International Journal of Mathematics Trends and Technology
ISSN: 2231 – 5373 /doi:10.14445/22315373/IJMTT-V66I10P504
Volume 66 Issue 10, 20-23, October 2020

Some Commutativity Results For s-Unital Rings Satisfying Polynomial Identities

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ABSTRACT:

H. E. Bell [2] proved that an n-torsion free ring with identity which satisfies the identity $(xy)^n = (yx)^n$ is necessarily commutative. More recently, Y. Hirano, M. Hongan and H.Tominaga has proved the same for s-unital rings [5]. On the other hand, H.Abu-Khuzam paper has proved that an (n + 1)n-torsion free ring with identity which satisfies the identity $(xy)^{n+1} = x^{n+1} y^{n+1}$ is commutative[1]. Our objective is to generalize Bell's result to s-unital rings satisfying weaker identities which are implied by the identity $(xy)^n = (yx)^n$ and to generalize the main theorem of [1] to s-unital rings. As stated in [5], if R is an s-unital ring, then for any finite subset F of R, there exists an element e in R such that ex = xe = e for all x in F. Such an element e will be called a pseudo-identity of F. Throughout, R will represent a ring with ceter C, and N will denote the set of all nilpotent elements of R. As usual, [x, y] will denote the commutator xy - yx. Our aim is to prove the following theorems.

KEYWORDS: *Commutator, s-Unital Ring, Torsion-free Ring, Center.*

INTRODUCTION

The study of associative and non- associative rings has evoked great interest and importance. The results on associative and non-associative rings in which one does assume some identities in the center have been scattered throughout the literature.

Many sufficient conditions are well known under which a given ring becomes commutative.Notable among them are some given by Jacobson, Kaplansky and Herstein. Many Mathematicians of recent years studied commutativity of certain rings with keen interest. Among the mathematicians Herstein, Bell, Johnsen, Outcalt, Yaqub, Quadri and Abu-khuram are the ones whose contributions to this field are outstanding

PRELIMINARIES

Non - Associative Ring:

If R is an abelian group with respect to addition and with respect to multiplication R is distributive over addition on the left as well as on the right.

For every elements x, y,z of R

$(x+y)z = xz+yz, z(x+y) = zx+zy$

Alternative rings, Lie rings and Torsian rings are best examples of these non-associative rings.

Commutator:

For every x, y in a ring R satisfying $[x,y]=xy-yx$ then $[x,y]$ is called a commutator.

Commutative Ring:

For every x, y in a ring R if $xy = yx$ then R is called a commutative ring.

Non commutative ring is split from the commutative ring, i.e., R is not commutative with respect to multiplication. i.e., we cannot take $xy = yx$ for every x,y in R as an axiom.

s-Unital Ring:

A ring R is called s-Unital Ring if for each x in R such that $x \in Rx \cap xR$.

Torsion-Free ring:

If R is m-torsion free ring, then mx=0 implies x=0 for positive integer m and x is in R.

Center:

In a ring R, the center denoted by Z(R) is the set of all elements $x \in R$ Such that xy=yx for all $X \in R$, It is important to note that this definition does not depend on the associative of multiplication and in fact, we shall have occasion to deal with derivation of non-associative algebras.

MAIN RESULTS

Theorem 1. Let n be a fixed positive integer. Let R be an s-unital ring in which every commutator is n-torsion free. If R satisfies the polynomial identities $[x^n ,y^n] = 0$ and $[x, (xy)^n - (yx)^n] = 0$, then R is commutative.

Theorem 2. Let n be a fixed positive integer. Let R be an s-unital ring in which every commutator is (n+1)n-torsion free. If R satisfies the polynomial identity $(xy)^{n+1} – x^{n+1} y^{n+1} = 0$, then R is commutative.

In preparation for the proof of our theorems, we establish the following lemmas.

Lemma 1:- Let m, n be fixed positive integers.

