Fair Restrained Domination in Graphs

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Abstract: *Let G be a connected simple graph. A dominating set $S \subseteq V(G)$ is a fair dominating set in G if every two distinct vertices not in S have the same number of neighbors from S, that is, for every two distinct vertices u and v from $V(G) \setminus S$, $|N(u) \cap S| = |N(v) \cap S|$. A fair dominating set $S \subseteq V(G)$ is a fair restrained dominating set if every vertex not in S is adjacent to a vertex in S and to a vertex in $V(G) \setminus S$. Alternately, a fair dominating set $S \subseteq V(G)$ is a fair restrained dominating set if $N[S] = V(G)$ and $\langle V(G) \setminus S \rangle$ is a subgraph without isolated vertices. The minimum cardinality of a fair restrained dominating set of G, denoted by $\gamma_{frd}(G)$, is called the fair restrained domination number of G. In this paper, we initiate the study of the concept and give some realization problems. In particular, we show that given positive integers k , m, and $n \geq 3$ such that $1 \leq k \leq m \leq n-2$, there exists a connected nontrivial graph G with $|V(G)| = n$ such that $\gamma_{fd}(G) = k$ and $\gamma_{frd}(G) = m$. Further, we show the characterization of the fair restrained dominating set in the join of two nontrivial connected graphs.*

Keywords: *dominating set, fair dominating set, restrained dominating set, fair restrained dominating set*

I. INTRODUCTION

This Domination in graph was introduced by Claude Berge in 1958 and Oystein Ore in 1962 [1]. Following an article [2] by Ernie Cockayne and Stephen Hedetniemi in 1977, the domination in graphs became an area of study by many researchers. A subset S of $V(G)$ is a dominating set of G if for every $v \in V(G) \setminus S$, there exists $x \in S$ such that $xv \in E(G)$, that is, $N[S] = V(G)$. The domination number $\gamma(G)$ of G is the smallest cardinality of a dominating set of G. Some studies on domination in graphs were found in the papers [3, 4, 5, 6, 7, 8, 9, 10, 11, 12,13,14,15,16].

In 2011, Caro, Hansberg and Henning [17] introduced fair domination and k-fair domination in graphs. A dominating subset S of $V(G)$ is a fair dominating set in G if all the vertices not in S are dominated by the same number of vertices from S, that is, $|N(u) \cap S| = |N(v) \cap S|$ for every two distinct vertices u and v from $V(G) \setminus S$ and a subset S of $V(G)$ is a k-fair dominating set in G if for every vertex $v \in V(G) \setminus S$, $|N(v) \cap S| = k$. The minimum cardinality of a fair dominating set of G, denoted by $\gamma_{fd}(G)$, is called the fair domination number of G. A fair dominating set of cardinality $\gamma_{fd}(G)$ is called γ_{fd}-set. Some studies on fair domination in graphs were found in the paper [18].

The restrained domination in graphs was introduced by Telle and Proskurowski [19] indirectly as a vertex partitioning problem. Accordingly, a set $S \subseteq V(G)$ is a restrained dominating set if every vertex not in S is adjacent to a vertex in S and to a vertex in $V(G) \setminus S$. Alternately, a subset S of $V(G)$ is a restrained dominating set if $N[S] = V(G)$ and $\langle V(G) \setminus S \rangle$ is a subgraph without isolated vertices. The minimum cardinality of a restrained dominating set of G, denoted by $\gamma_r(G)$, is called the restrained domination number of G. A restrained dominating set of cardinality $\gamma_r(G)$ is called γ_r-set. Restrained domination in graphs was also found in the papers [20, 21, 22, 23, 24, 25, 26, 27]. Motivated by fair domination and restrained domination in graphs, we introduce the study of fair restrained dominating set. A fair dominating set $S \subseteq V(G)$ is a fair restrained dominating set if every vertex not in S is adjacent to a vertex in S and to a vertex in $V(G) \setminus S$. The minimum cardinality of a fair restrained dominating set of G, denoted by $\gamma_{frd}(G)$, is called the fair restrained domination number of G. A fair restrained dominating set of cardinality $\gamma_{frd}(G)$ is called γ_{frd}-set.

