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A New Integral Transform for Solution of Convolution Type Volterra Integral Equation of First Kind

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Abstract - *In this paper, the convolution theorem and uniqueness theorem for the new integral transform is proved. Further, a new integral transform and convolution theorem is applied to solve convolution type Volterra integral equation of first kind.*

keywords: *Integral transform, Integral equation, Convolution theorem*

I. Introduction

There are many integral transforms which support the tools for solving ordinary and partial differential equations. A new integral transform which was introduced by Artion Kashuri and Akli Fundo [1] can also be used to solve ordinary and partial differential equations. Elzaki transform [2,3] found to be effective tool in solving convolution type Volterra integral equation of first kind and higher order ordinary differential equations. Agrawal and Gupta [4] applied Sumudu transform for the solution of Abel's integral equation. Janki Vashi & M. G. Timol [5] established the relationship between Laplace and Sumudu transform and also provided some applications of Sumudu transform in Physics and electric circuits. The main purpose of this paper is to show the applicability of a **New Integral Transform** to solve convolution type Volterra integral equation of first kind and Abel's integral equation.

II. Preliminaries

Definition. Consider the class of functions F [1], where

$$F = \left\{ f(t) \mid \exists\, M, k_1, k_2 > 0 \text{ such that } |f(t)| \le M e^{\frac{|t|}{k_i^2}}, \text{if } t \in (-1)^i \times [0,\infty) \right\} \tag{1}$$

For a given function in the set F, the constant M must be finite number, k_1, k_2 may be finite or infinite. A new integral transform denoted by the operator $\mathcal{K}(.)$ is defined by

$$\mathcal{K}[f(t)] = H(v) = \frac{1}{v}\int_0^\infty e^{\frac{-t}{v^2}} f(t)dt, t \ge 0, -k_1 < v < k_2 \tag{2}$$

New integral transform of some special functions

i) $\mathcal{K}[1] = v$

ii) $\mathcal{K}[t^n] = n!\, v^{2n+1}$

iii) $\mathcal{K}[e^{at}] = \frac{v}{1-av^2}$

iv) $\mathcal{K}[sinat] = \frac{av^3}{1+a^2v^4}$

v) $\mathcal{K}[cosat] = \frac{v}{1+a^2v^4}$

vi) $\mathcal{K}[\sinh(at)] = \frac{av^3}{1-a^2v^4}$

vii) $\mathcal{K}[\cosh(at)] = \frac{v}{1-a^2v^4}$

Here we provide the proof of two new integral transform for further reference

viii) $\mathcal{K}[t^{-\alpha}] = \Gamma(1-\alpha)v^{-2\alpha+1}$, where $\alpha > 0$

Proof. By definition of new integral transform, we have

$$\mathcal{K}[t^{-\alpha}] = \frac{1}{v}\int_0^\infty e^{\frac{-t}{v^2}}\, t^{-\alpha} dt$$
$$= \frac{1}{v}\int_0^\infty e^{-u}\,(uv^2)^{-\alpha} v^2 du, \quad \text{putting } \frac{t}{v^2} = u$$
$$= v^{-2\alpha+1}\int_0^\infty e^{-u}\, u^{-\alpha} du$$
$$= v^{-2\alpha+1}\int_0^\infty e^{-u}\, u^{(1-\alpha)-1} du$$
$$= \Gamma(1-\alpha) v^{-2\alpha+1} \quad (3)$$

ix) $\mathcal{K}[t^{\alpha-1}] = \Gamma(\alpha) v^{2\alpha-1}$

Proof. Again, by definition of new integral transform, we have

$$\mathcal{K}[t^{\alpha-1}] = \frac{1}{v}\int_0^\infty e^{\frac{-t}{v^2}}\, t^{\alpha-1} dt$$
$$= \frac{1}{v}\int_0^\infty e^{-u}\,(uv^2)^{\alpha-1} v^2 du$$
$$= v^{2\alpha}\int_0^\infty e^{-u}\, u^{\alpha-1} du \text{ , putting } \frac{t}{v^2} = u$$
$$= \Gamma(\alpha) v^{2\alpha-1} \quad (4)$$