(1) If $[a, [a, b]] = 0$ then $[a^n, b] = na^{n-1}[a, b]$, where $a, b \in R$.

(2) Let e be a pseudo-identity of $\{a, b\} \subseteq R$.If $a^m b = 0 = (a+e)^m b$ then $b = 0$.

(3) If R satisfies the polynomial identity $[x^n,y^m] = 0$ or $(xy)^{n+1} – x^{n+1} y^{n+1} = 0$ then the commutator ideal D(R) of R is contained in N.

(4) If R is an s-unital ring satisfying the polynomial identiy $[x^n ,y^n] = 0$, then there exists a positive integer k such that $k[x^n, y] = 0$.

Proof. (1) is well known. (3) is obvious by [4, Theorem] and [3, Theorem 1], and (4) is proved in [5, Lemma 10].

(2) We have

$0= a^{m-1}(a+e)^m b = a^{m-1}b$, and

$0 = (-1) a^m(a+e)^{m-1}a^m b = (-1)^m(a+e)^{m-1}\{-e+(a+e)\}^m b = (a+e)^{m-1}b$.

Continuing this process, we obtain eventually b = 0.

Lemma 2:-Let n be a fixed positive integer. Let R be an s-unital ring in which every commutator is n-torsion free.

(1) If R satisfies the polynomial identity $nx^m [x, y] = 0$ with a non-negative integer m, then R is commutative.

(2) If R satisfies the polynomial ideniiy $[x^n, y] = 0$, then R is commutative.

Proof. (1) Let a, b be arbitrary elements of R, and e a pseudo- identity of {a, b} . Since $na^m[a, b] = 0$ and $n(a+e)^m[a, b] = 0$, by Lemma 1 (2) we have n[a, b] = 0, and therefore[a, b] = 0.

(2) Since D(R) ⊆ N by Lemma 1(3), from the proof of [5, Lemma 9] it follows that N ⊆ C. Hence, by Lemma 1 (1), $nx^{n-1}[x, y] = [x^n,y] = 0$. Now, the commutativity of R is obvious by (1).

We are now ready to complete the proof of Theorem 1.

Proof of Theorem 1. First, we shall show that $[u, d^n] = 0$ for every $d \in R$ and every $u \in N$. Let f be a pseudo-identity of {d, u}. Since u is nilpotent, there exists a minimal positive integer m such that $[u^i,d^n] = 0$. For all integers i≥m**.** If m ≥ 2, then $0 = [(f+u^{m-1})^n, d^n] = [f^n +nu^{m-1} + u^{(m-1)n} ,d^n] = n[u^{m-1}, d^n]$, and hence $[u^{m-1}, d^n] = 0$, which contradicts the minimality of m. Thus m=1 and $[u^{m-1}, d^n] = 0$.

According to Lemma 1 (4), there exists a positive integer k such that $k[x^n,y] = 0$.Since D(R) ⊆ N by Lemma 1(3), it follows from what was just shown above that $[x^n, [x^n, y]] = 0$. Hence, by Lemma 1(1),$[x^{nk}, y] = kx^{n(k-1)} [x^n, y] = 0$. Now, let a,b be arbitrary elements of R, and let e be a pseudo-identity of {a, b}. Then, combining the above with the second polynomial identity, we have

$$0 = [a, (a^{nk}b)^n – (a^{nk-1} ba)^n] = [a, a^{n2k} b^n – a^{n2k-1} b^n a] = a^{n2k-1} [a, [a, b^n]].$$

Similarly, we have $0 = (a + e)^{n2k-1}[a, [a,b^n]]$. We obtain therefore $[a, [a, b^n]] = 0$ (Lemma 1 (2)) and na^{n-1} (Lemma 1 (1)). Again by Lemma 1 (2), $n[a, b^n] = 0$ and hence $[a, b^n] = 0$. Now, R is commutative by Lemma 2 (2).