For the general terminology in graph theory, readers may refer to [28]. A graph G is a pair $(V(G), E(G))$, where $V(G)$ is a finite nonempty set called the vertex-set of G and $E(G)$ is a set of unordered pairs $\{u, v\}$ (or simply uv) of distinct elements from $V(G)$ called the edge-set of G. The elements of $V(G)$ are called vertices and the cardinality $|V(G)|$ of $V(G)$ is the order of G. The elements of $E(G)$ are called edges and the cardinality $|E(G)|$ of $E(G)$ is the size of G. If $|V(G)| = 1$, then G is called a trivial graph. If $E(G) = \emptyset$, then G is called an empty graph. The open neighborhood of a vertex $v \in V(G)$ is the set

$N_G(v) = \{u \in V(G): uv \in E(G)\}$. The elements of $N_G(v)$ are called neighbors of v. The closed neighborhood of $v \in V(G)$ is the set $N_G[v] = N_G(v) \cup \{v\}$. If $X \subseteq V(G)$, the open neighborhood of X in G is the set $N_G(X) = \cup_{v\in X} N_G(v)$. The closed neighborhood of X in G is the set $N_G[X] = \cup_{v\in X} N_G[v] = N_G(X) \cup X$. When no confusion arises, $N_G[x]$ $[resp. N_G(x)]$ will be denoted by $N[x]$ [resp. $N(x)$].

II. RESULTS

Remark 2.1 [17] If $G \neq \overline{K}_n$, then $\gamma_{fd}(G) = min\{\gamma_{kfd}(G)\}$, where the minimum is taken over all integers k where $1 \leq k \leq |V(G)| - 1$.

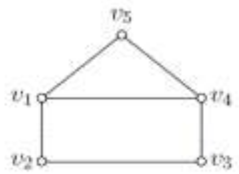

Figure 1: A graph G with $\gamma_{frd}(G) = 2$

Example 2.2 Consider the graph G in Figure 1. Then the set $S_1 = \{v_1, v_2\}$ is a 1-fair dominating set of G, while $S_2 = \{v_2, v_3, v_5\}$ is a 2-fair dominating set of G. It can be observed that S_1 is a γ_{1fd}-set of G while S_2 is a γ_{2fd}-set of G. Hence, $\gamma_{1fd}(G) = 2$ and $\gamma_{2fd}(G) = 3$, that is, $\gamma_{fd}(G) = 2$. Further, observed that S_1 is also a γ_r-set of G. Therefore, S_1 is a γ_{frd}-set of G. Hence, $\gamma_{frd}(G) = 2$.

Since $\gamma_{frd}(G)$ does not always exists in a connected nontrivial graph G, we denote $\mathcal{FR}(G)$ be a family of all graphs with fair restrained dominating set. Thus, for the purpose of this study, it is assumed that all connected nontrivial graphs considered belong to the family $\mathcal{FR}(G)$.

Remark 2.3 Let G be a connected graph. Then for any positive integer n, $K_{1,n} \notin \mathcal{FR}(G)$.

From the definition of a fair restrained domination number $\gamma_{frd}(G)$ of G, the following result is immediate.

Remark 2.4 Let G be any connected graph of order $n \geq 3$. Then

(i) $1 \leq \gamma_{frd}(G) \leq n - 2$ and
(ii) $\gamma(G) \leq \gamma_r(G) \leq \gamma_{frd}(G)$.

The next result says that the value of the parameter $\gamma_{frd}(G)$ ranges over all positive integers from $1, 2, \dots, n-2$ where n is the order of G.

Theorem 2.5 Let k, m, and $n \geq 3$ be positive integers such that $1 \leq k \leq m \leq n - 2$. Then there exists a connected nontrivial graph G with $|V(G)| = n$ such that $\gamma_{fd}(G) = k$ and $\gamma_{frd}(G) = m$.

