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Direct Product of BP-Algebra

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Abstract - *In this paper, the notion of direct product of BP-algebra are introduced and some of related properties are investigated. Also, the notion of direct product of 0-commutative BP-algebra and BP-homomorphism are studied. Then, the notion of direct product in BP-algebra is expanded to finite family of BP-algebra and some of its properties are investigated.*

Keyword - *BP-algebra, direct product, finite family, 0-commutative BP-algebra , BP-homomorphism*

I. INTRODUCTION

A *B*-algebra was introduced by Neggers and Kim [7] in 2002, which is a non-empty set X with a constant 0 and a binary operation "$*$" denoted by $(X; *, 0)$, satisfying the following axioms: $(B1)$ $x * x = 0$, $(B2)$ $x * 0 = x$, and $(B3)$ $(x * y) * z = x * (z * (0 * y))$ for all $x, y, z \in X$. Then, Ahn and Han [1] introduced the generalized of *B*-algebra called *BP*-algebra, which is a non-empty set X with binary operations $*$ and a constant 0, and satisfies the following axioms: (*B1*), $(BP1)$ $x * (x * y) = y$, $(BP2)$ $(x * z) * (y * z) = x * y$ for all $x, y, z \in X$. Ahn and Han [1] also provide the concept of 0-commutative *BP*-algebra, which is a *BP*-algebra (X; *, 0) is said to be 0-commutative if satisfying $x * (0 * y) = y * (0 * x)$ for all $x, y \in X$. The concept of homomorphism is also studied in abstract algebra. A map $\psi : A \to B$ is called a *BP*-homomorphism if $\psi(x * y) = \psi(x) * \psi(y)$ for all $x, y \in A$, where *A* and *B* are two *BP*-algebras. The kernel of ψ denoted by ker ψ is defined to be the set $\{x \in A : \psi(x) = 0_B\}$. A *BP*-homomorphism ψ is called a *BP*-monomorphism, *BP*-epimorphism, or *BP*-isomorphism if one-one , onto, or a bijection, respectively. The concepts of *B*-algebra and *BP*-algebra have been discussed by researchers, for instance the concept of direct product. The notion of direct product was first discussed in group and some of properties are obtained, such as the direct product of the group is a group and the direct product of the abelian group is also an abelian group. Then, the notion of direct product of group is applied in other algebraic structures. Lingcong and Endam [6] discuss the notion of direct product of *B*-algebra, 0-commutative B-algebra and B-homomorphism. The results define direct product of B-algebra and some of related properties are obtained. One of them is the direct product of two *B*-algebras is also a *B*-algebra. Then, they extend the concept of direct product of *B*-algebra to finite family *B*-algebra and some of related properties are investigated. Furthermore, Widianto et al. [9] discussed the concept of direct product in *BG*-algebra, 0-commutative *BG*-algebra and *BG*-homomorphism and some of related properties are investigated.

The objective of this paper is to construct the concept of direct product of *BP*-algebras, and then investigate direct product of 0-commutative *BP*-algebras and *BP*-homomorphism. Finally, we study direct product of finite family *BP*-algebra and some related properties are explored.

II. PRELIMINARIES

In this section, we recall the notion of *B*-algebra and *BP*-algebra and review some properties which we will need in the next section. Some definitions and theories related to direct product of *BP*-algebra that have been discussed by several authors [1, 5, 6, 7] will also be presented.

Definition 2.1. [7] A *B*-algebra is a non-empty set *X* with a constant 0 and a binary operation " $*$ " satisfying the following axioms: for all $x, y, z \in X$,

(*B1*) $x * x = 0$,
(*B2*) $x * 0 = x$,
(*B3*) $(x * y) * z = x * (z * (0 * y))$.

Definition 2.2. [5] A *B*-algebra $(X ; *, 0)$ is said to be 0-commutative if $x * (0 * y) = y * (0 * x)$, for any $x, y \in X$.

