Complex Basis For Spectral Analysis of Graph Signals

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Abstract — *The Signal Processing on Graph (SPG) is an emerging field of research aiming to develop accurate methods for big data analysis by combining graph theory and classical signal processing methods. One key method in signal processing on graph is the so-called Graph Fourier Transform (GFT) which is a generalization of the Classical Fourier Transform (defined for data lying on regular domains :1D for times series or 2D for images) to data lying on networks. Those network data are viewed like a set of N interrelated data points lying on a graph whose graph vertices map the data points and graph links encode the relationship between data. In the classical framework, the Fourier transform is a linear operator that performs the mapping of a vector from its initial representation domain to the frequency domain through the Fourier matrix which is an orthonormal basis formed by complex exponential vectors constructed from powers of the complex number $\omega = e^{2\pi i/N}$. Those vectors are of a key importance in the properties of the transform and its applications. However, for each graph Fourier transform proposed in the literature, although its graph Fourier matrix is orthonormal, its vectors are not complex as in the classical framework, limiting the extension and the use of some useful properties of the classical Fourier transform to the graph signals framework. In this work, we present a method to define a complex orthonormal basis for the graph Fourier transform that allows to perform spectral analysis for graph signals in the frequency domain. The graph Fourier basis we defined is identical to the Fourier basis when applied to graph signals defined on a regular domain. We applied the proposed method successfully to signal detection on an irregularly sampled sensor network.*

Keywords — *Fourier basis, linear operator, graph signal processing, Laplacian matrix*.

I. INTRODUCTION

The ongoing and unstoppable progress in microelectronics and computerized technologies has led to the *explosion of numeral data*, also known as *big data*, characterized by their higher volume, velocity and variety. In order to analyze such complex data, the signal processing on graph community researchers try to generalize the classical signal processing methods to data lying on irregular domains for application in various areas such as transportation network, social networks, biological networks, smart grids, internet of things, etc.

Graphs and their algebraic properties are pertinent tools to capture and analyze the irregular structure of data living on networks. Basically, a signal $x: V \to \mathbb{R}$ or $\mathbb{C}$ on a graph $G = (V, E)$ is defined such as: $x = [x_0, x_1, \ldots, x_{N-1}]^T$, is a set of N inter-related data points, lying on the N vertices of a graph G in which the links between vertices map the relations between data points. The graph $G = (V, E)$ is characterized by the set of vertices (or nodes) V, the number of vertices $n = card(V)$, the set of links (or edges) E, and the number of links $l = card(E)$. Two nodes v_i, v_j or simply i, j are connected $(i \sim j)$ if $\exists\, e_{ij} \in E \,/\, e_{ij} = w(i, j)$, where $w(i, j) \in W: V \times V \to \mathbb{R}$ is a scalar measuring the weight of the links i.e. how important is the relation between node i and node j in the data structure. The degree or the strength of a node i is the sum of its weighted relations with other nodes, given by $d_i = \sum_i w(i, j)$. A graph in which $e_{ij} = 1$ if $i \sim j$ and 0 otherwise is said binary. Furthermore, a graph with directed links is said directed, else the graph is said undirected. A pair of nodes can be connected by multiple links; indeed, if a link connects a node to itself, it's a self-loop. A graph is said simple if it has no multiple links nor self-loop. In this work we consider only simple, undirected and binary graphs since most networks encountered in real life usually meet these properties and the extension of the proposed scheme to other type of graph can be done naturally. Many important properties of a graph are encoded by their related matrices. Among them, the mostly used in graph signal processing are the adjacency matrix and the Laplacian matrix. The adjacency matrix A of a graph G of N vertices is a $N \times N$ squared matrix whose elements are: $a_{ij} = 1$ if $i \sim j$ and 0 otherwise. Then the Laplacian matrix is given by $L = D - A$, where $D = diag(\{d_i\}_{i=0,1,\ldots,N-1})$. The Laplacian matrix can be normalized by two ways to give: the normalized Laplacian matrix $\mathcal{L} = D^{-1/2} L D^{-1/2}$ and the random walk Laplacian $\mathcal{L}_{RW} = D^{-1} L = I - D^{-1} A$, where I is the $N \times N$ identity matrix. More information on graph theory can be found in [1],[2],[3].

The graph signal processing aims to generalized the classical signal processing concepts on data lying on irregular domain represented by graphs so that classical signals defined on regular domains can be viewed as particular graph signals [4]. This leads to a natural question: what graph represents the regular structure of a classical 1D signal? The answer comes from the observation that in the classical form, the inverses discrete Fourier transform $\check{x}$ of a signal x of N sample is N-periodic since the discrete Fourier transform periodizes the signal x[5]. So, the best way to tackle the Fourier transform is to consider a classical signal as a N-periodic 1D series, and the graph representing a periodic or a cyclic structure is the cyclic graph. Thus, a classical 1D signal of N samples is equivalent to a signal on graph consisting of a set of N data points lying on a cyclic graph of N vertices. Fig. 1 shows 64 samples of a sine wave in the 1D time domain (fig 1 a) and the same signal in the graph domain, embedded on 64 nodes of a cyclic graph (fig 1 b). Regarding these considerations, the generalization of a classical signal processing method to graph signals imply the results obtain in the classical framework should be similar to the signals defined on a cyclic graph. However, it's not always the case in related works concerning the graph Fourier transform. Thus, our main goal is to propose a method that match the above condition, and that will allow the embedding on the graph signal processing framework of some useful properties of the classical framework such spectral analysis and to see some of its applications.

II. RELATED WORKS

Many methods have been extended from the classical signal processing framework to analyze graph signals. In general, those methods can be categorized in two kinds. We have first, transformed-based methods such as: graph Fourier transform [6],[7],[8], graph wavelet transforms [9], [10,[11], multiscale decomposition [12],[13],[14]. Then, we have model-based methods including among others: neural network for graph signal analysis [15],[16],[17], supervised and semi-supervised learning [18],[19], [20,[21]. Although the two kinds of methods are sometimes interrelated or combined in some situations, we focus in this work on transformed-based methods. Among transform-based methods, the graph Fourier transform is of a great importance since it's the base to defined all other transform-based method. In fact, even in the classical framework, the Fourier transform is related to many other transforms [20] such as Gabor transform, wavelet transform, Radon transform.

