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Method of Lower and Upper Solutions for Differential Equations of Fractional Order with Integral Boundary Conditions

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Abstract-*System of Riemann-Liouville fractional differential equations with integral boundary conditions is considered. Method of lower and upper solutions is developed for system of Riemann-Liouville fractional differential equations with integral boundary conditions. Method is successfully employed to study existence and uniqueness results for the problem.*

Keywords: *Lower and upper solutions, fractional differential equations, integral boundary conditions, monotone technique, existence and uniqueness.*

I.INTRODUCTION

A wide range of applications of fractional differential equations attracted researchers in the last three decades. [1, 3, 17]. Numerous techniques have been developed to study fractional differential equations. The qualitative properties of solution of fractional differential equations [6] parallel to the well-known theory of ordinary differential equations [4] has been growing recently. Amongst various techniques, the monotone iterative technique is widely used in the study of fractional differential equations. Ladde et al.[4] have developed and extensively employed this method in the study of differential equations which arise in biological and physical problems. In the year 2008, Lakshmikantham and Vatsala [5, 6, 7, 9] developed the basic theory of fractional differential equation with Riemann-Liouville fractional derivative. In 2009, McRae [10] developed monotone method for Riemann-Liouville fractional differential equation with initial conditions and studied the qualitative properties of solutions of initial value problem (IVP). Vatsala, McRae, Vasundhara Devi, Zhang and author have developed monotone method for fractional differential equations [11, 12, 13, 14, 15]. Comparison results, existence and uniqueness of solution of ordinary differential equation with integral boundary condition was firstly obtained by Jankowski [2]. In 2012 , Nanware and Dhaigude [15] developed the monotone method for Riemann-Liouville fractional differential equations with integral boundary condition when the right hand side function is splited as sum of two non-decreasing and non-increasing function. Wang and Xie [18] developed the monotone method and obtained existence and uniqueness of solution of fractional differential equation with integral boundary condition.

The paper is organized as follows:

In section II, some definitions and basic results are given. In section III, monotone iterative technique is developed and it is applied to obtain existence and uniqueness of solutions for the system of Riemann-Liouville fractional differential equations with integral boundary conditions.

II. PRELIMINARIES

The Riemann-Liouville fractional derivative [14] of order q $(0 < q < 1)$ of function $u(t)$ is defined as

$$D^q u(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_0^t \frac{u(s)}{(t-s)^q} ds. \qquad (2.1)$$

Wang and Xie in 2008, have proved existence and uniqueness of solution of the following Riemann -Liouville fractional differential equation with integral boundary condition

$$D^q u(t) = f(t,u), \quad t \in J = [0,T], \quad T \geq 0, \quad u(0) = \lambda \int_0^T u(s)ds + d, \quad d \in R. \qquad (2.2)$$

where $0 < q < 1$, λ is 1 or -1 and $f \in C[J \times R, R]$, by monotone method. He has studied the above problem for any choice of $\lambda \geq 0$ and developed monotone iterative technique [19]. This motivates us to study the following system of Riemann-Liouville fractional differential equations with integral boundary conditions when $\lambda \geq 0$:

$$D^q u_i(t) = f_i(t, u_1(t), u_2(t)), \quad u_i(0) = \lambda \int_0^T u_i(s)ds + d_i, \quad i = 1,2 \qquad (2.3)$$

where $d_i \in R$, $t \in [0,T]$ f_1, f_2 in $C(J \times R^2, R)$, $J = [0,T]$, $0 < q < 1$.

We apply monotone technique, to prove the existence and uniqueness of solution of problem (2.3).

