx + ix) \\ &= (Tx, Tx) + (Tx, ix) + (ix, Tx) + (ix, ix) \\ &= \|Tx\|^2 - i(Tx, x) + i(Tx, x) + \|x\|^2 \\ \|Tx + ix\|^2 &= \|Tx\|^2 + \|x\|^2\end{aligned}$$

Example:

If A is self adjoint operator and B is bounded operator show that B^*AB is self adjoint.

Proof:

Given that A is self adjoint operator and B is bounded operator

Consider $(B^*AB)^* = B^*A^*(B^*)^*$

$= B^*AB$ (since A is self adjoint)

Hence B^*AB is self adjoint.

Definition: Orthogonality in Banach space

In a Banach space X, two elements a & b are said to be Orthogonal if and only if

$$\|a + \alpha b\| = \|a - \alpha b\| \ \forall\ \alpha \in R$$

Definition: Distance Orthogonality

The element a is said to be distance Orthogonal to b if

$$\inf_\lambda \|a-\lambda b\| = \|a\|$$

$$\inf_\lambda \|b-\lambda a\| = \|b\|$$

It is denoted by $a \perp^d b$.

Theorem:

Let X be a Banach space, u,v$\in$ X and $u \perp^d v$ then $au \perp^d bv$.

Proof:

Given that u & v $\in$ X and $u \perp^d v$

by the definition of distance Orthogonal we have

$$\inf_\lambda \|u-\lambda v\| = \|u\| \quad \&$$

$$\inf_\lambda \|v-\lambda u\| = \|v\|$$

now consider

$$\inf_\lambda \|au-\lambda b\| = \|au\|$$

$$= |a| \, \|u\| \qquad (1)$$

$$\inf_\lambda \|bv-\lambda a\| = \|bv\|$$

$$= |b| \, \|v\| \qquad (2)$$

From (1)&(2) we have

au is distance Orthogonal to bv

i.e.) $au \perp^d bv$

Theorem:

Let X be a Banach space $u, v \in X$ and $u \perp^d v$ then prove that $v \perp^d u$.

Proof:

Given that $u \perp^d v$

By the definition of distance Orthogonal we have

$$\inf_\lambda \|u-\lambda v\| = \|u\| \quad \&$$

$$\inf_\lambda \|v-\lambda u\| = \|v\|$$

consider $\inf_\lambda \|v-\lambda u\| = \|v\|$

$$\inf_\lambda \|u-\lambda v\| = \|u\|$$

Form this we can write

v is distance Orthogonal to u

i.e.) $v \perp^d u$

Definition: Projection Orthogonality

Let u and v in a Banach space X. If u and v are said to be projection Orthogonal if and only if $A : [u, v] \to [u]$ and $B : [u, v] \to [v]$ are contractive projections

i.e.) $\|A\|=1=\|B\|$.

Projection Orthogonality is denoted by $u \perp^p v$.

Theorem:

If $A = \{a_1,a_2,\ldots\}$ is a p-Orthonormal sequence then the sequence $\{a_1,a_2,\ldots\}$ is indepeent.

proof:

Let the P- Orthonormal sequence A = $\{a_1,a_2,\ldots.\}$.

To prove that, the sequence A is independent

it is enough we have to prove this theorem for n

i.e.) to prove $\{a_1,a_2,\ldots.a_n\}$ is independent

now we prove this theorem by induction method.

Let n = 1

Then $\{a_1\}$ is obviously independent.

Hence the theorem is true for n = 1

now assume that the theorem is true for n = k then

$\{a_1,a_2,\ldots.a_k\}$ is independent

Next we prove the theorem is true for n = k + 1

i.e.) to prove $\{a_1,a_2,\ldots.a_k,a_{k+1}\}$ is independent

without loss of generality we may assume that

$$a_{k+1} = \sum_{i=1}^{n} b_i a_i$$

consider P: $[a_1,a_2,\ldots.a_k,a_{k+1}] \rightarrow [a_1,a_2,\ldots.a_k]$

Assume that P is contractive projection.

$$\|P(w)\| \leq \|w\| \quad \text{for all } w \in [a_1,a_2,\ldots.a_k,a_{k+1}]$$

then

$$\left\|\sum_{i=1}^{n} x_i a_i\right\| \leq \left\|\sum_{i=1}^{k} x_i a_i + x_{i+1}a_{i+1}\right\|$$

choose $x_i = b_i$ for all $1 \leq i \leq k$ and $x_{i+1} = -1$

then

$$\sum_{i=1}^{k} x_i a_i + x_{i+1}a_{i+1} \neq 0$$

since by the assumption $\{a_1,a_2,\ldots.a_k\}$ is independent

but $\sum_{i=1}^{k} x_i a_i = 0$

this is a contradiction

Hence $\{a_1,a_2,\ldots.a_k,a_{k+1}\}$ is independent

Hence $\{a_1, a_2,\ldots..\}$ is independent.

Conclusion

In this paper we present types of Orthogonalities in Banach space with the basic concepts of each types and operators on Hilbert space. The reflection of such Orthogonalities to Orthogonal sequence and example are also discussed.

Reference

[1] Ahkiezer, N.I., the classical moment problem, Hafner, new york, N.Y., (1965)
[2] Askey, R., Ismail, M.E.H., Recurrence relations, continued fractions and Orthogonal polynomials, memoirs amer, math., soc ., 300(1984)
[3] Birikhoff, G.(1935) Orthogonality in linear metric spaces, duke math.J. (1), 169-172
[4] Bustoz, J., Ismail, M.E.H., the associated ultra-spherical polynomials and their q-analogous, candian journal of math. 34(1982)
[5] Conway J 1990 a course in functional analysis new york: springer – verlag
[6] Diestel, joseph(1943), sequences and series in Banach spaces, new york, springer – verlag
[7] James, R. C.(1945), Orthogonality in normal linear spaces, duke math. J. (12), 291-302
[8] Khalil, R. and alkhawalda, A, university of Jordan department of mathematics,
[9] Khalil, R.(1990) Orthogonality in Banach spaces math. J, of toyama university(13), 185-205
[10] L.Debnath, F.A. Shah, wavelet transformation and their application Jairo a. charris and luis a. gomez
[11] Saidi, F. (2002), characterization of Orthoganality in certain Banach spaces, bull. Austral. Math. Soc. (65), 93-104
[12] Singer, I.(1970), Bases in Banach spaces, springer-verlog, new york
[13] Szego, G., Orthogonal polynomials, amer. Math. Soc. Colloquium publications, vol. XXII, 4th ed., providence., R.I., 1975