Theorem 1 Let $G(v)$ be a new integral transform of $f(t)$, then

i) $\mathcal{K}[f'(t)] = \frac{\mathrm{G(v)}}{v^2} - \frac{f(0)}{v}$

ii) $\mathcal{K}[f''(t)] = \frac{\mathrm{G(v)}}{v^4} - \frac{f(0)}{v^3} - \frac{f\prime(0)}{v}$

Theorem 2[Duality relation] [1] Let $f(t) \in F$ with Laplce transform $F(s)$. Then a new integral transform $G(v)$ of $f(t)$ is given by:

$$G(v) = \frac{1}{v} f\left(\frac{1}{v^2}\right) \quad (5)$$

Theorem 3. [1] Let $f(t)$ and $g(t)$ be defined in F having Laplace transform $F(s)$ and $G(s)$ and a new integral transform $A(v)$ and $B(v)$. Then a new integral transform of the convolution of f and g:

$$(f * g)(t) = \int_0^t f(x)g(t-x)dx$$

is given by

$$\mathcal{K}(f * g)(t) = vA(v)B(v)$$

Proof: The Laplace transform of (f ∗g) is given by

$$L(f * g)(t) = F(s)G(s)$$

By the duality relation (5) we have

$$\mathcal{K}[(f * g)](t) = \frac{1}{v} L[(f * g)(t)]$$

and since

$$A(v) = \frac{1}{v} F\left(\frac{1}{v^2}\right), B(v) = \frac{1}{v} G\left(\frac{1}{v^2}\right)$$

Then

$$\mathcal{K}[(f * g)](t) = \frac{1}{v}\left[F\left(\frac{1}{v^2}\right) G\left(\frac{1}{v^2}\right)\right]$$

$$= \frac{1}{v}[vA(v)vB(v)]$$
$$= vA(v)B(v) \quad (6)$$

Which proves the theorem.

Theorem 4[Uniqueness theorem] If $A(v)$ and $B(v)$ are new integral transforms of $f(t)$ and $g(t)$ respectively, then:

$$A(v) = B(v) \Rightarrow f(t) = g(t)$$

Proof. Consider

$$A(v) = B(v)$$
$$\mathcal{K}[f(t);v] = \mathcal{K}[g(t);v]$$

$$\frac{1}{v}L\left[f(t);\frac{1}{v^2}\right] = \frac{1}{v}\ L\left[g(t);\frac{1}{v^2}\right]$$
$$L\left[f(t);\frac{1}{v^2}\right] = L\left[g(t);\frac{1}{v^2}\right]$$

By uniqueness of Laplace transform we obtain f(t) = g(t).

III. Main Result

Consider the convolution type Volterra integral equation of first kind

$$f(t) = \int_0^t k(t-x)u(x)\,dx$$

$$= k(t) * u(t) \qquad (7)$$

Let $\mathcal{K}[f(t)](v) = F(v), \mathcal{K}[k](v) = K(v), \mathcal{K}[u](v) = U(v)$

Applying the new integral transform on both sides of (7), we get

$$\mathcal{K}[f(t)] = \mathcal{K}[k(t) * u(t)]$$

$F(v) = vK(v)U(v)$, by convolution theorem

$$\Rightarrow U(v) = \frac{F(v)}{vK(v)} \qquad (8)$$

Applying the inverse of a new integral transform, we get

$$u(t) = \mathcal{K}^{-1}\left\{\frac{F(v)}{vK(v)}\right\} \qquad (9)$$

Now we solve Abel's integral equation

$$f(t) = \int_0^t \frac{u(x)}{(t-x)^{\alpha}}\,dx,\ 0 < \alpha < 1 \qquad (10)$$

Solution. The integral equation (10) can be written in convolution type as

$$f(t) = u(t) * t^{-\alpha} \qquad (11)$$

Taking new integral transform of both sides of (11), we obtain

$$\mathcal{K}[f(t)] = v\mathcal{K}[u(t)]\mathcal{K}[t^{-\alpha}]$$

$\Rightarrow F(v) = vU(v)\Gamma(1-\alpha)v^{-2\alpha+1}$, from(3)