The fundamental method that gives rise to the graph signal processing is the so-called graph Fourier transform. In the literature two types of graph Fourier transform (GFT) are given: one based on the adjacency matrix A [7], and another based on the Laplacian matrix L [23]. But the idea behind is the same: since the matrices A and L are both symmetric [24], the spectral theorem states that they are diagonsable in an orthonormal basis F which spans the N-diemsionnal space, indeed F is always invertible, so the defined transform too. The basis F is taken as the graph Fourier transform matrix to define a linear operator that maps a given signal on graph $x = (x_0, x_1, \ldots, x_{N-1})^T$ to its graph Fourier transform $\hat{x} = Fx$. The inverse graph Fourier transform is then given by $\check{x} = F^{-1}\hat{x} = x$. At this step, some observations can be done:

Remark 1. The basis F is defined on the Euclidean space, so the vectors are real and not complex as in the classical form, also the basis F is not unique, while it is in the classical form.

Let's now consider the most popular graph Fourier transform which is based on the Laplacian matrix L [23], [25]. The Laplacian matrix $L = D - A$ is symmetric because A and D are both symmetric. For a given graph signal x, the quadradic form

$$Q = x^T L x = \frac{1}{2}\sum_{i,j}|x_i - x_j|^2 \tag{1}$$

is viewed as a local variation of the signal on graph x. So, a high (resp. low) variation of two signal samples lying on two connected nodes i and j i.e. large (resp. small) $|x_i - x_j|^2$ will lead to a high (resp. low) value of Q corresponding to a high (resp. low) frequency. Indeed L is positive semidefinite, which implies that all its eigenvalues ($\mu_0 \leq \mu_1 \leq \ldots \leq \mu_{N-1}$) are non-negative and at least one is zero because $\det(L) = 0$ and the set of corresponding eigenvector $U = \{u_0, u_1, \ldots, u_{N-1}\}$ form an orthonormal basis of the Euclidean space. Orthogonality in very important in transform-based signal processing methods as stated by the following theorem 1.

Theorem 1. Lets $U = \{u_0, u_1, \ldots, u_{N-1}\}$ be an orthonormal subset of a vector space with the dot product $(V, \langle, \rangle)$, $\dim(V) = N$, then $\forall v, w \in V$

 a. U spans V

$$v = \sum_{i=0}^{N-1}\langle v, u_i \rangle u_i$$

b. *Parseval Theorem*

$$\langle v, w \rangle = \sum_{i=0}^{N-1} \langle v, u_i \rangle \langle u_i, w \rangle$$

The Parseval Theorem is the energy conservation property of an orthogonal linear transform: so, all the information content of the signal is present in the transform or the projection space and can be retrieved by the reverse transform.

c. *Plancherel theorem*

$$\|v\|^2 = \sum_{i=0}^{N-1} |\langle v, u_i \rangle|^2$$

The Plancherel theorem is the energy decomposition properties of an orthogonal linear transform: the orthonormal basis decomposes the signal energy in the N directions of the orthonormal basis vectors. So, this could allow to view some details in the transform space that are unreveals in the original space.

Proof of a. It's obvious, since $U = \{u_0, u_1, \ldots, u_{N-1}\}$ is orthonormal $\langle v, u_i \rangle u_i = v_i u_i$.

Proof of b. We have:

$$\langle v, w \rangle = \langle \sum_{i=0}^{N-1} \langle v, u_i \rangle u_i, w \rangle \; using \; the \; sesquilinerity \; property \; of \; the \; dot \; product \; i.e. \langle \sum_{i=0}^{N-1} \alpha_i v_i, w \rangle = \sum_{i=0}^{N-1} \alpha_i \langle v_i, w \rangle$$

$$we \; then \; have \; \langle v, w \rangle = \langle \sum_{i=0}^{N-1} \langle v, u_i \rangle u_i, w \rangle = \langle v, w \rangle = \sum_{i=0}^{N-1} \langle v, u_i \rangle \langle u_i, w \rangle$$

Proof of c.

$$\|v\|^2 = \langle v, v \rangle = \sum_{i=0}^{N-1} \langle v, u_i \rangle \langle u_i, v \rangle = \sum_{i=0}^{N-1} |\langle v, u_i \rangle|^2$$

Corollary 1. Any reversible transform based-method for data processing that preserves the information content of data fills the Parseval and the Plancherel theorems.

The GFT fills the Parseval and Plancherel Theorem. Moreover, the GFT definition is such that: local variation for constant signal is zero since $Q = 0 \; if \; x_i = x_j$. This is a nice property, since it matches definition of frequency in the classical Fourier transform framework, i.e. a constant signal corresponds to minimal local variation and thus zero frequency. On the other hand, let's consider a Dirac pulse δ_i localized on one vertex of the graph, the corresponding local variation is $Q = \delta_i^T L \delta_i = d_i$ i.e. the degree of node i [25], this result is far from the classical framework in which a Dirac pulse define a high localized variation corresponding to an infinite frequency. Indeed fig. 2 (columns 6 and 7) shows that the GFT give results which don't match the classical framework when applied to basic signals on the cyclic graph. This result is due to the fact that the graph Fourier matrix F lies in a Euclidean space, the resulting transformed signal $\hat{x}$ is real, there is no way to perform spectral amplitude and spectral phase analysis. Therefore, many properties of the classical Fourier transform which are due to the complex exponential form of the Fourier basis are then lost.

In summary, we want to perform a graph Fourier transform in which the Fourier basis lies in a Unitary space to allow spectral analysis of graph signals. Indeed, the proposed transform should give the same results as in the classical framework when applied to the cyclic graph. In the next paragraph we present some important preliminaries on which we based to build the method.

III. PRELIMINARIES

III.1. Background: The classical Discrete Fourier Transform (DFT). We begin by presenting some properties of the classical Fourier transform we have used to develop the proposed technique.

Definition 1. G iven a vector $x = (x_0, x_1, \ldots, x_{N-1})^T$ in the canonical basis of an N-subspace T of the unitary space, the DFT is a linear operator that maps the initial subspace T, which is usually the ''time

$$\begin{aligned} T &\to W \\ x &\to \hat{x} = F_N x \end{aligned} \qquad (2)$$

domain'' to another subspace W known as the ''frequency domain'' though the $N \times N$ Fourier matrix F_N:

$$\text{with } F_N = \frac{1}{\sqrt{N}} e^{-2\pi i \frac{nm}{N}} \begin{cases} n = 0, 1, \cdots, N-1 \\ m = 0, 1, \cdots, N-1 \end{cases} = \frac{1}{\sqrt{N}} (\omega^{(nm)})_{0 \le n,m \le N-1} = \frac{1}{\sqrt{N}} (v_0|v_1|\cdots|v_{N-1}) \tag{3}$$

The column vectors $v_k = (1, \omega^k, \omega^{2k}, \ldots, \omega^{(N-1)k}), k = 0, 1, \ldots, N-1$ of F_N are constructed from power of the complex numbers $\omega = e^{-2\pi i/N}$.