Definition 2.1 *A pair of functions* $x = (x_1, x_2)$ *and* $y = (y_1, y_2)$ *in* $C_p(J,R)$ *is continuous with exponent* $\lambda > q$. *A pair* (x, y) *of functions x and y is called ordered lower and upper solutions* $(x_1, x_2) \leq (y_1, y_2)$ *of problem* (2.3) *if*

$$D^q x_i(t) \leq f_i(t, x_1(t), x_2(t)), \qquad x_i(0) \leq \lambda \int_0^T x_i(s)ds + d_i$$

and

$$D^q y_i(t) \geq f_i(t, y_1(t), y_2(t)), \qquad y_i(0) \geq \lambda \int_0^T y_i(s)ds + d_i.$$

Definition 2.2 *A function* $f_i = f_i(t,u_1,u_2)$ *in* $C(J \times R^2, R)$ *is said to be quasi-monotone non-decreasing if for* $i,j = 1,2$ $f_i(t,u_1(t),u_2(t)) \leq f_i(t,v_1(t),v_2(t))$ *if* $u_i = v_i$ *and* $u_j \leq v_j, i \neq j$.

Lemma 2.1 *[5] Let* $m \in C_p(J,R)$ *be continuous with exponent* $\lambda > q$ *and for any* $t_1 \in (0,T]$ *we have* $m(t_1) = 0$ *and* $m(t) \leq 0$ *for* $0 \leq t \leq t_1$. *Then it follows that* $D^q m(t_1) \geq 0$.

Lemma 2.2 *[5] Let* $\{u_\epsilon(t)\}$ *be a family of continuous functions on J, for each* $\epsilon > 0$ *where* $D^q u_\epsilon(t) = f(t, u_\epsilon(t)), \quad u_\epsilon(0) = u_0$ *and* $|f(t, u_\epsilon(t))| \leq M$ *for* $0 \leq t \leq T$. *Then the family* $\{u_\epsilon(t)\}$ *is equi-continuous on* $[0,T]$.

Theorem 2.1 *[5] Suppose that*

(i) $x(t)$ *and* $y(t)$ *in* $C_p(J,R)$ *are continuous functions and f in* $C(J \times R, R)$

(ii) $x(t)$ *and* $y(t)$ *satisfies following inequalities for* $0 \leq t \leq T$:

$$D^q x(t) \le f(t, x(t))$$

$$D^q y(t) \ge f(t, y(t))$$

(iii) *f satisfies one-sided Lipschitz condition:*

$$f(t,x(t)) - f(t,y(t)) \le \frac{L}{1+t^q}(x-y) \qquad \text{when ever} \qquad x \ge y$$

$and\ 0 < LT^q \le \frac{1}{\Gamma(1-q)}$. *Then* $x(0) \le y(0)$ *implies that* $x(t) \le y(t), 0 \le t \le T$.

III. MAIN RESULTS

In this section we develop monotone technique for system of fractional differential equations with integral boundary conditions. We apply developed method to obtain existence and uniqueness of solution of problem (2.3).

Theorem 3.1 *Assume that*

(i) $f_i = f_i(t,u_1,u_2) \in C[J \times R^2, R]$ *is quasi-monotone non-decreasing,*

(ii) $x^0 = (x_1^0, x_2^0)$ *and* $y^0 = (y_1^0, y_2^0) \in C(J,R)$*, are ordered lower and upper solutions of problem* (2.3) *such that* $x_1^0(0) \le y_1^0(0), x_2^0(0) \le y_2^0(0)$ *on* $[0,T]$

(iii) $f_i \equiv f_i(t,u_1,u_2)$ *satisfies one-sided Lipschitz condition,*

$$f_i(t,u_1,\overline{u_2}) - f_i(t,u_1,u_2) \ge -M_i[\overline{u_i} - u_i] \qquad (3.1)$$

whenever $x^0_i(0) \le u_i \le y_i^0(0)$, $M_i \ge 0$, $\qquad x^0_i(0) \le u_i \le \overline{u_i} \le y_i^0(0)$

