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Orderings and Preorderings on Semihyperrings

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Abstract — *The main aim of this paper is to develop the study of semihyperrings in real algebra. By introducing the notions of semireal semihyperrings, preorderings and orderings on semihyperrings, we discuss the interplay between them. For the 0-regular semihyperrings, we establish analogous Artin-Schreier theory.*

Keywords — *semihyperrings, semireal semihyperrings, preorderings,orderings, 0-regular.*

I. INTRODUCTION

The research on hyperstructures can be traced back to 1934. F.Marty[8] introduced the concept of hyperoperation at the Eighth Congress of Scandinavian Mathematics Conference and defined hypergroups as a generalization of groups. In 1956, M.Krasner [9,10] introduced a kind of hyperrings, called Krasner hyperrings. Krasner hyperrings have two operations, where addition is a hyperoperation, and multiplication is an operation. This type of hyperring is widely studied [2,12,14]. In 2007, R.Ameri and H.Hedayati [13] introduced a kind of semihyperrings in the sense of Krasner and discussed some properties about the k-hyperideals of semihyperrings.

The concept of orderings on fields was first introduced by E.Artin and O.Schreier[4,5,6] for solving Hilbert's 17th Problem in 1927. Later, orderings were introduced to some other algebraic categories, such as ring category[16,17], module category[3,7] and semiring category [1,15]. In 2006, M.Marshall [11] introduced the concept of multirings and established partial Artin-Schreier theory in this category. The concept of multirings is similar to the concept of Krasner hyperrings, and the only difference between them is about the distributive property. In view of the importance of the study on the orderings(preorderings) of rings(fields) in real algebraic geometry, we introduce the concept of orderings(preorderings) in semihyperring category and develop analogous Artin-Schreier theory.

II. SEMIHYPERRINGS

In this section we quickly review some basic definitions and properties of semihyperrings which will be used in this paper.

Let H be a non-empty set. Recall that a hyperoperation $\circ$ on H is a map from $H\times H$ to $P^*(H)$, where $P^*(H)$ denotes the set containing all non-empty subsets of H.

A hypergroupoid $(H,\circ)$ is non-empty set H equipped with a hyperoperation $\circ$. For any two non-empty subsets A and B of H, we define

$$A\circ B=\bigcup_{a\in A,b\in B} a\circ b\,.$$

For simplicity, we write $A\circ x$, $x\circ B$ instead of $A\circ\{x\}$,$\{x\}\circ B$ respectively.

A hypergroupoid $(H,\circ)$ is called a semihypergroup if H satisfies associative law: $x\circ(y\circ z)=(x\circ y)\circ z$ for all $x,y,z\in H$, that is $x\circ\{y\circ z\}=\{x\circ y\}\circ z$. A semihypergroup H is said to be commutative if its hyperoperation is commutative, that is $x\circ y=y\circ x$ for all $x,y\in H$.

Definition 2.1[13]**.** A semihyperring $(S,+,\cdot)$ is an non-empty set S equipped with a hyperoperation + and a usual operation $\cdot$ which satisfy the following conditions:

(1) $(S,+)$ is a commutative semihypergroup;

(2) $(S,\cdot)$ is a semigroup;

(3) $\cdot$ is distributive with respect with +, i.e. , $\forall\ x,y,z\in S$, $x(y+z)=xy+xz$, $(x+y)z=xz+yz$.

A semihyperring S is called with zero element, if S contains an element 0 such that $x\cdot 0=0\cdot x=0$ and $x+0=\{x\}$ for all $x\in S$.

A semihyperring S is called with identity element, if S contains an element 1 such that $x\cdot 1=1\cdot x=x$ for all $x\in S$.

A semihyperring S is said to be commutative if its multiplicative operation is commutative, that is $x\cdot y=y\cdot x$ for all $x,y\in S$.

By convention,we write $0+x=x$, xy instead of $0+x=\{x\}$, $x\cdot y$ *respectively*.