Property 1. The classical Fourier transform matrix F unitary.

Proof. Let's write $F_N F_N^* = (f_{n,m})_{1 \le n,m \le N}$. Since $F^H = \frac{1}{\sqrt{N}} (\omega^{-nm})_{0 \le n,m \le N-1}$, then

$$f_{n,m} = \frac{1}{N} \sum_{k=1}^{N} \omega_N^{(n-1)(k-1)} \omega_N^{-(k-1)(m-1)} = \frac{1}{N} \sum_{k=1}^{N} \omega_N^{(n-m)(k-1)}$$

so, if $n = m$, then $f_{n,m} = 1$ and and $n \ne m$ then $f_{n,m} = 0$, thus $F_N F_N^* = F_N^* F_N = I_N$ where F_N^* is adjoint matrix of F_N and I_N the $N \times N$ identity matrix.

Corollary 2. The classical Fourier transform is invertible, and meets Parseval and Plancherel theorems.

Remark 2. The Fourier matrix F_N is a Vandermonde matrix defined from the vector v_k. So, if $p(x) = \sum_{i=0}^{N-1} a_i x^i$ is the polynomial associated to the vector x, then $\hat{x}$ is the vector whose components correspond to the evaluation of x at the Nth unity roots.

III.2 Relations with some particular operators. The interaction of the classical Fourier transform with some specific linear operators is important to understand the basic properties of the Graph Fourier transform.

The Fourier transform and the shift matrix. Let's consider the circular shift matrix and its powers:

$$P = \begin{pmatrix} 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, P^2 = \begin{pmatrix} 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 & 0 \end{pmatrix}$$

$$P^{N-1} = P^{-1} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}, P^N = P^0 = I \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & \vdots \\ \vdots & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}$$

Applied to a column vector $c = [c_0, c_1, c_2, \ldots, c_N]^T$, the circular shift matrix operate a circular shift by one position of the vector components so that $Pc = [c_{N-1}, c_0, c_1, \ldots, c_{N-2}]^T$

Lemma 1. The Fourier matrix F diagonalizes the circular shift matrix P : i.e. $FP = DF$ with $D = diag(1, \omega, \omega^2, \ldots, \omega^{N-1})$

Proof. The characteristic polynomial of P is $x^N - 1$, thus the eigenvalues of P are the Nth unity root and the eigenvectors are the column vectors of F since each column vector of F, is proportional N^{th} unity root, thus $P = F^* DF$ with $D = diag(1, \omega, \omega^2, \ldots, \omega^{N-1})$

A circulant matrix C is an $N \times N$ matrix with the form

$$C = \begin{pmatrix} c_0 & c_{N-1} & c_{N-2} & \cdots & c_1 \\ c_1 & c_0 & c_{N-1} & \cdots & c_2 \\ c_2 & c_1 & c_0 & \ddots & c_3 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ c_{N-1} & c_{N-2} & c_{N-3} & \cdots & c_0 \end{pmatrix}$$

In which the first column is the vector $c = [c_0, c_1, c_2, \ldots, c_{N-1}]^T$ and each column is a one position circular shift of the previous. In fact, $c_{jk} = c_{(j-k) \bmod N}$.

Lemma 2. Every circulant matrix C, defined by the column vector $c = [c_0, c_1, c_2, \ldots, c_{N-1}]^T$ is a polynomial of the shift matrix with c components as coefficients:

$$C = c_0 P^o + c_1 P^1 + \cdots + c_{N-1} P^{N-1} = \sum_{i=0}^{N-1} c_i P^i$$

Proof. Observing that
$$P^k e_j = e_{(j+k) \bmod N}, \text{ we then have: } (c_0 P^o + c_1 P^1 + \cdots + c_{N-1} P^{N-1}) e_j = c_0 + c_1 e_{j+1} + \cdots +$$
$$c_{N-j} e_N + c_{N-j+1} e_1 + \cdots + c_{n-1} e_{j-1} = P^{j-1} c = C e_j$$

thus the jth column of C and $c_0 P^o + c_1 P^1 + \cdots + c_{N-1} P^{N-1}$ are equal.

Theorem 2. Given a $N \times N$ circulant matrix C defined by its first column vector c, then it is diagonalizable by F and its eigenvalues are given by the Fourier transform of the vector c.
More precisely:
$$C = F^* D F \text{ with } D = diag(\hat{c} = Fc) \tag{4}$$

Proof.
$$FCF^* = F \left(\sum_{i=0}^{N-1} c_i P^i \right) F^* = \sum_{i=0}^{N-1} c_i (FPF^*)^i = \sum_{i=0}^{N-1} c_i D^i = diag(Fc) = diag(\hat{c})$$
with $D = diag(1, \omega, \omega^2, \ldots, \omega^{N-1})$ and $D^k = diag(1, \omega^k, \omega^{2k}, \ldots, \omega^{(N-1)k})$

Corollary 3. All circulant matrices share a common set of eigenvectors F, and any matrix of the form $F^* \Psi F$ is circulant.

Corollary 4. Given a circulant matrix $C(c)$ defined by its first column $c = [c_0, c_1, c_2, \ldots, c_{N-1}]^T$ and its Fourier transform $\hat{c} = Fc = [\hat{c}_0, \hat{c}_1, \hat{c}_2, \ldots, \hat{c}_{N-1}]^T$. The inverse of $C(c)$ is a circulant matrix $C^{-1}(c) = C(c^{\#})$ with:
$$c^{\#} = F^* \left(\frac{1}{\hat{c}_0}, \frac{1}{\hat{c}_1}, \cdots, \frac{1}{\hat{c}_{N-1}} \right)^T$$

Proof.
$C(c) = F^* DF, \text{ with } D = diag(Fc = \hat{c}), thus$
$$C^{-1}(c) = F^* D^{-1} F = F^* diag \left(\left(\frac{1}{\hat{c}_0}, \frac{1}{\hat{c}_1}, \cdots, \frac{1}{\hat{c}_{N-1}} \right)^T \right) F = F^* diag \left(FF^* \left(\frac{1}{\hat{c}_0}, \frac{1}{\hat{c}_1}, \cdots, \frac{1}{\hat{c}_{N-1}} \right)^T \right) F = C(c^{\#})$$

Corollary 4. The set of circulant matrices formed a cumulative algebra.
Proof.
Given two circulant matrices C, C' defined by two vectors $c \text{ and } c'$. It's obvious that $C + C'$ is circulant. Indeed
$$CC' = F^* DFF^* D' F = F^* DD' F = F^* D' DF = C' C,$$
$$\text{with } D = diag(Fc) \text{ and } D' = diag(Fc')$$
Thus $CC' \text{ and } C'C \text{ are both circulants and commutative}$.