Then there exists monotone sequences $\{x^n(t)\} = (x_1^n, x_2^n)$ *and* $\{y^n(t)\} = (y_1^n, y_2^n)$ *such that*

$$\{x^n(t)\} \to x(t) = (x_1,x_2) \quad \text{and} \quad \{y^n(t)\} \to y(t) = (y_1,y_2) \text{ as } n \to \infty$$

where $x(t)$ *and* $y(t)$ *are minimal and maximal solutions of problem* (2.3) *respectively.*

Proof. For any $\eta(t) = (\eta_1,\eta_2)$ in $C(J,R)$ such that for $x_1^0(0) \le \eta_1, x_2^0(0) \le \eta_2$ on $J = [0,T]$, consider the following system of linear fractional differential equations

$$D^q u_i^1(t) = f_i(t,\eta_1(t),\eta_2(t)) - M_i[u_i^1(t) - \eta_i(t)], \quad u_i^1(0) = \lambda \int_0^T u_i^0(s)ds + d_i. \tag{3.2}$$

Clearly, for every $\eta(t)$ there exists a unique solution $u^1(t) = (u_1^1(t), u_2^1(t))$ of the system (3.2) on J. For each $\eta(t)$ and $\mu(t)$ in $C(J,R)$ satisfying $x^0_i(0) \le \eta_i(t)$, $x^0_i(0) \le \mu_i(t)$, define a mapping A by $A[\eta,\mu] = u^1(t)$ where $u(t)$ is the unique solution of system (3.2). This mapping defines the sequences $\{x^n(t)\}$ and $\{y^n(t)\}$. Firstly, we prove that (A_1) $x^0 \le A[x^0,y^0], y^0 \ge A[y^0,x^0]$

(A_2) A possesses monotone property on segment

$$[x^0, y^0] = \{(u_1,u_2) \in C(J,R^2): x_i^0 \le u_i \le y_i^0\}, i = 1,2.$$

Set $A[x^0,y^0] = x^1(t)$, where $x^1(t) = (x_1^1, x_2^1)$ is the unique solution of system (3.2) with $\eta_i = x^0_i(0)$.

Setting $p_i(t) = x_i^0(t) - x^1_i(t)$ we see that

$$D^q p_i(t) = D^q x^0_i(t) - D^q x^1_i(t)$$

$$\le -M_i[x^0_i(t) - x^1_i(t)]$$

$$\le -M_i p_i(t) \text{ and } \qquad p_i(0) = 0.$$

By Theorem 2.1, it follows that $p_i(t) \le 0$ on $0 \le t \le T$ and hence $x_i^0(t) - x^1_i(t) \le 0$ which implies $x^0 \le A[x^0,y^0]$. Set $A[y^0,x^0] = y^1(t)$, where $y^1(t) = (y_1^1, y_2^1)$ is the unique solution of system (3.2) with $\mu_i = y_i^0(t)$.

Setting $p_i(t) = y_i^0(t) - y_i^1(t)$ we see that

$$
\begin{aligned}
D^q p_i(t) &= D^q y_i^0(t) - D^q y_i^1(t)\\
&\ge -M_i[y_i^0(t) - y_i^1(t)]\\
&\ge -M_i p_i(t)
\end{aligned}
$$

and $\quad p_i(0) = 0.$

By Theorem 2.1, it follows that $y_i^0(t) \ge y_i^1(t)$. Hence $y^0 \ge A[x^0,y^0]$.

To prove (A_2), let $\eta_1(t),\eta_2(t),\mu(t) \in [x^0,y^0]$ be such that $\eta_1(t) \le \eta_2(t)$.

Suppose that $A[\eta_1, \mu] = u_1(t) = (u_1^1, u_2^1), A[\eta_2, \mu] = u_2(t) = (u_1^2, u_2^2)$.