Proof: Consider the following cases:

Case 1. Suppose that $1 = k = m \leq n - 2$. Let $G = K_n$. Clearly, $|V(G)| = n$ and $\gamma_{fd}(G) = 1 = \gamma_{frd}(G)$.

Case 2. Suppose that $1 = k < m \leq n - 2$. Let $G = \langle v \rangle + \langle V(P_{n-r-1}) \cup V(\overline{K}_r) \rangle$ such that $m = r + 1$. Then $A = \{v\}$ is a fair dominating set of G and $B = \{v\} \cup V(\overline{K}_r)$ is the minimum fair restrained dominating set in G. Thus, $|V(G)| = 1 + (n - r - 1) + r = n$, $\gamma_{fd}(G) = |A| = 1 = k$, and $\gamma_{frd}(G) = |B| = 1 + r = m$.

Case 3. Suppose that $1 < k = m \leq n - 2$. Let $n = 2b$ where $b \equiv 1(mod\ 4)$, $b \neq 1$, and $4k - 2 = n$. Let G be a graph below (see Figure 2).

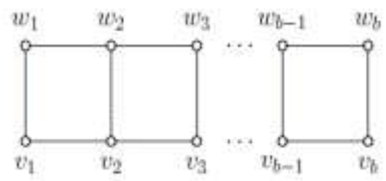

Figure 2: A graph G with $1 < k = m \leq n - 2$.

The set $A = \left\{v_{4i-3}: i = 1,2,3, \ldots, \frac{b+3}{4}\right\} \cup \left\{w_{4i-1}: i = 1,2,3, \ldots, \frac{b-1}{4}\right\}$ is a γ_{fd}-set and a γ_{frd}-set of G. Thus, $\gamma_{fd}(G) = |A| = \frac{b+3}{4} + \frac{b-1}{4} = \frac{2b+2}{4} = \frac{n+2}{4} = k = m = \gamma_{frd}(G)$ and $|V(G)| = 2b = n$.

Case 4. Suppose that $1 < k < m \leq n - 2$.

Let $V(P_r) = \{x_1, x_2, \ldots, x_r\}$, $V(P_s) = \{y_1, y_2, \ldots, y_s\}$ with $n - 2 = r + s = m$, and $r = s$. Then let G be a graph obtained from $H_1 = \langle v_1 \rangle + P_r$ and $H_2 = \langle v_2 \rangle + P_s$ with $v_1 v_2 \in E(G)$ (see Figure 3).

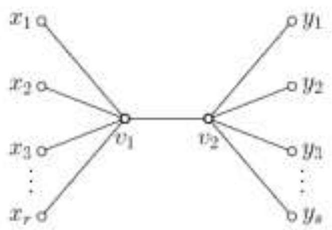

Figure 3: A graph G with $1 < k < m \leq n - 2$.

The set $A = \{v_1, v_2\}$ is a γ_{fd}-set and $B = V(P_r) \cup V(P_s)$ is a γ_{frd}-set of G. Thus, $|V(G)| = r + s + 2 = n$, $\gamma_{fd}(G) = |A| = 2 = k$, and $\gamma_{frd}(G) = |B| = r + s = n - 2 = m$.

This proves the assertion. ∎

The following result is a direct consequence of Theorem 2.5.

Corollary 2.6 The difference $\gamma_{frd}(G) - \gamma_{fd}(G)$ can be made arbitrarily large.

Proof: Let $n = 2r + 2$ and $k = n - 4$ where r is a positive integer and k is a nonnegative integer. By Theorem 2.5, there exists a connected graph G such that $\gamma_{frd}(G) = 2r$ and $\gamma_{fd}(G) = 2$. Thus, $\gamma_{frd}(G) - \gamma_{fd}(G) = 2r - 2 = (n-2) - 2 = n - 4 = k$, showing that $\gamma_{frd}(G) - \gamma_{fd}(G)$ can be made arbitrarily large. ∎

Remark 2.7 $\gamma_{frd}(G) \neq n - 1$ for any connected graph G of order $n \geq 4$.