Example 1. Let $A = \{0, a, b\}$ be a set with Cayley's table as follows:

Table 1: Cayley's table for $(A ; *, 0)$

*	0	a	b
0	0	b	a
a	a	0	b
b	b	a	0

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From Table 1 we get the value of main diagonal is 0, such that A satisfies $x * x = 0$, for all $x \in A$ (*B1* axiom). In the second column we see that for all $x \in A$, then $x * 0 = x$ (*B2* axiom) and it also satisfies $(x * y) * z = x * (z * (0 * y))$, for all $x, y, z \in A$. Hence, $(A ; *, 0)$ be a *B*-algebra. It easy to check $(A ; *, 0)$ satisfies $x * (0 * y) = y * (0 * x)$, for all $x, y \in A$. Hence, A be a 0-commutative *B*-algebra.

A non-empty subset S of *B*-algebra $(X ; *, 0)$ is called a subalgebra of X if $x * y \in S$, for all $x, y \in S$.

Example 2. Let $A = \{0, a, b, c, d, e\}$ be a set with Cayley's table as follows:

Table 2: Cayley's table for (X; ∗, 0)

∗	0	a	b	c	d	e
0	0	b	a	c	d	e
a	a	0	b	d	e	c
b	b	a	0	e	c	d
c	c	d	e	0	b	a
d	d	e	c	a	0	b
e	e	c	d	b	a	0

Then, $(X ; *, 0)$ is a *B*-algebra and the set $S = \{0, a, b\}$ is a subalgebra of X.

Definition 2.4. [6] Let $\boldsymbol{A} = (A; *, 0_A)$ and $\boldsymbol{B} = (B; *, 0_B)$ be *B*-algebras. Define the direct product of $\boldsymbol{A}$ and $\boldsymbol{B}$ to be the structure $\boldsymbol{A} \times \boldsymbol{B} = (A \times B; \circledast, (0_A, 0_B))$, where $A \times B$ is the set $\{(a, b) : a \in A,\ b \in B\}$ and whose binary operation $\circledast$ is given by as $(a_1, b_1) \circledast (a_2, b_2) = (a_1 * a_2, b_1 * b_2)$.

Definisi 2.5. [1] A *BP*-algebra is a non-empty set X with a constant 0 and a binary operation " $*$ " satisfying the following axioms: for all $x, y, z \in X$,
(*B1*) $x * x = 0$,
(*BP1*) $x * (x * y) = y$,
(*BP2*) $(x * z) * (y * z) = x * y$.

Definisi 2.6. [1] A *BP*-algebra $(X; *, 0)$ is said to be a 0-commutative if $x * (0 * y) = y * (0 * x)$ for any $x, y \in X$.

Example 4. Let $X = \{0, 1, 2, 3\}$ be a set with Cayley's table as follows:

Table 3: Cayley's table for $(X; *, 0)$

∗	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Then, from Table 3 it can be shown that $(X ; *, 0)$ is a *BP*-algebra.

Theorem 2.7. [1] If $(X; *, 0)$ a *BP*-algebra, then for all $x, y \in X$,

(i) $0 * (0 * x) = x$,
(ii) $0 * (y * x) = x * y$,
(iii) $x * 0 = x$,
(iv) If $x * y = 0$ then $x * y = x$,
(v) If $0 * x = 0 * y$ then $x = y$,
(vi) $0 * x = y$ then $0 * y = x$,
(vii) $0 * x = x$ then $x * y = y * x$.

Proof. The Theorem 2.7 has been proved in [1].

Let $(X; *, 0)$ and $(Y; *, 0)$ be two *BP*-algebras. A map $\psi: X \to Y$ is called a *BP*-homomorphism if $\psi(a * b) = \psi(a) * \psi(b)$ for any $a, b \in X$. The kernel of ψ denoted by ker ψ is defined to be the set $ker\ \psi = \{x \in X: \psi(x) = 0_Y\}$. A BP-homomorphism ψ is called a BP-monomorphism, BP-epimorphism, or BP-isomorphism if one-one , onto, or a bijection function, respectively.

III. DIRECT PRODUCT OF BP-ALGEBRA

By using the same idea in Lincong and Endam's research in [6] we get the definition of direct product in *BP*-algebra and its properties are obtained. The concept can be extended to the finite famiy *BP*-algebra. Then, we have some of the related properties.