The discrete Fourier transform and stationary operators. The DFT has a particular relationship with a class of linear operator said ''stationary''.

Definition 2. A linear operator T acting on a signal z is stationary if its action on z is independent on the time at which the signal is applied to T. If S_k is the translation operator of k position ($k \in \mathbb{Z}$), then the stationarity of an operator T is defined by:
$$T(S_k z) = S_k(Tz), \quad \forall z \in \mathbb{C}^N,$$
$i.e. T \text{ is stationary if it communtes with any translaton operator } S_k: T \circ S_k = S_k \circ T, \quad k \in \mathbb{Z}$

A stationary operator always has the following expression:
$$(Tz)(n) = \sum_{k=0}^{N-1} a_k z(n-k) = \sum_{k=0}^{N-1} a_k S_k z(n) \qquad n = 0, 1, \cdots, N-1$$

Theorem 3 Given a linear operator T, the following properties are all equivalents:
1) T is stationary
2) The matrix $C(c)$ of T in the canonical basis is circulant, with c being the first column of C
3) The vector c is the impulse response of T i.e. $c = T\delta = the\ first\ column\ of$ C
4) T is a convolution operator with the vector c i.e. $Tz = c * z = z * c$
5) T is a Fourier multiplicator by $\hat{c} = Fc$ i.e. $Tz = F^{-1}(\hat{c}.\hat{z})$
6) T is diagonalizable by the Fourier matrix and the eigenvalues of C are components of $\hat{c}$.

Proof. See [26].

Let's consider a N-periodic time series z. The first discrete derivative (the gradient operator) of z is given by:

$$T_1 z(n) = z(n+1) - z(n)$$

The second discrete derivative (the Laplacian operator) of z is:

$$T_2 z(n) = T_1 z(n) - T_1 z(n-1)$$

$$T_2 z(n) = z(n+1) - z(n) - \big(Z(n) - Z(n-1)\big) = z(n+1) - 2z(n) + z(n-1)$$

The matrices of T_1 and T_2 in the canonical basis are:

$$C_1 = \begin{pmatrix} -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \cdots & \vdots \\ \vdots & 0 & -1 & \ddots & 0 \\ 0 & \vdots & \ddots & \ddots & 1 \\ 1 & 0 & \cdots & 0 & -1 \end{pmatrix}, C_2 = \begin{pmatrix} -2 & 1 & 0 & \cdots & 1 \\ 1 & -2 & 1 & \cdots & 0 \\ 0 & 1 & -2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 \\ 1 & 0 & \cdots & 1 & -2 \end{pmatrix} \quad (5)$$

Thanks to theorem 2, The matrices C_1 *and* C_2 are stationary operators since their matrices are both circulant. Their spectra are given the du Fourier transform of the first matrix column:

$$(\lambda_1)_m = e^{2\pi i \frac{m}{N}} - 1, \quad m = 0,1,\cdots,N-1, for\ T_1 \quad (6)$$

$$(\lambda_2)_m = -2 + 2\cos\left(\pi\frac{m}{N}\right), \quad m = 0,1,\cdots,N-1, for\ T_2 \quad (7)$$

The spectrum of the second order derivative is real since the matrix C_2 is symmetric. In [26], it's shown that both first order and second order derivatives are high pass filter and that the second order derivation amplifies the high frequencies (resp. attenuates the low frequencies) two times the first order.

After this overview of the classical Fourier transform and its main properties, lets now study the spectra of some graphs of particular interest.

III.3 Spectrum of the graph Laplacian. The spectrum of a graph is the eigen structure of its matrices. The cyclic graph is of a key importance in our analysis since it's the graph that matches the 1D time series in the classical Fourier framework. We will show some important properties of its spectrum. *We focus on the spectrum of the Laplacian* matrix because it has more information than the other graph related matrix such as the adjacency matrix.

Given that the Laplacian matrix L of a graph is always symmetric, singular ($as\ \det(L) = 0$) and positive semidefinite, it's diagonalizable in an orthonormal basis U of the Euclidian space. The eigenvalues $((\mu_m)_{m=0,1,\cdots,N-1} = \mu_0 \leq \mu_1 \leq \ldots \leq \mu_{N-1})$ are all non-negative. The eigenvectors are all real and form the orthonormal basis $U = \{u_0, u_1, \ldots, u_{N-1}\}$ thus:

$$L = U^{-1}\Lambda U, where\ \Lambda = diag((\mu_m)_{m=0,1,\cdots,N-1}) \quad (8)$$

The smallest eigenvalue is $\mu_0 = 0$ and the corresponding normalized eigenvector is $u_0 = \frac{1}{\sqrt{N}}[1,1,\ldots\ldots,1]^T$.

Also, the sum of each eigenvector components is zero since it should be orthogonal to u_0. In the graph signal processing community, the matrix U is usually referred to as the graph Fourier matrix since it spans the Euclidean space and it preserves metrics.

Remark 3. The first column vectors are exactly the same in both the graph Fourier matrix U and the classical Fourier matrix F.

Remark 4: The matrix U is orthogonal and spans the Euclidean space, while the matrix F is unitary and spans the unitary space. So many properties and analysis performed in the classical Fourier framework can't be performed using the graph matric U since its vectors are all real instead of complex.

Let's now look in depth at the spectrum of a graph of a particular interest: the cyclic graph.

The spectrum of a cyclic graph. A cyclic graph is a graph that has a ring topology where each node on a circle is connected to its previous and its subsequent neighbor. Hence the cyclic graph is a special case of regular graph of degree 2, indeed it is also a small world graph [27]. A small world graph is a graph in which each node on a circle is connected to its k previous and its k subsequent neighbor. The Laplacian matrix of a cyclic graph is such as:

$$L = \begin{pmatrix} 2 & -1 & 0 & \dots & -1 \\ -1 & 2 & -1 & \dots & 0 \\ 0 & -1 & 2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & -1 \\ -1 & 0 & \dots & -1 & 2 \end{pmatrix} \qquad (9)$$

We can see, the Laplacian matrix of a cyclic graph is the opposite second order operator (eq. 5), it's a circulant and symmetric matrix. The set of its eigenvalues:

$$(\mu_m) = 2 - 2\cos\left(\pi \frac{m}{N}\right), \quad m = 0,1,\cdots, N-1 \qquad (10)$$

are real sine L is symmetric. The Laplacian matrix of the cyclic graph L is a stationary operator given that its matrix is circulant. So, the Fourier matrix F diagonalized it, it is a Fourier multiplicator and a convolution operator (theorem 2).