The path P_n of order $n \geq 1$ is the graph with distinct vertices $v_1, v_2, \ldots, v_n$ and edges $v_1v_2, v_2v_3, \ldots, v_{n-1}v_n$. In this case, P_n is also called a v_1-v_n path or the path $P(v_1, v_n)$.

Remark 2.8 Let $n \geq 4$. Then $\gamma_{frd}(P_n) = \begin{cases} \frac{n+2}{3} & if \quad n \equiv 1 (mod\ 3) \\ \frac{n+4}{3} & if \quad n \equiv 2 (mod\ 3) \\ \frac{n+6}{3} & if \quad n \equiv (mod\ 3) \end{cases}$

The cycle C_n of order $n \geq 3$, is the graph with distinct vertices $v_1, v_2, \ldots, v_n$ and edges $v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1$.

Remark 2.9 Let $n \geq 3$. Then $\gamma_{frd}(C_n) = \begin{cases} \frac{n}{3} & if \quad n \equiv 0(mod\ 3) \\ \frac{n+2}{3} & if \quad n \equiv 1(mod\ 3) \\ \frac{n+4}{3} & if \quad n \equiv 2(mod\ 3) \end{cases}$

The next result is the characterization of the graph G with a fair restrained domination number of one.

Theorem 2.10 Let G be a connected graph of order $n \geq 3$. Then $\gamma_{frd}(G) = 1$ if and only if $G = K_1 + H$ where H is a graph without isolated vertices.

Proof: Suppose that $\gamma_{frd}(G) = 1$. Let $S = \{v\}$ be a γ_{frd}-set in G. Then $\langle V(G) \setminus S \rangle$ is a subgraph without isolated vertices. Let $H = \langle V(G) \setminus S \rangle$. Then $G = K_1 + H$, where H is a graph without isolated vertices.

For the converse, suppose that $G = K_1 + H$, where H is a graph without isolated vertices. Let $S = V(K_1)$. Then S is a fair dominating set of G. Moreover, since $\langle V(G) \setminus S \rangle = H$ has no isolated vertices, it follows that S is a restrained dominating set of G. Thus, $\gamma_{frd}(G) = 1$. ∎

The complete graph of order $n \geq 1$, denoted by K_n, is the graph in which every pair of its distinct vertices are joined by an edge.

The fan of order $n + 1$, denoted by F_n, is the graph $K_1 + P_n$ where $n \geq 1$.

The wheel of order $n + 1$, denoted by W_n, is the graph $K_1 + C_n$ where $n \geq 3$.

The following result is an immediate consequence of Theorem 2.10.

Corollary 2.11 Let G be a connected graph of order $n \geq 3$. Then the following is satisfied.

(i) $\gamma_{frd}(K_n) = 1$
(ii) $\gamma_{frd}(F_n) = 1$
(iii) $\gamma_{fsd}(W_n) = 1$

A graph G is called a bipartite graph if its vertex-set $V(G)$ can be partitioned into two nonempty subsets V_1 and V_2 such that every edge of G has one end in V_1 and one end in V_2. The sets V_1 and V_2 are called the partite sets of G. If each vertex in V_1 is adjacent to every vertex in V_2, then G is called a complete bipartite graph. If $|V_1| = m$ and $|V_2| = n$, then the complete bipartite graph is denoted by $K_{m,n}$.

Remark 2.12 Let $m \geq 2$ and $m \geq 2$. Then $\gamma_{frd}(K_{m,n}) = 2$.

The join of two graphs G and H is the graph $G + H$ with vertex-set $V(G + H) = V(G) \cup V(H)$ and edge-set $E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$.

Remark 2.13 Let G and H be connected graphs. Then $V(G)$ and $V(H)$ are fair dominating sets of $G + H$.

We need the following Lemma for our next Theorem.

Lemma 2.14 Let G and H be nontrivial connected graphs. If $S = S_G \cup S_H$ where S_G is an r-fair dominating set of G, S_H is an s-fair dominating set of H, and $r - s = |S_G| - |S_H|$ then S is a fair restrained dominating set of $G + H$.