$$= U(v)\Gamma(1-\alpha)v^{-2\alpha+2}$$

$$\Rightarrow U(v) = \frac{F(v)}{\Gamma(1-\alpha)v^{-2\alpha+2}}$$

$$= \frac{F(v)v^{2\alpha-2}}{\Gamma(1-\alpha)}$$

$$= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \{\Gamma(\alpha)F(v)v^{2\alpha-2}\}$$

$$= \frac{sin\pi\alpha}{\pi} \{\Gamma(\alpha)F(v)v^{2\alpha-2}\}$$

$$= \frac{1}{v}\frac{sin\pi\alpha}{\pi} \{\Gamma(\alpha)F(v)v^{2\alpha-1}\}$$

$$= \frac{1}{v^2}\frac{sin\pi\alpha}{\pi} \{v\Gamma(\alpha)F(v)v^{2\alpha-1}\}$$

$$= \frac{1}{v^2}\frac{sin\pi\alpha}{\pi}\, \mathcal{K}\left[f(t) * t^{(\alpha-1)}\right], \text{ from (4)}$$

$$\Rightarrow \mathcal{K}[u(t)] = \frac{1}{v^2}\frac{sin\pi\alpha}{\pi}\, \mathcal{K}\left[\int_0^t (t-x)^{\alpha-1} f(x)dx\right]$$

$$= \frac{1}{v^2}\frac{sin\pi\alpha}{\pi}\, \mathcal{K}[G(t)] \tag{12}$$

Where $[G(t)] = \int_0^t (t-x)^{\alpha-1} f(x)dx, G(0) = 0$. Let $\mathcal{K}[G(t)] = H(v)$

We know that

$$\mathcal{K}[G'(t)] = \frac{H(v)}{v^2} - \frac{G(0)}{v} = \frac{H(v)}{v^2}$$

$$= \frac{\pi v^2 \mathcal{K}[u(t)]}{sin\pi\alpha}\frac{1}{v^2} = \frac{\pi\mathcal{K}[u(t)]}{sin\pi\alpha}$$

$$\Rightarrow \mathcal{K}[u(t)] = \frac{sin\pi\alpha}{\pi}\, \mathcal{K}[G'(t)]$$

$$\Rightarrow u(t) = \frac{sin\pi\alpha}{\pi}\, [G'(t)]$$

$$\Rightarrow u(t) = \frac{sin\pi\alpha}{\pi}\frac{d}{dt}\, [G(t)]$$

$$\Rightarrow u(t) = \frac{sin\pi\alpha}{\pi}\frac{d}{dt}\left[\int_0^t (t-x)^{\alpha-1} f(x)dx\right]$$

Which is the solution of Abel's integral equation.

IV. Applications

In this section we solve particular Abel's integral equation and provide the verification by Laplace transform.

Example 1. Solve the integral equation

$$\int_0^t \frac{u(x)}{(t-x)^{\frac{1}{3}}}\, dx = t(1+t) \tag{13}$$

Solution. Equation (13) can be written as

$$u(t) * t^{-\frac{1}{3}} = t(1+t) \tag{14}$$

Taking new integral transform

$$\mathcal{K}\left[u(t) * t^{-\frac{1}{3}}\right] = \mathcal{K}[t] + \mathcal{K}[t^2]$$

$$v\mathcal{K}[u(t)]\mathcal{K}\left[t^{-\frac{1}{3}}\right] = v^3 + 2v^5$$

$$v\mathcal{K}[u(t)]\Gamma\left(1-\frac{1}{3}\right)v^{\frac{1}{3}} = v^3 + 2v^5$$

$$\mathcal{K}[u(t)] = \frac{v^3 + 2v^5}{\Gamma\left(\frac{2}{3}\right)v^{\frac{4}{3}}}$$

$$\mathcal{K}[u(t)] = \frac{1}{\Gamma\left(\frac{2}{3}\right)}\left(v^{\frac{5}{3}} + 2v^{\frac{11}{3}}\right)$$