Remark 5. As seen in the paragraph 3.2, this matrix acts like a high filter which amplify (resp. attenuate) high (resp. low) frequencies.

Considering the remarks 1, 2, 3, 4 and 5, we can summarize by saying that, the graph Fourier transform in its actual form is a stationary operator that acts like a high frequency filter on graph signals and more over it can't allow spectral analysis (amplitude, phase and power spectrum analysis for deterministic signal, power density spectrum of the autocorrelation function for stochastic signals). Because of these observations, when applying the graph Fourier transform on common basic signals (rectangular pulse, sinus cardinal, step function, Dirac pulse, sinus function) on a cyclic graph, the results obtain are very different from the classical framework (see fig. 2). So, the aim of the next paragraph is to define a graph Fourier matrix (formed by complex orthonormal vector in the unitary space) that will give the same results as those of the classical framework when applied on the cyclic graph.

IV. PROPOSED METHOD

To setup the proposed method, we should first understand the relation between complex basis and real basis.

IV.1. From a complex to a real basis.

Theorem 3. Every linear operator A in the Unitary space can be represented in the form:

$$A = A_1 + iA_2$$

where A_1 and A_2 are the hermitian operator of A

Proof

if $A = A_1 + iA_2$, then $A^* = A_1 - iA_2$, therefore $A_1 = \frac{1}{2}(A + A^*)$ and $A_2 = \frac{1}{2}(A - A^*)$

The representation of a linear operator A in the form $A = A_1 + iA_2$, is an analogue to the representation of a complex number $z = z_1 + iz_2$, where z_1 and z_2 are real.

Corollary 5. Since the Fourier matrix $F = \frac{1}{\sqrt{N}}\{v_k = (1, \omega^k, \omega^{2k}, \ldots, \omega^{(N-1)k})\}_{k=0,1,\cdots,N-1}$ is a basis for the Laplacian matrix L of the cyclic graph, the Basis R_F formed by the real part of F and the basis I_F formed by the imaginary part of F are also orthonormal basis of L. :

$$R_F = \{R_e(v_k)\} = \frac{1}{\sqrt{N}}\left\{(1, \cos\frac{2k\pi}{N}, \cos\frac{4k\pi}{N}, \cdots, \cos\frac{2(N-1)k\pi}{N}\right\} \quad k = 0,1,\cdots, N-1 \qquad (11)$$

$$I_F = \{I_m(v_k)\} = \frac{1}{\sqrt{N}}\left\{(0, \sin\frac{2k\pi}{N}, \sin\frac{4k\pi}{N}, \cdots, \sin\frac{2(N-1)k\pi}{N}\right\} \quad k = 0,1,\cdots, N-1 \qquad (12)$$

As shown in [28], the equal pair of eigenvalues $\mu_k = \mu_{N-k}$ gives the two eigenvectors $R_F(k)$ and $I_F(k)$. The first vectors of both bases are: $R_F(0) = \frac{1}{\sqrt{N}}[1,1,\ldots,1]^T$ and $I_F(0)$ is the null vector. For N even we have $R_F(\mu_{N/2}) = \frac{1}{\sqrt{N}}[1,-1,\ldots,1,-1]^T$ and $I_F(\mu_{N/2})$ is the null vector. It's such expressions with null vectors and constant vector, that make the real classical Fourier transform less attractive that the complex form

in which vectors are oscillating sine and cosine waves capturing information variations in the data of interest. **For the same reason, we think that a complex basis for signal defined graph can be more attractive that a real basis**.

The analyses developed in this paragraph show that it's always possible and quite simple to move from a complex basis to a real basis with the same properties, but what about the inverse problem: how to move from a real basis to a complex basis with the same properties.

IV.2. Form a real to a complex basis: extension of a Euclidean linear operator to a unitary space. We are seeking a way to extend a linear operator in a Euclidean space $\mathbb{E}$ to a unitary space $\mathbb{H}$. As shown is [29], this extension is made by the following way:

1. The vectors of $\mathbb{E}$ are 'real' vectors
2. We introduce 'complex' vector $z = x + iy, with\ x \in \mathbb{E},\ y \in \mathbb{E}$
3. The operation of addition of complex vectors and multiplication by a number are defined in the natural way. Then the set of all complex vectors forms an N-diemsional vector space $\mathbb{H}$ over the field of complex numbers which contain $\mathbb{E}$ as subspace
4. In $\mathbb{H}$ we introduce the Hermitian metric given in the following way:

$If\ z = x + iy\ and\ w = u + iv\ (x, y, u, v) \in \mathbb{E}, then$

$\langle zw \rangle = \langle xu \rangle + \langle yv \rangle + i(\langle yv \rangle - \langle xu \rangle)$

$Setting\ \bar{z} = x - iy\ and\ \bar{w} = u - iv, we\ have\ \langle \bar{z}w \rangle = \langle z\bar{w} \rangle$

Theorem 4. Every linear operation A in $\mathbb{E}$ extends uniquely to a linear operator in $\mathbb{H}$:

$$A(x + iy) = Ax + iAy$$

The proof is obvious since A is a linear.

In a real basis, real operators are determined by real matrices, i.e. matrices with real elements.

We consider a real operator A with the eigenvalues:

$\xi_{2k-1} = \alpha_k + i\beta_k, \qquad \xi_{2k} = \alpha_k - i\beta_k, \qquad \xi_l = \alpha_l, \qquad (k = 1,2, \dots, q; l = 2q + 1, \dots, N)$

Where $\alpha_k, \beta_k, \alpha_l$ are real and $\alpha_l \neq 0,\ \ (k = 1,2, \dots, q)$

Then the eigenvectors corresponding to these eigenvalues can be chosen such that:

$z_{2k-1} = x_k + iy_k, \qquad z_{2k} = x_k - iy_k, \qquad z_l = x_l, \qquad (k = 1,2, \dots, q; l = 2q + 1, \dots, N)$

The vectors

$x_1, y_1, x_2, y_2, \dots, x_q, y_q, x_{2q+1}, \dots, x_N, \qquad (k = 1,2, \dots, q; l = 2q + 1, \dots, N)$ (13)

form a basis of the Euclidian space $\mathbb{E}$. Here

$$Ax_k = \alpha_k x_k - \beta_k y_k$$
$$Ay_k = \beta_k x_k + \alpha_k y_k$$
$$Ax_l = \alpha_l x_l$$
(14)