Throughout the paper, all semihyperrings are assumed to be commutative and contain 0 and $1\neq 0$ unless otherwise stated.

Let A be a non-empty subset of semihyperring S. The negative set of A, denoted $-A$, is the set $\{b\in S|0\in a+b$ for some $a\in A\}$. By convention, we write $-x$ instead of $-\{x\}$.

It easy to check that $-A = \cup_{a\in A} -a$. Notice that the negative set of a subset of *S* does not always exist. However, it follows by definition that if $-A$ is not empty and $x \in -A$, then $-x$ is not empty too. Indeed, $a \in -x$ implies that $x \in -a$.

Proposition 2.2. Let *S* be a semihyperring. Assume that the following negative sets always exist. For all $x, y, z, w \in S$,
(1) zero element is unique;
(2) identity element is unique;
(3) $0 = -0$;
(4) $x \in -(-x)$;
(5) $-x - y \subseteq -(x + y)$;
(6) $x(-y) \subseteq -xy,\ (-x)y \subseteq -xy$;
(7) $(x + y)(z + w) \subseteq xz + xw + yz + yw.$

Remark 2.3. In general, the doubly distributive property is not hold for semihypering, i.e., the (7) in the Proposition 2.2 cannot be rewritten as $(x + y)(z + w) = xz + xw + yz + yw.$

Definition 2.4. Let *S* be a semihyperring. A non-empty subset *T* of *S* is called a subsemihyperring of *S* if *T* contains 0,1 and for all $x, y \in T$, $x + y \subseteq T$ and $xy \in T$.

Definition 2.5. Let *S* be a semihyperring. A nonempty subset *I* of *S* is a hyperideal if:
(1) $\forall x, y \in I, x + y \subseteq I;$
(2) $\forall x \in I, r \in S, xr \in I.$

Definition 2.6. Let *S* be a semihyperring. A hyperideal $I \subset S$ is prime if for all $x, y \in S$ with $xy \in I$, we have $x \in I$ or $y \in I$.

Definition 2.7. Let *S* be a semihyperring and *a* be an element of *S*. *S* is called *a*-regular if *S* satisfies the following property: $a \in b + c$, where $b, c \in S$, implies that $b + c = \{a\}$. Let *A* be subset of *S*. *S* is called *A*-regular if *S* is *a*-regular for all $a \in A$.

Obviously, if a semihyperring *S* is regular with respect to all elements of it, then *S* is a semiring.

Example 2.8. Let $S = \{a \mid a \geq 0, a \text{ is a real number}\}$. The multiplicative operation is the usual multiplication of real numbers. The (hyper)addition is given as follows:

$$x \oplus y = \begin{cases} x & \text{if } y = 0 \\ y & \text{if } x = 0 \\ [\max\{x, y\}, +\infty) & \text{if } x > 0, y > 0 \end{cases}$$

It is easy to check that *S* is a semihyperring and *S* is 0-regular. Notice that for any non-zero element *a* of *S*, *S* is not *a*-regular.

Definition 2.9. An element *a* in a semihyperring *S* is invertible if there exists $b \in S$, such that $ab = ba = 1$. Also we call an invertible element a unit. The unit set of *S*, denoted *U*(*S*), is the set of all units of *S*.

Proposition 2.10. Let *S* be a semihyperring and *a* be an element of *S*. If *S* is *a*-regular, then *S* is *ua*-regular for all $u \in U(S)$. In particular, if *S* is *u*-regular for some $u \in U(S)$, then *S* is *U*(*S*)-regular.

Proof. Suppose that $ua \in b + c$ with $b, c \in S$, then $a \in u^{-1}b + u^{-1}c$. Hence $a = u^{-1}b + u^{-1}c$ since *S* is *a*-regular. It follows that $ua = b + c$. Therefore, *S* is *ua*-regular. Suppose that *S* is *u*-regular for some $u \in U(S)$. Notice that, for any $v \in U(S)$, $v = (vu^{-1})u$ and $vu^{-1} \in U(S)$. This means that *S* is *v*-regular, and thus, *S* is *U*(*S*)-regular.