Proof: Since S_G is an r-fair dominating set of G, for every $u \in V(G) \setminus S_G$, $|N_G(u) \cap S_G| = r$. Since S_H is an s-fair dominating set of H, for every $v \in V(H) \setminus S_H$, $|N_H(v) \cap S_H| = s$. Now, $S_G \subset V(G)$ implies that $V(G) \setminus S_G \neq \emptyset$. Let $u \in V(G) \setminus S_G$. Then $u \in V(G + H) \setminus S$, $|(N_{G+H}(u) \cap S_H)| = |S_H|$, and

$$\begin{aligned} |N_{G+H}(u) \cap S| &= |N_{G+H}(u) \cap (S_G \cup S_H)| \\ &= |(N_{G+H}(u) \cap S_G) \cup (N_{G+H}(u) \cap S_H)| \end{aligned}$$

$$\begin{aligned} &= |(N_{G+H}(u) \cap S_G)| + |(N_{G+H}(u) \cap S_H)| \\ &= |(N_G(u) \cap S_G)| + |S_H| \\ &= r + |S_H|. \end{aligned}$$

Similarly, since $S_H \subset V(H)$, $V(H) \setminus S_H \neq \emptyset$. Let $v \in V(H) \setminus S_H$. Then $v \in V(G+H) \setminus S$, $|(N_{G+H}(v) \cap S_G)| = |S\_G|$, and

$$\begin{aligned} |N_{G+H}(v) \cap S| &= |N_{G+H}(v) \cap (S_G \cup S_H)| \\ &= |(N_{G+H}(v) \cap S_G) \cup (N_{G+H}(v) \cap S_H)| \\ &= |(N_{G+H}(v) \cap S_G)| + |(N_{G+H}(v) \cap S_H)| \\ &= |S_G| + |(N_H(v) \cap S_H)| \\ &= |S_G| + s \\ &= |S_G| + (r - |S_G| + |S_H|) \\ &= r + |S_H|. \end{aligned}$$

Thus, for every $u, v \in V(G+H) \setminus S$, $|N_{G+H}(u) \cap S| = |N_{G+H}(v) \cap S|$. Hence, S is a fair dominating set of $G+H$. Now, let $u \in V(G) \setminus S_G$ and $v \in V(H) \setminus S_H$. Then $u, v \in V(G+H) \setminus S$ and $uv \in E(G+H)$. Since S is a dominating set, there exists $x \in S$ such that $xu \in E(G+H)$ or $xv \in E(G+H)$. Thus, every vertex in $V(G+H) \setminus S$ is adjacent to a vertex in S and to another vertex in $V(G+H) \setminus S$. Hence, S is a fair restrained dominating set of $G+H$. ∎

The following result is the characterization of the fair restrained domination in the join of two connected non-complete graphs.

Theorem 2.15 Let G and H be nontrivial connected graphs. Then a nonempty proper subset S of $V(G+H)$ is a fair restrained dominating set of $G+H$ if and only if one of the following statement is satisfied.

(i) $S = V(G)$ or S is a k-fair dominating set of G where $k = |S|$.
(ii) $S = V(H)$ or S is a k-fair dominating set of H where $k = |S|$.
(iii) $S = S_G \cup S_H$ where $S_G \subset V(G)$ is an r-fair dominating set of G, and $S_H \subset V(H)$ is an s-fair dominating set of H, and $r - s = |S_G| - |S_H|$.

Proof: Suppose a nonempty proper subset S of $V(G+H)$ is a fair restrained dominating set of $G+H$. Consider the following cases:

Case 1. Consider that $S \cap V(H) = \emptyset$. Then $S \subseteq V(G)$. If $S = V(G)$, then the proof of statement (i) is satisfied. Suppose that $S \neq V(G)$. Let $u \in V(G) \setminus S$ and $v \neq u$ such that $u, v \in V(G+H) \setminus S$. Then, $|N_{G+H}(u) \cap S| = |N_{G+H}(v) \cap S|$ since S is a fair dominating set of $G+H$. If $v \in V(G) \setminus S$, then $|N_G(u) \cap S| = |N_G(v) \cap S| = k$ for some positive integer k. This implies that S is a k-fair dominating set of G. If $v \in V(H)$, then $k = |N_G(u) \cap S| = |N_{G+H}(u) \cap S| = |N_{G+H}(v) \cap S| = |S|$. This complete the proof of statement (i).

