International Journal of Mathematics Trends and Technology Volume 66 Issue 10, 1-7, October 2020
ISSN: 2231 – 5373 /doi:10.14445/22315373/IJMTT-V66I10P501 

# Volatility Swap Pricing and Variance Swap Pricing under the Mean-Reverting Gaussian Model

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***Abstract :*** *In the past few decades, stochastic volatility models have been very popular in the pricing of financial derivatives.Volatility derivatives are a special kind of financial derivatives. A volatility swap and a variance swap discussed in this section are both volatility derivatives.Their essence is a forward contract whose value depends on the future volatility level of the underlying asset. This paper mainly uses the risk-neutral pricing principle to derive the pricing formulas of a volatility swap and a variance swap under the mean-reverting Gaussian volatility model.*

***Keywords:*** *mean-reverting Gaussian model, the risk-neutral pricing, volatility swap,variance swap.*

## I. Background Introduction

The stochastic volatility model was first proposed by Hull-White[1].After that, Merton[2] and Heston[3] did a more in-depth study on the related theories of stochastic volatility.Besides,we can refer to stochastic volatility literature [4,5]about the recent survey.

Whether it is a variance swap or a volatility swap, they are all new derivatives that provide risk exposure for volatility.One method of studying volatility derivatives is to assume that the volatility of asset prices is subject to a random process, and then derive an analytical formula for pricing volatility derivatives under this assumption. This paper mainly uses this method to obtain analytical formulas for volatility swap pricing and variance swap pricing under the mean-reverting Gaussian volatility model.

For the mean-reverting Gaussian volatility model, the price of the underlying asset $S_t$ and its volatility $\sigma_t$ satisfy the following stochastic differential equation:

$$\begin{cases} \dfrac{dS_t}{S_t} = rdt + \sigma_t dW_t^1 \\ d\sigma_t = \kappa(\theta - \sigma_t)dt + \nu dW_t^2 \end{cases} \qquad (1)$$

where $r$ represents the expected return rate of the asset, $\theta$ represents the long-term average value of volatility, $\kappa$ represents the mean recovery speed parameter of volatility, $\nu$ represents the volatility of volatility (also known as the coefficient of variation of volatility) and it is used to describe volatility uncertainty. $W_t^1$ and $W_t^2$ represent two standard Brownian motions[10] with a correlation coefficient of $\rho$ .

## II. Preliminary

In this chapter, we will briefly introduce the definition of a volatility swap,a variance swap ,Brownian motion, Ito process and Ito-Deblin formula about the Ito process.

***A . Definition [6]*** Volatility swap is sometimes called a realized volatility forward contract.Its profit and loss on the maturity date is $(\sigma_R^{vol} - K^{vol})N$. $\sigma_R^{vol}$ represents the realized volatility of stock returns over the entire contract period, $K^{vol}$ represents the delivery price of the contract. N represents the value of currency represented by the volatility of each point.

Variance swap is a forward contract based on the variance of stock returns. Its profit and loss on the maturity date is $(\sigma_R^2 - K^{vol})N$. $\sigma_R^2$ represents the variance of stock returns over the entire contract period, $K^{vol}$ represents the delivery price of the contract. N represents the value of currency represented by the square of the volatility of each point.

***B . Definition [7]*** Assuming (Ω,F, P) is a probability space.For every ω∈Ω,we suppose that there is a continuous function that depends on ω and satisfies $W(0) = 0$. We use E[·] to represent the mathematical expectation and Var[·] to represent variance.If increments $W(t_1) = W(t_1) - W(t_0), W(t_2) - W(t_1), \cdots, W(t_m) - W(t_{m-1})$ are independent mutually for all $0 = t_0 < t_1 < \cdots < t_m$, each increment obeys a normal distribution and $\mathrm{E}[W(t_{i+1}) - W(t_i)] = 0$, $Var[W(t_{i+1}) - W(t_i)] = t_{i+1} - t_i$,

then $W(t)(t \geq 0)$ is a Brownian motion.

C . Definition [8] Assuming $W(t), \quad t \geq 0$ is Brownian motion, F(t) is the corresponding basin flow. The Ito process is a random process of the following form:

$$X(t) = X(0) + \int_0^t \Delta(u)dW(u) + \int_0^t \Theta(u)du .$$

where $X(0)$ is not random, $\Delta(u)$ and $\Theta(u)$ are adaptive stochastic processes.