Proposition 2.11. Let *S* be a 0-regular semihyperring and *a* be an element of *S*. If $-a$ is not empty, then $-a$ is a singleton.

Proof. If $b, c \in -a$, then $0 \in a + b$ and $0 \in a + c$. It follows that $0 = a + b$ and $0 = a + c$, since *S* is 0-regular. Hence, we have

$$b = b + 0 = b + (a + c) = (b + a) + c = 0 + c = c .$$

Therefore, $-a$ is a singleton.

By this Proposition, some results of Proposition 2.2 can be rewrote as follows:

Proposition 2.12. Let *S* be a 0-regular semihyperring. And let $x, y \in S, A, B \subseteq S$. If their negative sets exist, then

(1) $x = -(-x)$;
(2) $-x - y = -(x + y)$;
(3) $x(-y) = -xy = (-x)y$;
(4) $(-x)(-y) = xy$;
(5) $(-A)(-B) \subseteq AB$.

III. PREORDERINGS AND ORDERINGS ON SEMIHYPERRINGS

In this section, we first introduce the concepts of preorderings and orderings on semihyperrings, and then give some necessary and sufficient conditions for a semihyperring to possess preorderings and orderings.

Let *S* be a semihyperring. Then we obtain two multiplicatively closed subsets of *S* as follows:

$$\sum S^2 := \left\{ \sum_{i=1}^{n} a_i^2 \,\middle|\, n \text{ is a positive integer, and } a_i \in S \text{ for } i = 1, 2, \cdots, n \right\};$$

$$1 + \sum S^2 := \left\{ 1 + t \,\middle|\, t \in \sum S^2 \right\}.$$

Definition 3.1. A semihyperring *S* is called a semireal semihyperring if $0 \notin 1 + \Sigma S^2$.

Definition 3.2. A semihyperring *S* is called a real semihyperring if *S* satisfies the following property: for any positive integer *n*, if $0 \in \sum_{i=1}^{n} x_i^2$,then $x_1 = x_2 = \cdots x_n = 0$.

We clearly have that *S* is real implies *S* is semereal.

Definition 3.3. Let *S* be a semihyperring. A subset $T \subseteq S$ is called a preordering on *S* if *T* satisfies the following conditions:

(1) $T + T \subseteq T$;
(2) $T \cdot T \subseteq T$;
(3) $x^2 \in T$ for all $x \in S$;
(4) $0 \notin 1 + T$ (or equivalently, $1 \notin -T$).

Theorem 3.4. Let *S* be a semihyperring. Then *S* is semireal if and only if *S* possesses a preordering.

Proof. If *S* is semireal, set $T := \Sigma S^2$, then *T* is a preordering on *S*. Conversely, assume that *S* possesses a preordering *T*. By Definition 3.3 (1) and (3), we have $\Sigma S^2 \subseteq T$. It follows by Definition 3.3 (4) that $0 \notin 1 + \Sigma S^2$. Hence, *S* is semireal.

Now we consider 0-regular semihyperrings.

Lemma 3.5. Let *S* be a 0-regular semihyperring and *P* a multiplicatively closed set of *S*. If $x, y \in -P$, then $xy \in P$.

Proof. If $x, y \in -P$, then there exist $p, q \in P$, such that $x \in -p$ and $y \in -q$. Hence, $p \in -x$ and $q \in -y$. By Proposition 2.11, we have $p = -x$ and $q = -y$. It follows by Proposition 2.12 that $xy = (-x)(-y) = pq \in P$.

Proposition 3.6. Let *T* be a preordering on a 0-regular semihyperring *S*. If $T \cup (-T) = S$, then $T \cap (-T)$ is a hyperideal of *S*.

Proof. Notice that *T* is multiplicatively closed. It is follows by Lemma 3.5 that this Proposition holds.