Case 2. Consider that $S \cap V(G) = \emptyset$. Then $S \subseteq V(H)$. If $S = V(H)$, then the proof of statement (ii) is satisfied. Suppose that $S \neq V(H)$. Let $u \backslash in\ V(H) \setminus S$ and $v \neq u$ such that $u, v \in V(G+H) \setminus S$. Then, $|N_{G+H}(u) \cap S| = |N_{G+H}(v) \cap S|$ since S is a fair dominating set of $G+H$. If $v \in V(H) \setminus S$, then $|N_H(u) \cap S| = |N_H(v) \cap S| = k$ for some positive integer k. This implies that S is a k-fair dominating set of H. If $v \in V(G)$, then $k = |N_H(u) \cap S| = |N_{G+H}(u) \cap S| = |N_{G+H}(v) \cap S| = |S|$. This complete the proof of statement (ii).

Case 3. Consider that $S \cap V(G) \neq \emptyset$ and $S \cap V(H) \neq \emptyset$. Let $S_G = S \cap V(G)$ and $S_H = S \cap V(H)$. Then $S = S_G \cup S_H$ where $S_G \subset V(G)$ and $S_H \subset V(H)$. Suppose that to the contrary, S_G is not a fair dominating set of G. Then there exists distinct vertices u and v in $V(G) \setminus S_G$ such that $|N_G(u) \cap S_G| \neq |N_G(v) \cap S_G|$. Thus,

$$\begin{aligned} |N_{G+H}(u) \cap S| &= |N_{G+H}(u) \cap (S_G \cup S_H)| \\ &= |(N_{G+H}(u) \cap S_G) \cup (N_{G+H}(u) \cap S_H)| \end{aligned}$$

$$
\begin{aligned}
&= |(N_G(u) \cap S_G) \cup S_H|, \text{ since } u \in V(G) \setminus S \\
&= |N_G(u) \cap S_G| + |S_H| \\
&\neq |N_G(v) \cap S_G| + |S_H| \\
&= |(N_G(v) \cap S_G) \cup S_H| \\
&= |(N_{G+H}(v) \cap S_G) \cup (N_{G+H}(v) \cap S_H)|, \text{ since } v \in V(G) \setminus S \\
&= |N_{G+H}(v) \cap (S_G \cup S_H)| \\
&= |N_{G+H}(v) \cap S|
\end{aligned}
$$

This contradict to our assumption that S is a fair dominating set of $G + H$. Therefore, S_G must be a fair dominating set of G. Similarly, S_H is a fair dominating set of H. Thus, for every vertex $u \in V(G) \setminus S_G$, $|N_G(u) \cap S_G| = r$ for some positive integer r, and for every vertex $v \in V(H) \setminus S_H$, $|N_H(v) \cap S_H| = s$ for some positive integer s. This implies that S_G is an r-fair dominating set of G and S_H is an s-fair dominating set of H. Now, let $u \in V(G) \setminus S_G$ and $v \in V(H) \setminus S_H$. Then,

$$
\begin{aligned}
|N_{G+H}(u) \cap S| &= |N_{G+H}(u) \cap (S_G \cup S_H)| \\
&= |(N_{G+H}(u) \cap S_G) \cup (N_{G+H}(u) \cap S_H) \\
&= |(N_G(u) \cap S_G) \cup S_H| \\
&= |(N_G(u) \cap S_G| + |S_H| \\
&= r + |S\_H| \text{ and,}
\end{aligned}
$$

$$
\begin{aligned}
|N_{G+H}(v) \cap S| &= |N_{G+H}(v) \cap (S_G \cup S_H)| \\
&= |(N_{G+H}(v) \cap S_G) \cup (N_{G+H}(v) \cap S_H)| \\
&= |S_G \cup (N_H(v) \cap S_H)| \\
&= |S_G| + |N_H(v) \cap S_H| \\
&= |S_G| + s.
\end{aligned}
$$

Since S is a fair dominating set of $G + H$, \$$|N_{G+H}(u) \cap S| = |N_{G+H}(v) \cap S|$, that is, $r + |S_H| = |S_G| + s$. Hence, $r - s = |S_G| - |S_H|$ proving statement (iii).