Let $\boldsymbol{A} = (A; *, 0_A)$ and $\boldsymbol{B} = (B; *, 0_B)$ be *BP*-algebras. Define the direct product of $\boldsymbol{A}$ and $\boldsymbol{B}$ to be the structure $\boldsymbol{A} \times \boldsymbol{B} = (A \times B; \circledast, (0_A, 0_B))$, where $A \times B$ is the set $\{(a, b) : a \in A,\ b \in B\}$ and whose binary operation $\circledast$ is given by as $(a_1, b_1) \circledast (a_2, b_2) = (a_1 * a_2, b_1 * b_2)$. By definition of direct product of *BP*-algebra we obtain Theorem 3.1.

Theorem 3.1. The direct product of two *BP*-algebras is also a *BP*-algebra.
Proof. Let $\boldsymbol{P} = (P; *, 0_P)$ and $\boldsymbol{Q} = (Q; *, 0_Q)$ are two BP-algebras, then the direct product of P and Q is a structure $\boldsymbol{P} \times \boldsymbol{Q} = (P \times Q; \circledast, (0_P, 0_Q))$ for all $(p_1, q_1) \in P \times Q$ we have

$$(p_1, q_1) \circledast (p_1, q_1) = (p_1 * p_1,\ q_1 * q_1) = (0_P, 0_Q),$$

then the axiom *B1* is satisfied. For any $(p_1, q_1),\ (p_2, q_2) \in P \times Q$ obtained

$$\begin{aligned}(p_1, q_1) \circledast ((p_1, q_1) \circledast (p_2, q_2)) &= (p_1, q_1) \circledast (p_1 * p_2, q_1 * q_2)\\ &= \big(p_1 * (p_1 * p_2), q_1 * (q_1 * q_2)\big)\\ (p_1, q_1) \circledast ((p_1, q_1) \circledast (p_2, q_2)) &= (p_2, q_2),\end{aligned}$$

then the axiom *BP1* is satisfied. Then, for any $(p_1, q_1),\ (p_2, q_2),\ (p_3, q_3) \in P \times Q$ we get

$$\begin{aligned}((p_1, q_1) \circledast (p_3, q_3)) \circledast ((p_2, q_2) \circledast (p_3, q_3)) &= (p_1 * p_3, q_1 * q_3) \circledast (p_2 * p_3, q_2 * q_3)\\ &= \big((p_1 * p_3) * (p_2 * p_3), (q_1 * q_3) * (q_2 * q_3)\big)\\ &= (p_1 * p_2, q_1 * q_2)\\ ((p_1, q_1) \circledast (p_3, q_3)) \circledast ((p_2, q_2) \circledast (p_3, q_3)) &= (p_1, q_1) \circledast (p_2, q_2),\end{aligned}$$

then the axiom *BP2* is satisfied. Since $\boldsymbol{P} \times \boldsymbol{Q}$ satisfies all axioms of *BP*-algebra, hence $\boldsymbol{P} \times \boldsymbol{Q}$ is a *BP*-algebra.

The concept of direct product in *BP*-algebra is extended to finite family of *BP*-algebra. Let $I_n = \{1, 2, \dots, n\}$ and let $\{\boldsymbol{P_i} = (P_i; *,\ 0_i) : i \in I_n\}$ be a finite family of *BP*-algebra. Define the direct product of BP-algebras $\boldsymbol{P_1}, \dots, \boldsymbol{P_n}$ to be the structure

$$\prod_{i=1}^{n} \boldsymbol{P_i} = \left(\prod_{i=1}^{n} P_i\, ; \circledast, (0_1, \dots, 0_n) \right),$$

where

$$\prod_{i=1}^{n} P_i = P_1 \times \dots \times P_n = \{(p_1, \dots, p_n) :\ p_i \in P_i, i \in I_n\},$$

and whose operation $\circledast$ is given by $(p_1, \dots, p_n) \circledast (q_1, \dots, q_n) = (p_1 * q_1, \dots, p_n * q_n)$ for all $p_i, q_i \in P_i, i \in I_n$. Obviously, $\circledast$ is a binary operation on $\prod_{i=1}^{n} \boldsymbol{P_i}$.