From the proof of [5, Theorem 3], one will easily see that if R is an s-unital ring in which every commutator is a-torsion free then the polynomial identity $(xy)^n = (yx)^n = 0$ implies $[x^n, y^n] = 0$. Hence, Theorem1 implies the following

Corollary 1. Let R be an s-unital ring in which every commutator is n-torsion free. If R satisfies the identity $(xy)^n = (yx)^n$ then R is commutative.

Proof of Theorem 2. First, we shall show that $[u, d^{n+1}] = 0$ for every $d \in R$ and every $u \in N$. Let f be a pseudo-identity of {d, u}.If u_0, is the quasi-inverse of u, then $fu_0 = u_0f= u_0$ and the map $\sigma : R \to R$ defined by $x \to x-u_0x – xu + u_0xu$ is a ring automorphism of R. By hypothesis,

$$0 = (f – u)^{n+1} \{(f – u_0)^{n+1} d^{n+1} (f – u)^{n+1}\} (f – u_0) – d^{n+1} (f – u)^n$$

$$= (f – u)^{n+1} \sigma (d)^{n+1} (f – u_0) - d^{n+1} (f – u)^n$$

$$= (f – u)^{n+1} \sigma (d^{n+1})(f – u_0) - d^{n+1} (f – u)^n$$

$$= (f – u)^n d^{n+1} - d^{n+1}(f – u)^n = [(f – u)^n,d^{n+1} d^{n+1}]$$

Choose the minimal positive integer m such that $[u^i, d^{n+1}] = 0$ for all i≥m. Suppose m>1. Then, by the above, $[(f-u^{m-1})^n, d^{n+1}] = 0$.Combining this with $[u^i, d^{n+1}] = 0$ (i≥m) we get $n[u^{m-1}, d^{n+1}]$ and hence $[u^{m-1}, d^{n+1}] = 0$. But this contradicts the minimality of m. Thus, k=1, and hence $[u, d^{n+1}] = 0$.

Let R* be the subring generated by all (n+1)-th powers of elements of R. Then, it follows from what was just shown above that the set N^* of nilpotent elements of R* is contained in the center C* of R*. Moreover, by Lemma 1 (3), $D(R^*)\subseteq N^*\subseteq C^*$. Let a*, b* be arbitrary elements of R*. Then both $[a^*, b^{*n+1}]$ are in C*. Hence, by Lemma 1 (1),

$$na^{*n+1}[a^*, b^{*n+1}] = a^{*2}[a^{*n},b^{*n+1}] = a^*[a^{*n},b^{*n+1}]a^*$$

$$= a^{*n+1}b^{*n+1}a^* - a^*b^{*n+1}a^{*n+1}$$

$$= (a^*b^*)^{n+1}a^* - a^*(b^*a^*)^{n+1} = 0.$$

According to Lemma 1(2), it follows then that $n[a^*, b^{*n+1}] = 0$, and therefore $[a^*, b^{*n+1}] = 0$. Now, by Lemma 2 (2), $[a^*, b^*] = 0$. and hence for all x,y in R, $[x^{n+1}, y^{n+1}] = 0$. Combining this with the polynomial identity $(xy)^{n+1}-x^{n+1}y^{n+1} = 0$, we obtain $(xy)^{n+1} = (yx)^{n+1}$. Hence, R is commutative by Corollary 1.

REFERENCES

[1] H. ABU-KHUZAM : A commutativity theorem for rings, Math. Japonica, to appear.
[2] H.E. BELL : On rings with commuting powers, Math. Japonica 24 (1979), 473—478.
[3] I.N. HERSTEIN : Power maps in rings, Michigan Math. J. 8 (1961), 29—32.
[4] I.N. HERSTEIN : A commutativity theorem, J. Algebra 38 (1976), 112—118.
[5] Y. HIRANO, M. HONGAN and H. TOMINAGA : Commutativity theorems for certain rings, Math. J. Okayama Univ. 22 (1980), 65-72