For the converse, suppose that statement (i) is satisfied. If $S = V(G)$, then S is a fair dominating set of $G + H$ by Remark 2.13 and clearly, a restrained fair dominating set of $G + H$. Suppose that S is a k-fair dominating set of G where $k = |S|$. Let $u \in V(G + H) \setminus S$. If $u \in V(G) \setminus S$, then $|N_G(u) \cap S| = |N_G(u) \cap S| = k = |S|$. If $u \in V(H)$, then $|N_{G+H}(u) \cap S| = |S|$. Thus, for any $u, v \in V(G + H) \setminus S$, $|N_{G+H}(u) \cap S| = |N_{G+H}(v) \cap S|$, that is, S is a fair dominating set of $G + H$. Let $x \in V(G + H) \setminus S$. If $x \in V(H)$, then there exists $v \in S$ such that $xv \in E(G + H)$ since $S \subset V(G)$ and there exists $y \in V(H)$ such that $xy \in E(H) \subset E(G + H)$ since H is nontrivial connected graph. If $x \in V(G) \setminus S$, then there exists $v \in S$ such that $xv \in E(G) \subset E(G + H)$ since S is dominating and $xy \in E(G + H)$ for all $y \in V(H)$. Thus, S is a fair restrained dominating set of $G + H$. Similarly, if statement (ii) is satisfied, then S is a fair restrained dominating set of $G + H$. Finally, if statement (iii) is satisfied, then S is a fair restrained dominating set of $G + H$ by Lemma 2.14. ■

The next result is a consequence of Theorem 2.15.

Corollary 2.16 Let G and H be a nontrivial connected graphs. If S_G is an r-fair dominating set of G or S_H is an s-fair dominating set of H with $|S_G| - |S_H| = r - s$, then $\gamma_{frd}(G + H) \leq min\{|S_G|, |S_H|, |S_G| + |S_H|\}$.

Proof: Suppose that S_G is an r-fair dominating set of G. If $r = |S\_G|$, then S_G is a fair restrained dominating set of $G + H$ by Theorem 2.15(i). Thus, $\gamma_{frd}(G + H) \leq |S_G|$. If $r \neq |S_G|$, then consider that S_H is an s-fair dominating set of H. If $s = |S_H|$, then S_H is a fair restrained dominating set of $G + H$ by Theorem 2.15(ii). Thus, $\gamma_{frd}(G + H) \leq |S_H|$. If $s \neq |S_H|$, then let $S = S_G \cup S_H$. Since $S_G \subset V(G)$ is an r-fair dominating set of G, $S_H \subset V(H)$ is an s-fair dominating set of H, and $r - s = |S_G| - |S_H|$, it follows that S is a fair restrained dominating set of $G + H$ by Theorem 2.15 (iii). Thus, $\gamma_{frd}(G + H) \leq |S| = |S_G \cup S_H| = |S_G| + |S_H|$. Hence, $\gamma_{frd}(G + H) \leq min\{|S_G|, |S_H|, |S_G| + |S_H|\}$. ■

III. CONCLUSIONS

In this work, we introduced a new parameter of domination of graphs - the fair restrained domination of graphs. The fair restrained domination in the join of two graphs was characterized. The exact fair restrained domination number resulting from this binary operation of two graphs was computed. This study will pave a way to new research such bounds and other binary operations of two graphs. Other parameters involving fair restrained domination in graphs may also be explored. Finally, the characterization of a fair restrained domination in graphs and its bounds is a promising extension of this study.

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