Taking inverse of a new integral transform, we have

$$u(t) \;= \frac{1}{\Gamma\left(\frac{2}{3}\right)}\left\{\mathcal{K}^{-1}\left[v^{\frac{5}{3}}\right] + \mathcal{K}^{-1}\left[v^{\frac{11}{3}}\right]\right\}$$

$$= \frac{1}{\Gamma\left(\frac{2}{3}\right)}\left\{\left[\frac{t^{\frac{4}{3}-1}}{\Gamma\left(\frac{4}{3}\right)}\right] + 2\left[\frac{t^{\frac{7}{3}-1}}{\Gamma\left(\frac{7}{3}\right)}\right]\right\}$$

$$= \frac{1}{\Gamma\left(\frac{2}{3}\right)}\left[\frac{t^{\frac{1}{3}}}{\frac{1}{3}\Gamma\left(\frac{1}{3}\right)} + 2\frac{t^{\frac{4}{3}}}{\left(\frac{4}{3}\right)\left(\frac{1}{3}\right)\Gamma\left(\frac{1}{3}\right)}\right]$$

$$= \frac{3t^{\frac{1}{3}}}{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)}\left(1 + \frac{3t}{2}\right)$$

$$= \frac{3\sqrt{3}}{4\pi}t^{\frac{1}{3}}(2 + 3t)$$

Verification by Laplace transform:

Taking Laplace transform on both sides of equation (14), we have

$$L\left[u(t) * t^{-\frac{1}{3}}\right] = L[t] + L[t^2]$$

$$L[u(t)]\frac{\Gamma\left(\frac{2}{3}\right)}{p^{\frac{2}{3}}} = \frac{1}{p^2} + \frac{2}{p^3}$$

$$L[u(t)] = \frac{1}{\Gamma\left(\frac{2}{3}\right)}\left[\frac{1}{p^{\frac{4}{3}}} + \frac{2}{p^{\frac{7}{3}}}\right]$$

Taking inverse Laplace transform, we have

$$u(t) = \frac{1}{\Gamma\left(\frac{2}{3}\right)}\left[L^{-1}\left(\frac{1}{p^{\frac{4}{3}}}\right) + 2L^{-1}\left(\frac{1}{p^{\frac{7}{3}}}\right)\right]$$

$$= \frac{1}{\Gamma\left(\frac{2}{3}\right)}\left[\frac{t^{\frac{1}{3}}}{\frac{1}{3}\Gamma\left(\frac{1}{3}\right)} + 2\frac{t^{\frac{4}{3}}}{\left(\frac{4}{3}\right)\left(\frac{1}{3}\right)\Gamma\left(\frac{1}{3}\right)}\right]$$

$$= \frac{3\sqrt{3}}{4\pi}t^{\frac{1}{3}}(2 + 3t)$$

Example 2. Solve the integral equation

$$\int_0^t e^{t-x} u(x)\, dx = t \tag{15}$$

Solution. Equation (15) can be written as

$$u(t) * e^t = t \tag{16}$$

Applying new integral transform, we get

$$v^3 = v \frac{v}{1 - v^2} \mathcal{K}[u(t)]$$

$$\Rightarrow \mathcal{K}[u(t)] = v - v^3$$

Applying inverse of a new integral transform, we get

$$u(t) = 1 - t$$

Verification by Laplace transform

Applying Laplace transform on both sides of (16), we get

$$\frac{1}{p-1} L[u(t)] = \frac{1}{p^2}$$

$$\Rightarrow L[u(t)] = \frac{1}{p} - \frac{1}{p^2}$$

Applying inverse Laplace transform, we get

$$u(t) = 1 - t$$

V. Conclusion

Author found that the new integral transform is the powerful mathematical tool to solve convolution type Volterra integral equation of first kind.

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