Now, we discuss some related properties of direct product of finite family in *BP*-algebra.

Corollary 3.2. If $\{\boldsymbol{P_i} = (P_i; *,\ 0_i) : i \in I_n\}$ is a family *BP*-algebra, then $\prod_{i=1}^{n} \boldsymbol{P_i}$ is a *BP*-algebra.
Proof. Let $\{\boldsymbol{P_i} = (P_i; *,\ 0_i) : i \in I_n\}$ is a family *BP*-algebra. The direct product of $\boldsymbol{P_1}, \dots, \boldsymbol{P_n}$ to be a structure $\prod_{i=1}^{n} \boldsymbol{P_i} = \big(\prod_{i=1}^{n} P_i\, ; \circledast, (0_1, \dots, 0_n)\big)$ for each $\big(p_1, \dots, p_n\big) \in \prod_{i=1}^{n} P_i$ we have

$$\big(p_1, \dots, p_n\big) \circledast \big(p_1, \dots, p_n\big) = (p_1 * p_1, \dots, p_n * p_n) = (0_1, \dots, 0_n),$$

then the axiom *B1* is satiesfied. For each $\big(p_1, \dots, p_n\big), \big(q_1, \dots, q_n\big) \in \prod_{i=1}^{n} P_i$ obtained

$$\begin{aligned}(p_1, \dots, p_n) \circledast ((p_1, \dots, p_n) \circledast (q_1, \dots, q_n)) &= (p_1, \dots, p_n) \circledast (p_1 * q_1, \dots, p_n * q_n)\\ &= \big(p_1 * (p_1 * q_1), \dots, p_n * (p_n * q_n)\big)\end{aligned}$$

$(p_1, \dots, p_n) \circledast ((p_1, \dots, p_n) \circledast (q_1, \dots, q_n)) = (q_1, \dots, q_n)$,
then the axiom *B1* is fulfilled. For each $(p_1, \dots, p_n), (q_1, \dots, q_n), (r_1, \dots, r_n) \in \prod_{i=1}^{n} P_i$ we get

$$\begin{aligned}\big((p_1, \dots, p_n) \circledast (r_1, \dots, r_n)\big) \circledast \big((q_1, \dots, q_n) \circledast (r_1, \dots, r_n)\big) &= (p_1 * r_1, \dots, p_n * r_n) \circledast (q_1 * r_1, \dots, q_n * r_n) \\ &= \big((p_1 * r_1) * (q_1 * r_1), \dots, (p_n * r_n) * (q_n * r_n)\big) \\ &= (p_1 * q_1, \dots, p_n * q_n) \\ \big((p_1, \dots, p_n) \circledast (r_1, \dots, r_n)\big) \circledast \big((q_1, \dots, q_n) \circledast (r_1, \dots, r_n)\big) &= (p_1, \dots, p_n) \circledast (q_1, \dots, q_n),\end{aligned}$$

then the axiom *B1* is fulfilled. Since $\prod_{i=1}^{n} \boldsymbol{P_i}$ satisfies all *BP*-algebra axioms, such that $\prod_{i=1}^{n} \boldsymbol{P_i}$ is a *BP*-algebra.

Next, we get the properties of the direct product in the 0-commutative *BP*-algebra given in Theorem 3.3 and Corollary 3.4.

Theorem 3.3. Let $\boldsymbol{P} = (P; *, 0_P)$ and $\boldsymbol{Q} = (Q; *, 0_Q)$ are two *BP*-algebras. Then each $\boldsymbol{P}$ and $\boldsymbol{Q}$ is 0-commutative if and only if $\boldsymbol{P} \times \boldsymbol{Q} = (P \times Q; \circledast, (0_P, 0_Q))$ is 0-commutative .