Lemma 3.7. Let *T* be a preordering on a 0-regular semihyperring *S*, and let $a, b \in S$. If $ab \in -T$, then at least one of $T[a] := T + aT$, $T[b] := T + bT$ is a preordering on *S*.

Proof. It is easy to check that $T[a], T[b]$ satisfy the conditions (1),(2) and (3) in Definition 3.3. If neither $T[a]$ nor $T[b]$ is a preordering, then we would have $0 \in T[a]$ and $0 \in T[b]$. Hence, there exist $t_i \in T, i = 1, 2, 3, 4$, such that $0 \in 1 + t_1 + at_2$ and $0 \in 1 + t_3 + at_4$. It follows that $at_2 \in -(1 + t_1)$ and $bt_4 \in -(1 + t_3)$. Therefore, by Proposition 2.12 and by Proposition 2.2, we have

$$abt_2t_4 = (at_2)(bt_4) \in [-(1 + t_1)][-(1 + t_3)] \subseteq (1 + t_1)(1 + t_3) \subseteq 1 + (t_1 + t_3 + t_1t_3) \subseteq 1 + T.$$

Since $ab \in -T$, there exists $t_5 \in T$ such that $ab = -t_5$. Hence, there exists $t_6 \in T$, such that $-t_2t_4t_5 \in 1 + t_6$. It follows that $0 \in (-t_2t_4t_5) + t_2t_4t_5 \subseteq 1 + t_6 + t_2t_4t_5 \subseteq 1 + T$. Thus, $1 \in -T$, a contradiction.

Definition 3.8. Let *S* be a semihyperring and let *A* be a subset of *S*. The support set of *A*, denoted *supp*(*A*), is the set $A \cap (-A)$.

(1) *A* is called prime if $ab \in supp(A)$, where $a, b \in S$, implies that $a \in supp(A)$ or $b \in supp(A)$;

(2) *A* is called *N*-prime if $ab \in -A$, where $a, b \in S$, implies that $a \in A$ or $b \in A$;

(3) *A* is called *C*-prime if $ab \in supp(A)$, where $a, b \in S$, implies that $a \in A$ or $b \in A$.

Remark 3.9. Clearly, *A* is prime ⇒ *A* is *C*-prime, *A* is *N*-prime ⇒ *A* is *C*-prime.

Definition 3.10. A preordering *T* on a semihyperring *S* is called an ordering on *S* if it satisfies the following conditions:

(1) $T \cup (-T) = S$;

(2) *T* is *C*-prime.

Proposition 3.11. Let *T* be an ordering on a 0-regular semihyperring *S*. Then *T* is *N*-prime.

Proof. Let $ab \in -T$. Assume that $a \notin T$ and $b \notin T$. Since $T \cup (-T) = S$, we have $a \in -T$ and $b \in -T$, and thus, $a \in -s, b \in -t$ with some $s, t \in T$. It follows by Proposition 2.12 (4) that $ab \in (-s)(-t) = \{st\} \subseteq T$. This means that $ab \in T \cap (-T) = supp(T)$. Since *T* is *C*-prime, we have $a \in T$ or $b \in T$, a contradiction.

For a semihyperring *S*, we set

$$N(S) := \{x \in S \mid -x \text{ is not empty}\}, M(S) := S \setminus N(S) = \{x \in S \mid -x \text{ is empty}\}.$$

Proposition 3.12. Let *S* be a 0-regular semihyperring, and let *T* be a preordering on *S*. If $T \cup (-T) = S$, and *N*(*S*) satisfies the property: $ab \in N(S)$ implies $a \in N(S)$ and $b \in N(S)$, then the following statements are equivalent.

(1) *T* is prime;

(2) *T* is *N*-prime;

(3) *T* is *C*-prime.

Proof. By Remark 3.9 and Proposition 3.11, it only remains to prove (2) ⇒ (1). Assume that *T* is *N*-prime. If $ab \in supp(T)$, then $a \in T$ or $b \in T$. For proving that *T* is prime, there are two cases to consider.

