Self Adjoint Operator In Functional Analysis

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Abstract

In this paper, we study its possible to construct an self-adjoint for operators on banach space. Firstly, the necessary mathematical background namely, banach space, inner product space is reviewed. secondly we show the relationship between self-adjoint operator and other operator. self-adjoint operator on a finite dimensional complex vector space V with inner product <..,.> is a linear map P (from V to itself) that is its own adoint. self-adjoint operators are used in functional analysis and quantum mechanics.

Keywords

Banach space, self-adjoint operator, positive operator, eigen values, normal operator, problems

I. INTRODUCTION

Functional analysis is a branch of mathematical analysis, the core of which is formed by the study of vector spaces endowed with some kind of limit related structure [e.g. Norm, Inner product, topology, etc] and the linear functions defined on their spaces and structures in suitable sense. It is the study of certain topological-algebraic structures and the methods by which knowledge of these structures can be applied to analytic problems. The basic and historically first class of spaces studied in functional analysis are complete normed vector space over the real or complex numbers such spaces are called Banach space. A complete inner product spaces are known as the Hilbert spaces. In mathematics, an operator is generally a mapping that acts on the space to produce other elements of the same space.

Self-adjoint operators on infinite dimensional Hilbert spaces essentially resembles the finite dimensional case that is to say, operator are self-adjoint if and only if they are unitarily equivalent to real valued multiplication operators. In mathematics, operator is generally a mapping that acts on the space to produce other elements of the same space. In this chapter we discuss relationship between self-adjointoperator in inner product space with other operator in banach space.

Definition: Banach space

A complete normed linear space is called Banach space

Definition: Inner product space

An Inner product space is a vector space V over the field F together with a Inner product (i.e) with a map $\langle \cdot , \cdot \rangle : V \times V \longrightarrow F$ that satisfies 3 axioms for all vectors x,y,z ϵ V and all scalersα ϵ F

i. Conjugate symmentry:

$$\langle x,y\rangle = \overline{\langle y,x\rangle}$$

ii. Linearity:

$$\langle \alpha x,y\rangle = \alpha\langle x,y\rangle$$

$$\langle x+y,z\rangle = \langle x,z\rangle + \langle y,z\rangle$$

iii. Positive

$$\langle x,x\rangle \geq 0 \text{ and } \langle x,x\rangle = 0 \iff x = 0$$

Definition: Self-adjoint operator

If $P = P^*$ that is $\langle Px,y\rangle = \langle x,Py\rangle\ \forall\ x, y \in H$ then P is called self-adjoint operator

Definition: Positive operator

An operator P is called positive if it self-adjoint and

$$\langle Px,x\rangle \geq 0 \quad \forall\, x \in H$$

Definition: Normal operator

An operator P on Hilbert space H is said to be normal if

$$PP^* = P^*P$$

Theorem 1:

let H be a Hilbert space and let $P \in B(H)$ is given then P is self-adjointiff$\langle Px,x\rangle \in R$, $\forall x,y \in H$.

Proof:

consider P is self-adjoint$\Rightarrow P = P^*$

for any $x,x \in H$ we have

$$\overline{\langle Px,x\rangle} = \langle x,Px\rangle$$
$$= \langle P^*x,x\rangle$$
$$\overline{\langle Px,x\rangle} = \langle Px,x\rangle$$
$$\therefore \overline{\langle Px,x\rangle} \text{ is real}$$

now, assume that $\langle Px,x\rangle$ is real for all $x,x \in H$

choosex,y $\in H$ then

$$\langle P(x+y), x+y\rangle = \langle Px,x\rangle + \langle Px,y\rangle + \langle Py,x\rangle + \langle Py,y\rangle$$

since $\langle P(x+y), x+y\rangle$, $\langle Px,x\rangle$, $\langle Py,y\rangle$ is real

we conclude that

$$\langle Px,y\rangle + \langle Py,x\rangle \text{ is real}$$

hence its equals its own conjugate complex

$$\langle Px,y\rangle + \langle Py,x\rangle = \overline{\langle Px,y\rangle + \langle Py,x\rangle}$$
$$= \langle y,Px\rangle + \langle x,Py\rangle \quad \text{-------- (1)}$$

similarly after examine the equation

$$\langle P(x+iy), (x+iy)\rangle = \langle Px,x\rangle + \langle Px,iy\rangle + \langle iPy,x\rangle + \langle iPy,iPy\rangle$$
$$= \langle Px,x\rangle - i\langle Px,y\rangle + i\langle Py,x\rangle + \langle iPy,iPy\rangle$$

we conclude that

$$\langle Px,y\rangle - \langle Py,x\rangle = -\langle y,Px\rangle + \langle x,Py\rangle \quad \text{-------- (2)}$$

adding (1) and (2)

$$2\langle Px,y\rangle = 2\langle x,Py\rangle$$
$$\Rightarrow \langle Px,y\rangle = \langle x,Py\rangle$$
$$\Rightarrow \langle Px,y\rangle = \langle P^*x,y\rangle$$

since this is true for every x and y we conclude $P=P^*$

Problem 1:

If $P \in B(H)$ are self-adjoint and $\langle Px,x\rangle = 0$ for every x, P=0

Solution:

$$\overline{\langle Px,x\rangle} = \langle x,Px\rangle$$
$$= \langle P^*x,x\rangle$$

since$\langle Px,x\rangle = 0$ then $\overline{\langle Px,x\rangle} = 0$

$\langle P^*x,x\rangle = 0$

$\Rightarrow \quad P^* = 0$

since P is self-adjoint$\Rightarrow P = 0$

Problem 2:

If P is a self-adjoint and $P \neq 0$ then $P^* \neq 0$ for all $n \in n$

Solution:

Given P is a self-adjoint and $P \neq 0$

by the definition of self-adjoint $P = P^*$

since $P \neq 0 \Rightarrow P^* \neq 0$

Problem 3:

Show that if $P \in B(H)$ are self-adjoint then $P+P^*$, PP^*, P^*P and $PP^* - P^*P$ are all self-adjoint

Solution:

$$\begin{aligned} <(P+P^*)x,y> &= <x,(P+P^*)^*y> \\ &= <x,(P^*+P)y> \\ &= <x,(P+P^*)y> \end{aligned}$$

$\Rightarrow$ $(P+P^*)$ is self-adjoint

$$\begin{aligned} <PP^*x,y> &= <x,(PP^*)^*y> \\ &= <x,P^*(P^*)^*y> \\ &= <x,P^*Py> \\ &= <x,PP^*y> \end{aligned}$$

$\Rightarrow$ PP^* is self-adjoint

similarly, P^*P is self-adjoint

$<(PP^*- P^*P)x,y> = <x, (PP^* - P^*P)^*y>$

since PP^* and P^*P are self-adjoint

$<(PP^*- P^*P)x,y> = <x, (PP^* - P^*P)y>$

$\Rightarrow$ $(PP^*- P^*P)$ is self-adjoint

Problem 4:

show that if $P,Q \in B(H)$ are self-adjoint then PQP and QPQ are self-adjoint

Solution:

$$\begin{aligned} <PQPx,y> &= <Px,(QP)^*y> \\ &= <Px,Q^*P^*y> \\ &= <x,P^*QPy> \\ &= <x,PQPy> \end{aligned}$$

hence PQP is self-adjoint

similarly, QPQ is also self-adjoint

Problem 5:

let $P \in B(H)$ be given show that if $P \in B(H)$ is a positive operator then all eigen values of P are real and non negative

Solution:

P is a positive operator $\Rightarrow <Px,x> \geq 0$

If $Pu = \lambda u$ then

$$\begin{aligned} <\lambda u,u> &= <Pu,u> \\ &= <u,Pu> \\ &= \overline{<Pu,u>} \\ &= \overline{<\lambda u,u>} \\ <\lambda u,u> &= \bar{\lambda}<u,u> \end{aligned}$$

As $\|u\| = <u,u> \geq 0$ then $\lambda \geq \bar{\lambda}$

$\Rightarrow$ P is real and non negative

Theorem 2:

if $P \in B(H)$ show the P is normal iff $\|Px\| = \|P^*x\|$ for every $x \in H$

Proof:

consider P is normal

$$\begin{aligned} \|Px\|^2 &= <Px,Px> \\ &= <x,P^*Px> \\ &= <x,PP^*x> \\ &= <P^*x,P^*x> \\ \|Px\|^2 &= \|P^*x\|^2 \end{aligned}$$

conversely, $\|Px\| = \|P^*x\|$

$<Px,Px> = <P^*x,P^*x>$

$<x,P^*Px> = <x,PP^*x>$

$\Rightarrow P^*P = PP^*$

Conclusion

In this paper we discussed about self-adjoint operator and other operators with inner product space in Banach space

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