In the basis (11), there corresponds to the operator A the real quasi-diagonal matrix:

$$\left\{ \left\| \begin{matrix} \alpha_1 & \beta_1 \\ -\beta_1 & \alpha_1 \end{matrix} \right\|, \cdots, \left\| \begin{matrix} \alpha_q & \beta_q \\ -\beta_q & \alpha_q \end{matrix} \right\|, \alpha_{q+1}, \cdots, \alpha_N \right\}$$
(15)

Thus, for every operator A of simple structure in a Euclidean space, there exists a basis in which A correspond to a matrix of the form (13). Hence it follows that: a real matrix is real-similar to a canonical matrix of the form (13):

$$A = P \left\{ \left\| \begin{matrix} \alpha_1 & \beta_1 \\ -\beta_1 & \alpha_1 \end{matrix} \right\|, \cdots, \left\| \begin{matrix} \alpha_q & \beta_q \\ -\beta_q & \alpha_q \end{matrix} \right\|, \alpha_{q+1}, \cdots, \alpha_N \right\} P^{-1}\ (P = \bar{P})$$

The transpose operator A^T of A in $\mathbb{E}$ upon extension becomes the adjoint operator A^* *of* A in $\mathbb{H}$. Therefore: Normal, symmetric, skew-symmetric, and orthogonal operators in $\mathbb{E}$ after extension become normal, Hermitian, Hermitian multiplied by i, and unitary operators in $\mathbb{H}$.

All the eigenvalues of a symmetric operator S in a Euclidian space are real, since after the extension, the operator becomes Hermitian. For a symmetric operator, we must set $q = 0$, in (13). Then we obtain:

$$Sx_l = \alpha_l x_l\ [\langle x_k x_l \rangle = \delta_{kl}; k, l = 1,2, \dots, N$$

A symmetric operator S in a Euclidean space always has an orthonormal system of eigenvectors with real eigenvalues.

A real symmetric matrix is always real-similar and orthogonally-similar to a diagonal matrix:

$$S = P\{\alpha_1, \alpha_2, \cdots, \alpha_N\}P^{-1} \quad (P = (P^T)^{-1} = \bar{P})$$

All the eigenvalues of a skew-symmetric operator K in a Euclidean space are pure imaginary (after the extension the operator is i times a Hermitian operator). For a skew-symmetric operator, we must set in (13):

$$\alpha_1, = \alpha_2 = \cdots = \alpha_q = \alpha_{q+1} = \cdots = \alpha_N = 0$$

then the formulas take the form:

$$Kx_k = -\beta_k y_k$$
$$AK = \beta_k x_k$$
$$Kx_l = 0 \quad (k = 1,2,\ldots,q; l = 2q+1,\ldots,N)$$

Since K is a normal operator, the basis (13) can be assumed to be orthogonal. Thus, real skew-symmetric matrix is real-similar and orthogonally-similar to a canonical skew-symmetric matrix:

$$K = P\left\{\left\|\begin{matrix} 0 & \beta_1 \\ -\beta_1 & 0 \end{matrix}\right\|, \cdots, \left\|\begin{matrix} 0 & \beta_q \\ -\beta_q & 0 \end{matrix}\right\|, 0, \cdots, 0\right\}P^{-1} \quad (P = (P^T)^{-1} = \bar{P})$$

All the eigenvalues of an orthogonal operator O in a Euclidean space are of modulus 1 (upon extension the operator becomes unitary)

Therefore, in the case of an orthogonal operator we must set in (14):

$$\alpha_k^2 + \beta_k^2 = 1, \alpha_l = \pm 1, (k = 1,2,\ldots,q; l = 2q+1,\ldots,N)$$

For this basis (13), can be assumed to be orthogonal. The formulas (12) can be represented in the form

$$Ox_k = x_k \cos\varphi_k - y_k \sin\varphi_k$$
$$Oy_k = x_k \cos\varphi_k + y_k \sin\varphi_k$$
$$Ox_l = \pm x_l$$

From what we have shown, it follows that: every orthogonal matrix is real-similar and orthogonally-similar to a canonical orthogonal matrix:

$$O = P\left\{\left\|\begin{matrix} \cos\varphi_1 & \sin\varphi_1 \\ -\sin\varphi_1 & \cos\varphi_1 \end{matrix}\right\|, \cdots, \left\|\begin{matrix} \cos\varphi_q & \sin\varphi_q \\ -\sin q & \cos\varphi_q \end{matrix}\right\|, 0, \cdots, 0\right\}P^{-1} \quad (P = (P^T)^{-1} = \bar{P}) \quad (16)$$

IV.3 The algorithm. From equations (13), (14), (15) and (16), we can derive the algorithm for building a complex unitary basis from a Euclidean orthogonal basis. The implementation of algorithm uses some function of the interesting GSP toolbox [30].

We have used the proposed method to compute the graph Fourier transform of some signals over some well-known graph structures. Moreover, we used the proposed method to detect a signal of interest on a sensor network. The results are presented in the next paragraph.

V. NUMERICAL RESULTS

Algorithm 1. The complex graph Fourier basis.
Input: The graph structure G
Output: The complex graph Fourier matrix F
L←*Laplacian(G);*
[V,D]←*eigenstructure(L);*
while *i<m*
V(:,i)←*(V(:,i)+jV(:,i+1));*
V(:,i+1)←$\overline{V(:,i)}$ *;*
i←*i+2;*
end
for *i*←*1:m*
H(:,i)←*H(:,i)*×$H(1,i)^{-1}$*);*
end

When applying the algorithm above on a cyclic graph, we obtain the classical Fourier matrix. Thus, the graph Fourier transform performed on a cyclic graph using our method is exactly the same as the one performed in the classical framework (figure 2). So, this is the proof that the proposed graph Fourier transform is a generalization of the classical framework on irregular structure. In the following section, we have applied the proposed method to other but well-known graph structures with common signals, to measure how the irregular structure of a graph can affect the spectrum of its signal (figure 3, and figure 4). Finally, we have successfully

tested the proposed method in the detection of an event that the occurrence is sensing by an irregularly sampled sensor network (figure 5).

V.1 Spectral analysis of different graph signals. Figure 2 above shows a comparison of the spectral graph Fourier transform (SGFT) and the graph Fourier transform (GFT) defined in [30], which is the most popular. The comparison focuses on common signal patterns highlighted in the lines of the graphic: the first line is devoted to the Dirac pulse, the second to the step function, the third to the sine cardinal, the fourth to the rectangular pulse, and the fifth to the sine wave. Column 1 and column 3 are the representations of the same signal, but in column 1 the signal is represented in the time domain, whereas in column 3, it is represented in the graph domain. The field of comparisons are: the spectrum of the Fourier transform in the classical framework (CF): column 2, the spectrum of the SGFT in the graph domain: column 4, the spectrum of the SGFT in the 1D regular domain: column 5, the spectrum of the GFT in the graph domain: column 6, and the spectrum of the GFT in the 1D domain: column 7. The first observation is that graphics in columns 2 and 4 are the same on each line, but different from those in column 7. This is because the SGFT gives the same results as the classical framework, given that both have the same basis, but the GFT has a real basis.