Proof. By Theorem 3.1 we have $\boldsymbol{P} \times \boldsymbol{Q}$ is a *BP*-algebra. Let $(p_1, q_1), (p_2, q_2) \in P \times Q$, then $p_1, p_2 \in P$, $q_1, q_2 \in Q$. Since $\boldsymbol{P}$ and $\boldsymbol{Q}$ be 0-commutative *BP*-algebra, then $p_1 * (0_P * p_2) = p_2 * (0_P * p_1)$ and $q_1 * (0_Q * q_2) = q_2 * (0_Q * q_1)$, thus

$$\begin{aligned}(p_1, q_1) \circledast \big((0_p, 0_Q) \circledast (p_2, q_2)\big) &= (p_1, q_1) \circledast (0_P * p_2, 0_Q * q_2) \\ &= (p_1 * (0_P * p_2), q_1 * (0_Q * q_2)) \\ &= (p_2 * (0_P * p_1), q_2 * (0_Q * q_1)) \\ &= (p_2, q_2) \circledast (0_P * p_1, 0_Q * q_1) \\ (p_1, q_1) \circledast \big((0_p, 0_Q) \circledast (p_2, q_2)\big) &= (p_2, q_2) \circledast \big((0_p, 0_Q) \circledast (p_1, q_1)\big).\end{aligned}$$

Therefore, $\boldsymbol{P} \times \boldsymbol{Q}$ is 0-commutative. Conversely, let $\boldsymbol{P} \times \boldsymbol{Q} = (P \times Q; \circledast, (0_P, 0_Q))$ be 0-commutative. If $p_1, p_2 \in P$, $q_1, q_2 \in Q$, then $(p_1, q_1), (p_2, q_2) \in P \times Q$ and

$$(p_1, q_1) \circledast \big((0_p, 0_Q) \circledast (p_2, q_2)\big) = (p_2, q_2) \circledast \big((0_p, 0_Q) \circledast (p_1, q_1)\big).$$

Thus,

$$\begin{aligned}\big(p_1 * (0_P * p_2), q_1 * (0_Q * q_2)\big) &= (p_1, q_1) \circledast (0_P * p_2, 0_Q * q_2) \\ &= (p_1, q_1) \circledast \big((0_p, 0_Q) \circledast (p_2, q_2)\big) \\ &= (p_2, q_2) \circledast \big((0_p, 0_Q) \circledast (p_1, q_1)\big) \\ &= (p_2, q_2) \circledast (0_P * p_1, 0_Q * q_1) \\ \big(p_1 * (0_P * p_2), q_1 * (0_Q * q_2)\big) &= \big(p_2 * (0_P * p_1), q_2 * (0_Q * q_1)\big).\end{aligned}$$

Then $p_1 * (0_P * p_2) = p_2 * (0_P * p_1)$ and $q_1 * (0_Q * q_2) = q_2 * (0_Q * q_1)$. It is show that each $\boldsymbol{P}$ and $\boldsymbol{Q}$ is 0-commutative. ■

Based on Theorem 3.3, the following 3.4 result is obtained.

Corollary 3.4. Let $\{\boldsymbol{P_i} = (P_i; *, 0_i) : i \in I_n\}$ is a family *BP*-algebra. $\boldsymbol{P_i}$ is 0-commutative if and only if $\prod_{i=1}^{n} \boldsymbol{P_i} = \big(\prod_{i=1}^{n} P_i ; \circledast, (0_1, \dots, 0_n)\big)$ is 0-commutative.

Proof. From Corollary 3.2 we have $\prod_{i=1}^{n} \boldsymbol{P_i}$ is *BP*-algebra. Let $(p_1, \dots, p_n), (q_1, \dots, q_n) \in \prod_{i=1}^{n} P_i$, then $p_i, q_i \in P_i$. Since $\boldsymbol{P_i}$ is 0-commutative *BP*-algebra, then $p_i * (0_i * q_i) = q_i * (0_i * p_i)$ for each $i \in I_n$, such that

$$\begin{aligned}(p_1, \dots, p_n) \circledast ((0_1, \dots, 0_n) \circledast (q_1, \dots, q_n)) &= (p_1, \dots, p_n) \circledast (0_1 * q_1, \dots, 0_n * q_n) \\ &= (p_1 * (0_1 * q_1), \dots, p_n * (0_n * q_n)) \\ &= (q_1 * (0_1 * p_1), \dots, q_n * (0_n * p_n)) \\ &= (q_1, \dots, q_n) \circledast (0_1 * p_1, \dots, 0_n * p_n) \\ (p_1, \dots, p_n) \circledast ((0_1, \dots, 0_n) \circledast (q_1, \dots, q_n)) &= (q_1, \dots, q_n) \circledast ((0_1, \dots, 0_n) \circledast (p_1, \dots, p_n)).\end{aligned}$$