Then setting $p_i(t) = u_1^i(t) - u_2^i(t)$ we find that

$$
\begin{aligned}
D^q p_i(t) &= D^q u^i{}_1(t) - D^q u^i{}_2(t)\\
&= f_i(t,\eta_1(t),u_2(t)) - f_i(t,\eta_2(t),u_2(t)) - M_i[u^i{}_1(t) - \eta_1(t)] + M_i[u^i{}_2(t) - \eta_2(t)]\\
&\le -M_i[\eta_1(t) - \eta_2] - M_i[u^i{}_1(t) - \eta_1(t)] + M_i[u^i{}_2(t) - \eta_2(t)]\\
&\le -M_i[u^i{}_1(t) - u^i{}_2(t)]\\
&\le -M_i p_i(t) \text{ and } \quad p_i(0) = 0.
\end{aligned}
$$

As before in (A_1), we have $A[\eta_1,\mu] \le A[\eta_2,\mu]$.

Similarly, if we let $\eta(t),\mu_1(t),\mu_2(t) \in [v^0,w^0]$ be such that $\mu_1(t) \le \mu_2(t)$. Suppose that $A[\eta,\mu_i] = u_i(t)$ we prove that $A[\eta,\mu_1] \ge A[\eta,\mu_2]$.Thus it follows that mapping A possesses monotone property on the segment $[x^0,y^0]$. In view of (A_1) and (A_2), define the sequences

$$x_i^n(t) = A[x_i^{n-1}, y_i^{n-1}], \quad x_i^n(t) = A[y_i^{n-1}, x_i^{n-1}]$$

on the segment $[x^0,y^0]$ by

$$D^q x_i^n(t) = f_i(t, x_1^{n-1}, x_2^{n-1}) - M_i[x_i^n - x_i^{n-1}], \quad x_i^n(0) = \int_0^T x_i^{n-1}(s)ds + d$$

$$D^q y_i^n(t) = f_i(t, x_1^{n-1}, y_2^{n-1}) - M_i[y_i^n - y_i^{n-1}], \quad x_i^n(0) = \int_0^T y_i^{n-1}(s)ds + d.$$

From (A_1), we have $x^0{}_i(t) \le x^1{}_i(t), \quad y_i^0(t) \ge y_i^1(t)$.

Assume that $x_i^{k-1}(t) \le x_i^k(t), \quad y_i^{k-1}(t) \ge y_i^k(t)$. To prove $x_i^k(t) \le x_i^{k+1}(t), \quad y_i^k(t) \ge y_i^{k+1}(t)$ and $x^k{}_i(t) \ge y_i^k(t)$, we define $p_i(t) = x_i^k(t) - x_i^{k+1}(t)$. Thus

$$
\begin{aligned}
D^q p_i(t) &= D^q x_i^k(t) - D^q x_i^{k+1}(t)\\
&= f_i(t, x_1^{k-1}(t), x_2^{k-1}(t)) - M_i[x_1^k(t) - x_1^{k-1}(t)]\\
&\qquad - \left\{ f_i(t, x_1^k(t), x_2^k(t)) - M_i[x_1^{k+1}(t) - x_1^k(t)] \right\}\\
&\le M_i[x_i^k(t) - x_i^{k-1}(t)] - M_i[x_i^k(t) - x_i^{k-1}(t)] + M_i[x_1^{k+1}(t) - x_1^k(t)]\\
&\le -M_i p_i(t)
\end{aligned}
$$

and $\quad p_i(0) = 0.$

Applying Theorem 2.1, we have $p_i(t) \le 0$. This gives $x_i^k(t) \le x^{k+1}_i(t)$.

Similarly we can prove $x_i^k(t) \ge y_i^{k+1}(t)$ and $y_i^k(t) \ge y_i^k(t)$. By induction, it follows that

$x^0_i(t) \le x^1_i(t) \le x^2_i(t) \le ... \le x^n_i(t) \le y_i^n(t) \le y_i^{n-1}(t) \le ... \le y_i^1(t) \le y_i^0(t)$. (3.3)