In figure 3 above, we have plotted a Dirac pulse and its spectrum on five graph patterns: a full connected graph (line a), a grid graph (line b), a path graph (c), a sensor graph(d) and a community graph (e). We have chosen the Dirac function because it's the most basic signal and its Fourier transform reveals information on the structure of the signal. In the classical framework, the Fourier transform of a Dirac function is the transfer function, thus the model of the system through which the signal propagates. So, the response of a system to a Dirac pulse gives its signature or its nature. We should keep in mind that the Dirac transform of a regular 1D domain is a constant signal, thus a regular 1D domain (or the cyclic graph in the graph signal processing framework) is a constant support or the neutral element that does not affect the signal propagating through it. Then the above figure 3, shows how the irregular structure of different graphs patterns can affect the signal on it. We can see, when observing the spectra of different graph signal in figure 3, that the grid graph and the path graph give responses that tends to be regular, in opposite of the other graph patterns.

V.2 Application to event detection on a sensor network.We have tried to show the application of the method on graph sgnal detection. Signal detection is widely applied in the classical framwork. We supposed we have an irregularly sampled sensor network that surveys the occurrence of a critical phenomenon, such as a seism or a volcano eruption. The signal of interest that indicate the normal activity on the sensor network is assumed to be a sine wave $x = \sin(240\pi t)$ of frequency $120Hz$ plotted on the network sensor (figure. 4.a) and on the 1D regular domain (figure 4.b). This signal can be spread into a random noise signal indicating the occurrence of an abnormal activity on the sensor network. The noisy signal is assumed to be white gaussian with standard deviation 2: $n = 2rand(t)$, so that the recorded signal is: $y = x + n$ (figure 3.d and 3.e) we can observe that it is difficult to separate the signal of inetrest x, from the noisy signal y (figure 3.e). The power sepectral density of the signal x is plotted in figure 3.c, and the one of y in figure 3.f using the proposed method. In figure 3.f and 3.g, we clearly identify the signal of interest and the noise thanks to the proposed scheme.