Thus, it is show that $\prod_{i=1}^{n} \boldsymbol{P_i}$ is 0-commutative. Conversely, let $\prod_{i=1}^{n} \boldsymbol{P_i}$ is 0-commutative. If p_i, $q_i \in P_i$ for each $i \in I_n$, then $(p_1, \dots, p_n), (q_1, \dots, q_n) \in \prod_{i=1}^{n} P_i$, and

$$(p_1, \dots, p_n) \circledast ((0_1, \dots, 0_n) \circledast (q_1, \dots, q_n)) = (q_1, \dots, q_n) \circledast ((0_1, \dots, 0_n) \circledast (p_1, \dots, p_n)).$$

Thus, obtained

$$\begin{aligned}(p_1 * (0_1 * q_1), \dots, p_n * (0_n * q_n)) &= (p_1, \dots, p_n) \circledast (0_1 * q_1, \dots, 0_n * q_n) \\ &= (p_1, \dots, p_n) \circledast ((0_1, \dots, 0_n) \circledast (q_1, \dots, q_n))\end{aligned}$$

$$= (q_1, \ldots, q_n) \circledast ((0_1, \ldots, 0_n) \circledast (p_1, \ldots, p_n))$$
$$= (q_1, \ldots, q_n) \circledast (0_1 * p_1, \ldots, 0_n * p_n)$$
$$(p_1 * (0_1 * q_1), \ldots, p_n * (0_n * q_n)) = (q_1 * (0_1 * p_1), \ldots, q_n * (0_n * p_n)).$$

Then, it can be concluded that $p_i * (0_i * q_i) = q_i * (0_i * p_i)$ for each $i \in I_n$ So, it is proved that $\boldsymbol{P_i}$ is 0-commutative.

The direct product concept in *BP*-algebra can also be developed on *BP*-homomorphism and the following properties are obtained.

Theorem 3.5. Let $\psi_1: P_1 \to Q_1$ and $\psi_2: P_2 \to Q_2$ are two *BP*-homomorphisms. If ψ is the map $\psi: P_1 \times P_2 \to Q_1 \times Q_2$ given by $(p_1, p_2) \mapsto (\psi_1(p_1), \psi_2(p_2))$ then

(i) ψ is a *BP*-homomorphism,

(ii) $ker\ \psi = ker\ \psi_1 \times ker\ \psi_2$,

(iii) $\psi(P_1 \times P_2) = \psi_1(P_1) \times \psi_2(P_2)$.

Proof.

(i) Let $\psi_1: P_1 \to Q_1$ and $\psi_2: P_2 \to Q_2$ are two *BP*-homomorphisms and $\psi: P_1 \times P_2 \to Q_1 \times Q_2$ given by $(p_1, p_2) \mapsto \big(\psi_1(p_1), \psi_2(p_2)\big)$. If $(p_1, p_2), (r_1, r_2) \in P_1 \times P_2$, then

$$\psi\big((p_1, p_2) \circledast (r_1, r_2)\big) = \psi(p_1 * r_1, p_2 * r_2)$$
$$= \big(\psi_1(p_1 * r_1), \psi_2(p_2 * r_2)\big)$$
$$= \big(\psi_1(p_1) * \psi_1(r_1), \psi_2(p_2) * \psi_2(r_2)\big)$$
$$= (\psi_1(p_1), \psi_2(p_2)) \circledast (\psi_1(r_1), \psi_2(r_2))$$
$$\psi\big((p_1, p_2) \circledast (r_1, r_2)\big) = \psi(p_1, p_2) \circledast \psi(r_1, r_2).$$

Thus, it is show that ψ is a *BP*-homomorphism. The converse this theorem to be true in general.