Thus the sequences $\{x^n(t)\}$ and $\{y^n(t)\}$ are bounded from below and bounded from above respectively and monotonically nond-ecreasing and monotonically non-increasing on $[0,T]$. Hence pointwise limit exists and are given by $\lim_{n\to\infty} x^n_i(t) = x_i(t)$, $\lim_{n\to\infty} y^n_i(t) = y_i(t)$ on $[0,T]$. Using corresponding Volterra fractional integral equations

$$x_i^n(t) = x_i^0 + \frac{1}{\Gamma(q)}\int_0^T (t-s)^{q-1}\left\{f_i(s, x_1^n(s), x_2^n(s)) - M_i[x_i^n - x_i^{n-1}]\right\}ds$$

$$y_i^n(t) = y_i^0 + \frac{1}{\Gamma(q)}\int_0^T (t-s)^{q-1}\left\{f_i(s, y_1^n(s), y_2^n(s)) - M_i[y_1^n - y_1^{n-1}]\right\}ds, \tag{3.4}$$

it follows that $x(t)$ and $y(t)$ are solutions of system (3.2).

Lastly we prove $x(t)$ and $y(t)$ are the minimal and maximal solution of the system (3.2). Let $u(t) = (u_1,u_2)$ be any solution of (2.3) different from $x(t)$ and $y(t)$, so that there exists k such that $x^k_i(t) \le u_i(t) \le y_i^k(t)$ on $[0,T]$ and set $p_i(t) = x^{k+1}_i(t) - u_i(t)$ so that

$$D^q p_i(t) = f_i(t,x^k_1(t),x^k_2(t)) - M_i[x^{k+1}_i(t) - x_i^k(t)] - f_i(t,u_1(t),u_2(t))$$
$$\ge -M_i[x^{k+1}_i(t) - u_i(t)]$$
$$\ge -M_i p_i(t) \text{ and } p_i(0) = 0.$$

Thus $x_i^{k+1}(t) \le u_i(t)$ on $[0,T]$. Since $x^0_i(t) \le u_i(t)$ on $[0,T]$, by induction it follows that $x^k_i(t) \le u_i(t)$ for all k. Similarly, we can prove $u_i(t) \le y_i^k(t)$ for all k on $[0,T]$. Thus $x^k_i(t) \le u_i(t) \le y_i^k(t)$ on $[0,T]$. Taking limit as $n \to \infty$, it follows that $x_i(t) \le u_i(t) \le y_i(t)$ on $[0,T]$. Thus the theorem.

The uniqueness of solution of the system (2.3) is proved in the following theorem

Theorem 3.2 *Assume that*

(i) $f_i = f_i(t,u_1,u_2)$ *in* $C[J \times R, R^2]$ *is quasimonotone nondecreasing,*

(ii) $x = (x_1,x_2)$ *and* $y = (y_1,y_2)$ *in* $C(J,R)$ *are ordered lower and upper solutions of* (2.3) *on* $[0,T]$

(iii) $f_i = f_i(t,u_1,u_2)$ *satisfies Lipschitz condition,*

$$|f_i(t,u_1,u_2) - f_i(t,\overline{u}_1, \overline{u}_2)| \le -M_i|u_i - \overline{u}_i| \quad (3.5)$$

(iv) $\lim_{n\to\infty}\|y^n - x^n\| = 0$, *where the norm is defined by* $\|f\| = \int_0^T |f(s)|ds$*,then the solution of system* (2.3) *is unique.*

Proof. We need to prove $v(t) \ge w(t)$, since $x(t) \le y(t)$.. Set $p_i(t) = y_i(t) - x_i(t)$, we find that

$$D^q p_i(t) = D^q y_i(t) - D^q x_i(t)$$
$$\le -M_i[y_i(t) - x_i(t)]$$
$$\le -M_i p_i(t) \text{ and } p_i(0) = 0$$

By Theorem 2.1, it follows that $p_i(t) \le 0$. This gives $y_i(t) \le x_i(t)$. Hence $x(t) = y(t)$ is the unique solution of system (2.3) on $[0,T]$. This proves the uniqueness of solution of the system (2.3) on $[0,T]$.

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