• Samples in the time domain   b) Samples in the graph domain
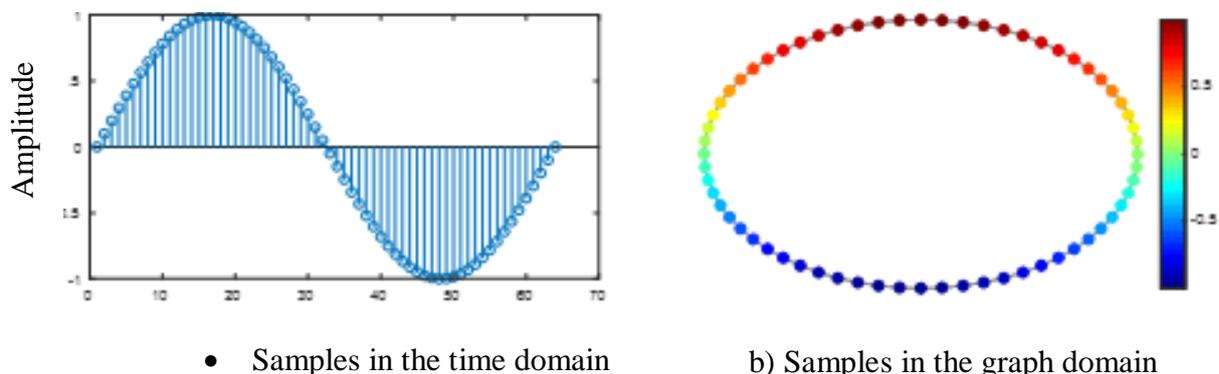
Figure 1: a classical sine wave in the time domain (a) and in the graph domain (b)
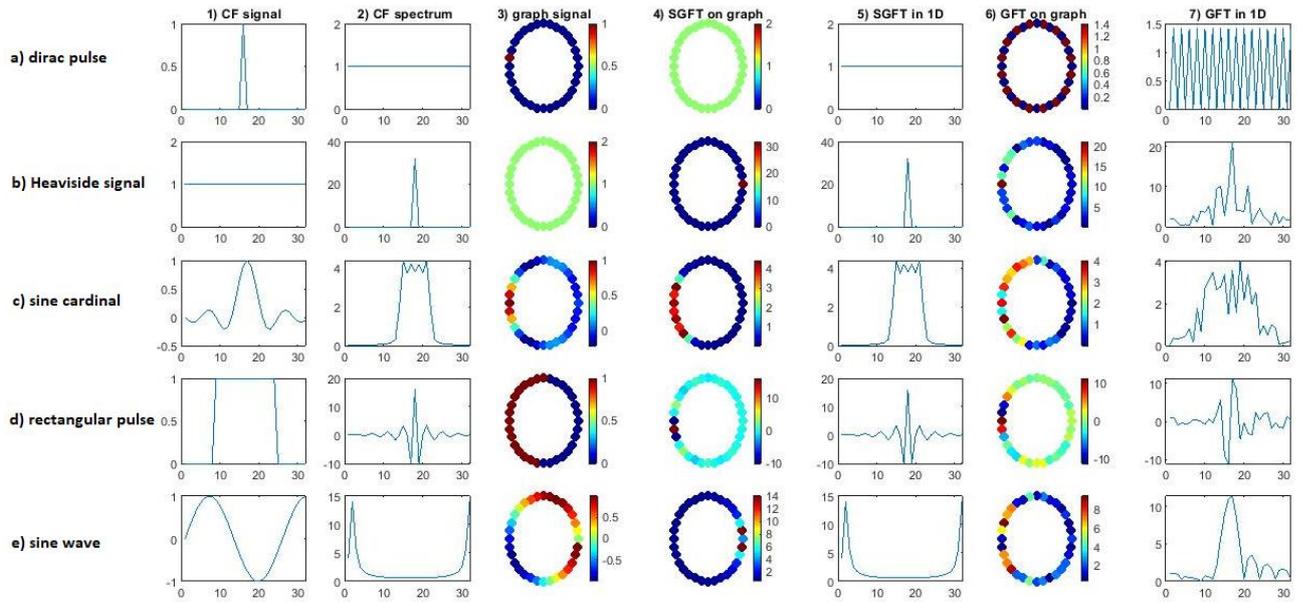
Figure 2: Comparing the spectral graph Fourier transform (SGFT) and the graph Fourier transform (GFT)
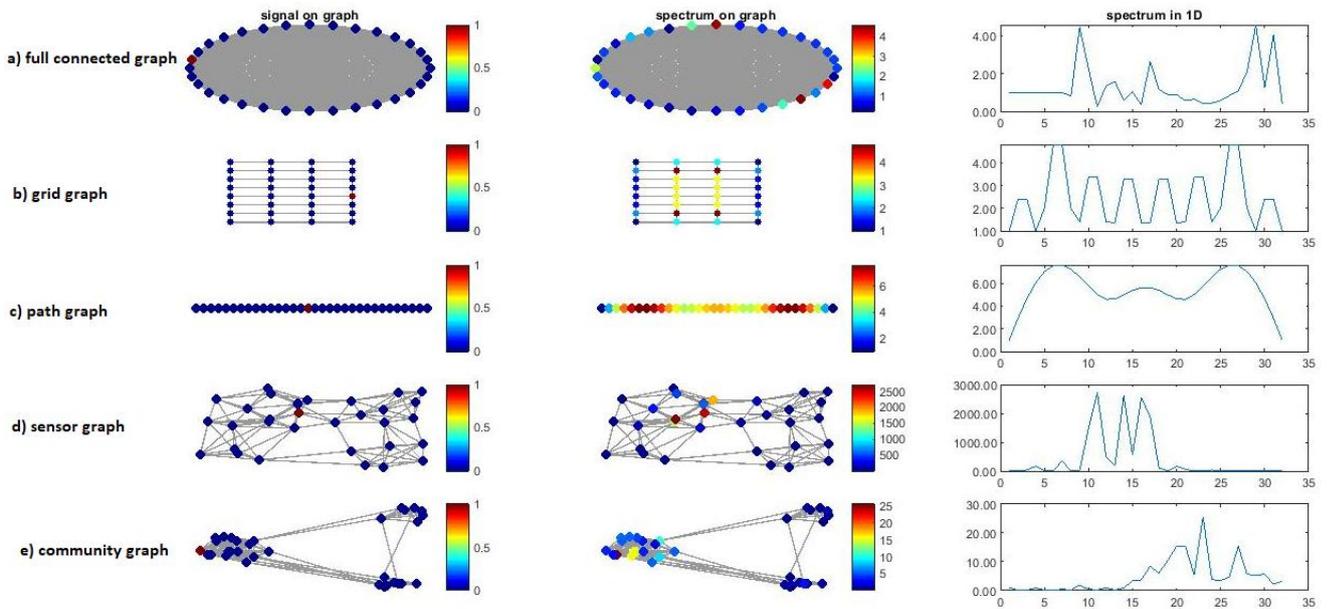
Figure 3: SGFT on different graph patterns
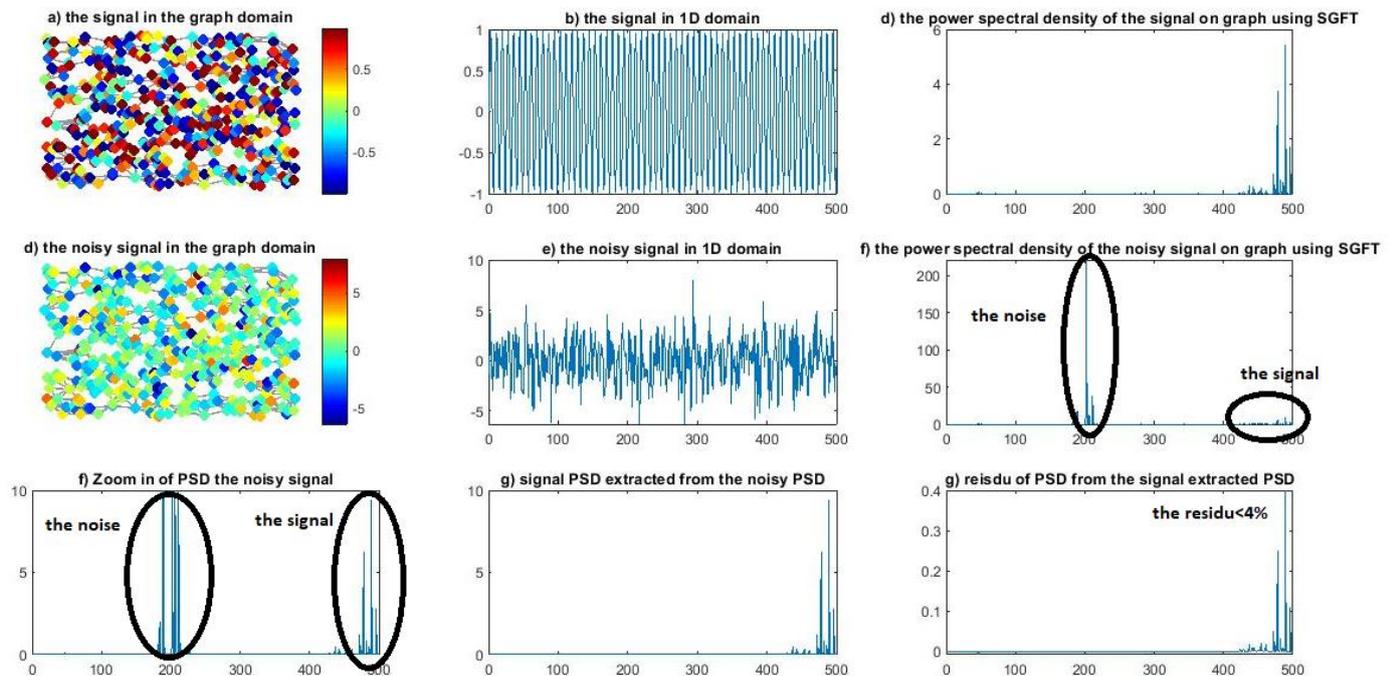
Figure 4: Spectral analysis on a sensor network graph signal

VI. CONCLUSIONS

We have presented in this work , a novel method for the generalization of the classical fourier transform to graph signal processing. The proposed method gives the same results as those obtained in the classical Fourier framework, when appying it in a regular domain modelled by a cyclic graph. The application of the proposed scheme allows to detect the occurrence of an alarm signal on a sensor network by analysing the power sptectral density of the recorded signal on graph. The proposed scheme raises many future works. It can be used first in applications such as spectral analysis of determistic graph signals, and stationnary stochasctiks graph signals. And secondly, it can also be used to define other related graph signals transform such as wavelet transform on graph signals, Gabor transform on graph signals, Radon transform on graph signals, and other multiscale graph transforms linked with the graph Fourier transform.

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