***D .Theorem [9]*** Let $X(t), \quad t \geq 0$ be an Ito process. The partial derivatives $f_t(t,x), f_x(t,x)$ and $f_{xx}(t,x)$ of the function $f(t,x)$ are defined and continuous, then for each $T \geq 0$:

$$f(T, X(T)) = f(0, X(0)) + \int_0^T f_t(t, X(t))\,dt + \int_0^T f_x(t, X(t))\,dX(t)$$

$$+ \frac{1}{2}\int_0^T f_{xx}(t, X(t))\,d[X, X](t).$$

The above formula is called the Ito-Deblin formula.If the differential notation is used, the above formula can be rewritten as

$$df(t, X(t)) = f_t(t, X(t))dt + f_x(t, X(t)) + \frac{1}{2} f_{xx}(t, X(t))dX(t)dX(t).$$

In fact, $f(t, X(t))$ takes the Taylor expansion respect to variables t and $X(t)$ ,then the quadratic variation of variable t is 0 and the quadratic variation of $X(t)$ is non-zero, so we expand t to the first order and expand $X(t)$ to the second order to get the above formula.

## III. PRICING ISSUES UNDER THE TWO SWAPS

In this chapter, we assume that volatility obeys the mean-reverting Gaussian model.Through a series of calculations, we can respectively derive pricing formulas for a volatility swap and a variance swap using the theory of risk-neutral pricing. Suppose $\mathrm{E}[\bullet]$ represents the mathematical expectation under the risk-neutral measure, $r$ represents the risk-free interest rate, and denote $\mathrm{E}[\bullet \mid F_t] = E_t[\bullet]$.

### *A .Volatility Swap*

***Theorem*** Suppose the price process of the underlying asset and the volatility process satisfies (1), then the value of a volatility swap at time t is:

$$V_t = \frac{1}{T} e^{-r(T-t)} [\int_0^t \sigma_s ds + \frac{1}{\kappa} e^{\kappa(t-T)}(\theta - \sigma(t)) + \frac{1}{\kappa}(\sigma(t) - \theta) - \frac{1}{\kappa}(\theta - \sigma(0)(e^{-\kappa T} - 1) - T\theta].$$

***Proof.*** The price process of the underlying asset $S_t$ satisfies

$$\frac{dS_t}{S_t} = rdt + \sigma_t dW_t. \qquad (2)$$

The volatility process $\sigma_t$ can be described as

$$d\sigma_t = \kappa(\theta - \sigma_t)dt + v\sigma_t dW_\sigma. \qquad (3)$$

Consider the value of a volatility swap at any moment $t(0 \le t \le T)$. We use $\sigma_R^{vol}$ to represent the realized volatility of the return, $K^{vol}$ to represent the delivery price, then the profit and loss of the volatility swap on the maturity date T is as follows:

$$\sigma_R^{vol} - K^{vol} = \frac{1}{T}\int_0^T \sigma_t dt - K^{vol} .$$

Before time t, the volatility has been achieved, so the above formula can be decomposed as

$$\sigma_R^{vol} - K^{vol} = \frac{1}{T}(\int_0^t \sigma_s ds + \int_t^T \sigma_s ds) - K^{vol} .$$

According to the risk-neutral pricing principle, the value of the volatility swap at time t can be obtained:

$$V_t = \mathrm{E}\left[e^{-r(T-t)}(\sigma_R^{vol} - K^{vol}) \mid F_t\right]$$

$$= \mathrm{E}[e^{-r(T-t)}\frac{1}{T}(\int_0^t \sigma_s ds + \int_t^T \sigma_s ds - K^{vo}) \mid F_t]$$

$$= e^{-r(T-t)}[\frac{1}{T}(\int_0^t \sigma_s ds + \int_t^T \mathrm{E}_t[\sigma_s]ds) - K^{vo}] \qquad (4)$$