Case 1. $a \in T$ and $b \in T$. In this case, it is enough to show that $a \in -T$ or $b \in -T$. Since $ab \in supp(T) \subseteq -T \subseteq N(S)$, by assumption, we have $a \in N(S)$ and $b \in N(S)$. Therefore, $(-a)(-b) = ab \in supp(T) \subseteq -T$. It follows by *T's* *N*-primeness that $-a \in T$ or $-b \in T$, and thus, $a \in -T$ or $b \in -T$. This implies that $a \in supp(T)$ or $b \in supp(T)$.

Case 2. Exactly one of *a*, *b* belongs to *T*. Without loss of generality, we may assume that $a \in T$ but $b \notin T$. Since $(-a)b = -ab \in -T$, and *T* is *N*-prime, we have $-a \in T$. Hence $a \in -T$. This implies that $a \in supp(T)$. This completes the proof.

In commutative ring category, the condition that $ab \in N(S)$ implies $a \in N(S)$ and $b \in N(S)$ is obviously satisfied. Therefore, by Proposition 3.6, the condition (2) in Definition 3.10 is equivalent to (2') supp(T) is a prime ideal. This is consistent with the definition of orderings in the commutative ring category[16, Definition 3.1].

The following result gives another characterization of an ordering on 0-regular semihyperring.

Theorem 3.13. A preordering *T* on a 0-regular semihyperring *S* is an ordering on *S* if and only if it satisfies the following conditions:

(1) $M(S) \subseteq T$;

(2) *T* is *N*-prime.

Proof. Let *T* be an ordering on *S*. *T's* *N*-primeness follows from Proposition 3.11. For (1), since $M(S) \subseteq S = T \cup (-T)$, and $M(S) \cap (-T) = \varnothing$, we have $M(S) \subseteq T$.

Conversely, since (2) holds, *T* is *C*-prime. Let $x \in S = N(S) \cup M(S)$. If $x \notin T$, then $x \notin M(S)$. This means that $x \in N(S)$, and thus, $-x$ is not empty. Since $(-x)x = -x^2 \subseteq -T$ and $x \notin T$, and by condition (2), we have $-x \in T$. Hence, $x \in -T$. This implies that $x \in T \cup (-T)$. Therefore, $T \cup (-T) = S$. This completes the proof.

Theorem 3.14. Let *S* be a 0-regular semihyperring, and let *T* be a maximal preordering on *S*. Then *T* is an ordering on *S*.

Proof. We will prove this Theorem by using Theorem 3.13. First we claim that T is N-prime. Indeed, assume that $ab \in -T$, by Lemma 3.7, at least one of $T[a]$, $T[b]$ is a preordering on S. Since T is maximal, we have $T[a] = T$ or $T[b] = T$. Hence, either $a \in T$ or $b \in T$.

It only remains to prove that $M(S) \subseteq T$. Assume, instead, that $M(S) \not\subseteq T$. Then there exists an element $x \in M(S)$ but $x \notin T$. Now we consider the set $T[x]$. Notice that $T[x]$ satisfies the conditions (1),(2), and (3) of Definition 3.3, and $x \in T[x] \setminus T$. By the maximality of T, we have $0 \in 1 + T[x]$. Hence there exist $t_1, t_2 \in T$, such that $0 \in 1 + t_1 + xt_2$. Thus, $0 = x \cdot 0 \in x + xt_1 + x^2 t_2$. This means that $-x$ is not empty, that is , $x \notin M(S)$, a contradiction. Therefore, $M(S) \subseteq T$. This completes the proof.

Corollary 3.15. Any preordering T on a 0-regular semihyperring is contained in an ordering.

Proof. The proof is similar to the classical case.

Now we are able to establish the Artin-Schrier theorem in semihyperring category.

Theorem 3.16. Let S be a 0-regular semihyperring. Then the following statements are equivalent.
(1) S is semireal;
(2) S has a preordering;
(3) S has an ordering.

Proof. Follows from Theorem 3.4 and Corollary 3.15.

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