According to (1), the volatility $\sigma_t$ satisfies

$$d\sigma_t = \kappa(\theta - \sigma_t)dt + \nu dW_t^2 \qquad (5)$$

In order to get the following closed form solution of the volatility process

$$\sigma_t = e^{-\kappa t}\sigma_0 + \theta(1 - e^{-\kappa t}) + \nu e^{-\kappa t}\int_0^t e^{\kappa s} dW_s^2 , \qquad (6)$$

we use Ito-Deblin formula to calculate the differential at the right end of the above formula.Let $g(t,x) = e^{-\kappa t}\sigma_0 + \theta(1 - e^{-\kappa t}) + \nu e^{-\kappa t}x$ and $X(t) = \int_0^t e^{\kappa s} dW_s^2$. Then $dX(t) = e^{\kappa t}dW_t^2$ and the right end of (6) can be written as $g(t, X(t))$. The following proves that $g(t, X(t)) = \sigma_t$. So we divide the right end of (6) into two parts: a function of the usual variables t and x (without randomness) and an Ito process X(t) (with all randomness).According to Ito formula,

$$dg(X(t)) = g_t(t, X(t))dt + g_x(t, X(t))dX_t + \frac{1}{2}g_{xx}(t, X(t))dX_t dX_t$$

$$= \kappa(\theta - g(t, x))dt + \nu dW_t^2 \qquad (7)$$

(7) shows that $g(t, X(t))$ satisfies the stochastic differential equation (5) that determines $\sigma_t$ and $g(0, X(0)) = \sigma_0$. Since $g(t, X(t))$ satisfies the stochastic differential equation that determines $\sigma_t$ and has the same initial conditions as $\sigma_t$, we have $g(t, X(t)) = \sigma_t$ for all $t \geq 0$. After getting the closed form solution of $\sigma_t$, for all $s \geq t$ we have

$$\sigma_s = e^{-\kappa(s-t)}\sigma_t + \theta(1 - e^{-\kappa(s-t)}) + \nu e^{-\kappa s}\int_t^s e^{\kappa u} dW_u^2$$

$$\mathrm{E}_t\sigma(s) = e^{-\kappa(s-t)}\sigma_t + \theta(1 - e^{-\kappa(t-s)}). \qquad (8)$$

Substituting (8) into (4),we have

$$V_t = e^{-r(T-t)}[\frac{1}{T}(\int_0^t \sigma_s ds + \theta(T-t) + \frac{\sigma_t - \theta}{\kappa}(1 - e^{-\kappa(T-t)})) - K^{vol}]. \qquad (9)$$

The above formula is the volatility swap pricing formula under the mean-reverting Gaussian model at any time t. According to the principle of no arbitrage[8], $V_0 = 0$ ,then $K_{vol} = \theta + \frac{\theta - \sigma_0}{T\kappa}(e^{-\kappa T} - 1)$.

Substituting $K_{vol}$ into (9),then the value of a volatility swap at time t is:

$$V_t = \frac{1}{T}e^{-r(T-t)}[\int_0^t \sigma_s ds + \frac{1}{\kappa}e^{\kappa(t-T)}(\theta - \sigma(t)) + \frac{1}{\kappa}(\sigma(t) - \theta) - \frac{1}{\kappa}(\theta - \sigma(0)(e^{-\kappa T} - 1) - T\theta].$$

**B .Variance Swap**

**Theorem** Suppose the price process of the underlying asset and the volatility process satisfies (1), then the value of a variance swap at time t is:

$$V_t = e^{-r(T-t)}[\frac{1}{T}(\int_0^t \sigma_s^2 ds + \frac{2\theta^2}{\kappa}(e^{\kappa(t-T)} - 1) + \theta^2(T-t) - (\frac{\sigma_t^2}{2\kappa} + \frac{\theta^2}{2\kappa})(e^{2\kappa(t-T)} - 1))$$

$$- \frac{2\theta^2}{T\kappa}(e^{-\kappa T}) - 1) - \theta^2 + (\frac{\sigma_0^2}{2T\kappa} + \frac{\theta^2}{2T\kappa})(e^{-2\kappa T} - 1)].$$

**Proof:** The profit and loss of the variance swap on the maturity date T is as follows:

$$\sigma_R^2 - K^{vol} = \frac{1}{T}\int_0^T \sigma_t^2 dt - K^{vol}.$$

We use $\sigma_R^2$ to represent the realized variance of the return, $K^{vol}$ to represent the delivery price.According to the principle of risk-neutral pricing, the value of the time variance swap can be obtained:

$$V_t = \mathrm{E}\left[e^{-r(T-t)}(\sigma_R^2 - K^{vol}) \mid F_t\right]$$

$$= \mathrm{E}[e^{-r(T-t)} \frac{1}{T}(\sigma_R^2 - K^{vol}) \mid F_t]$$

$$= e^{-r(T-t)}[\frac{1}{T}(\int_0^t \sigma_s^2 ds + \int_t^T \mathrm{E}_t[\sigma_s^2]ds) - K^{vol}]. \qquad (10)$$

Since it is the same as the model in Section 3.1, the closed form solution of $\sigma_t$ is also $\sigma_t = e^{-\kappa t}\sigma_0 + \theta(1 - e^{-\kappa t}) + \nu e^{-\kappa t}\int_0^t e^{\kappa s} dW_s^2$.

After getting the closed form solution of $\sigma_t$, for all $s \geq t$ we have

$$\sigma_s = e^{-\kappa(s-t)}\sigma_t + \theta(1 - e^{-\kappa(s-t)}) + \nu e^{-\kappa(s-t)}\int_t^s e^{\kappa u} dW_u^2.$$

According to Theorem C, the expected value of the Ito integral of the non-random integrand is 0, and the expected value of $\sigma_s^2$ is:

$$\mathrm{E}_t[\sigma_s^2] = e^{2\kappa(t-s)}\sigma_t^2 + \theta^2(1 - e^{\kappa(t-s)})^2.$$

Substitute the above formula into (10),

$$V_t = e^{-r(T-t)}[\frac{1}{T}(\int_0^t \sigma_s^2 ds + \int_t^T (e^{2\kappa(t-s)}\sigma_t^2 + \theta^2(1 - e^{\kappa(t-s)})^2)ds) - K^{vol}].$$

$$= e^{-r(T-t)}[\frac{1}{T}(\int_0^t \sigma_s^2 ds + (\frac{2\theta^2}{\kappa}e^{\kappa(t-s)} + \theta^2 s - \frac{\sigma_t^2 e^{2\kappa(t-s)}}{2\kappa} - \frac{\theta^2 e^{2\kappa(t-s)}}{2\kappa})\Big|_t^T) - K^{vol}]$$

$$= e^{-r(T-t)}[\frac{1}{T}(\int_0^t \sigma_s^2 ds + \frac{2\theta^2}{\kappa}(e^{\kappa(t-T)} - 1) + \theta^2(T-t) - (\frac{\sigma_t^2}{2\kappa} + \frac{\theta^2}{2\kappa})(e^{2\kappa(t-T)} - 1)) - K^{vol}].$$

According to the principle of no arbitrage, the initial value of variance swap is 0. Therefore, let $t = 0$ to get

$$\frac{1}{T}[\int_0^t \sigma_s^2 ds + \frac{2\theta^2}{\kappa}(e^{\kappa(t-T)} - 1) + \theta^2(T-t) - (\frac{\sigma_t^2}{2\kappa} + \frac{\theta^2}{2\kappa})(e^{2\kappa(t-T)} - 1)] - K^{vol} = 0,$$

$$K^{vol} = \frac{2\theta^2}{T\kappa}(e^{-\kappa T} - 1) + \theta^2 - (\frac{\sigma_0^2}{2T\kappa} + \frac{\theta^2}{2T\kappa})(e^{-2\kappa T} - 1).$$

Therefore, the final variance swap pricing formula is as follows:

$$V_t = e^{-r(T-t)}[\frac{1}{T}(\int_0^t \sigma_s^2 ds + \frac{2\theta^2}{\kappa}(e^{\kappa(t-T)} - 1) + \theta^2(T - t) - (\frac{\sigma_t^2}{2\kappa} + \frac{\theta^2}{2\kappa})(e^{2\kappa(t-T)} - 1))$$

$$- \frac{2\theta^2}{T\kappa}(e^{-\kappa T} - 1) - \theta^2 + (\frac{\sigma_0^2}{2T\kappa} + \frac{\theta^2}{2T\kappa})(e^{-2\kappa T} - 1)].$$

## IV. SUMMARY AND OUTLOOK

In this article, our discussions are mainly based on the mean-reverting Gaussian model.Under this model, we use the risk-neutral pricing theory to give two pricing formulas of a volatility swap and a variance swap.So far, we have discussed the pricing of a volatility swap and a variance swap under a stochastic volatility model. In addition, we have other related issues to study, such as: the pricing of other volatility derivatives under the mean-reverting Gaussian model. Of course, we can also promote the mean-reverting Gaussian model according to the actual situation,and discuss the pricing of volatility swaps ,variance swaps ,Gamma swaps and other volatility derivatives under extended models.These are all the problems we need to solve urgently in the future.

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