(ii) Let $\big(p_1, p_2\big) \in ker\ \psi$, then

$$\psi(p_1, p_2) = (0_1, 0_2)$$
$$\big(\psi_1(p_1), \psi_2(p_2)\big) = (0_1, 0_2).$$

This statement shows that $\psi_1(p_1) = 0_1$ and $\psi_2(p_2) = 0_2$, then $p_1 \in ker\ \psi_1$ and $p_2 \in ker\ \psi_2$, such that $(p_1, p_2) \in ker\ \psi_1 \times ker\ \psi_2$. Therefore, it is proved that $ker\ \psi = ker\ \psi_1 \times ker\ \psi_2$.

(iii) Let $(r_1, r_2) \in \psi(P_1 \times P_2)$, then there exist $(p_1, p_2) \in P_1 \times P_2$, such that

$$(r_1, r_2) = \psi(p_1, p_2) = (\psi_1(p_1), \psi_2(p_2)).$$

There is $p_1 \in P_1$ so that $r_1 = \psi_1(p_1) \in \psi(P_1)$ and there is $p_2 \in P_2$, such that $r_2 = \psi_2(p_2) \in \psi(P_2)$, then $\big(r_1, r_2\big) \in \psi_1\big(P_1\big) \times \psi_2\big(P_2\big)$. Therefore, it is proved that $\psi(P_1 \times P_2) = \psi_1(P_1) \times \psi_2(P_2)$.

Based on Theorem 3.5 the following result is obtained.

Corollary 3.6. Let $\{\psi_i: P_i \to Q_i : i \in I_n\}$ is a family *BP*-homomorphism. If ψ is a map $\prod_{i=1}^n P_i \to \prod_{i=1}^n Q_i$ given by $(p_1, \ldots, p_n) \mapsto (\psi_1(p_1), \ldots, \psi_n(p_n))$, then

(i) ψ is a *BP*-homomorphism,

(ii) $ker\ \psi = \prod_{i=1}^n ker\ \psi_i$,

(iii) $\psi(\prod_{i=1}^n P_i) = \prod_{i=1}^n \psi_i(P_i)$.

Proof.

(i) Let $\psi_i: P_i \to Q_i : i \in I_n\}$ is a family *BP*-homomorfism and $\prod_{i=1}^n P_i \to \prod_{i=1}^n Q_i$ given by $(p_1, \ldots, p_n) \mapsto (\psi_1(p_1), \ldots, \psi_n(p_n))$. If $(p_1, \ldots, p_n), (r_1, \ldots, r_n) \in \prod_{i=1}^n P_i$, then

$$\psi((p_1, \ldots, p_n) \circledast (r_1, \ldots, r_n)) = \psi(p_1 * r_1, \ldots, p_n * r_n)$$
$$= \big(\psi_1(p_1 * r_1), \ldots, \psi_n(p_n * r_n)\big)$$
$$= \big(\psi_1(p_1) * \psi_1(r_1), \ldots, \psi_n(p_n) * \psi_n(r_n)\big)$$
$$= (\psi_1(p_1), \ldots, \psi_n(p_n)) \circledast (\psi_1(r_1), \ldots, \psi_n(r_n))$$
$$\psi((p_1, \ldots, p_n) \circledast (r_1, \ldots, r_n)) = \psi(p_1, \ldots, p_n) \circledast \psi(r_1, \ldots, r_n).$$

Hence, ψ is a *BP*-homomorphism. ■

The converse of Corollary 3.6 need to be true in general.

(ii) Let $(p_1, \ldots, p_n) \in ker\,\psi$, then

$$\psi(p_1, \ldots, p_n) = (0_1, \ldots, 0_n)$$
$$\big(\psi_1(p_1), \ldots, \psi_n(p_n)\big) = (0_1, \ldots, 0_n).$$

This shows that $\psi_i(p_i) = 0_i$ each $i \in I_n$, then $p_i \in ker\,\psi_i$ each $i \in I_n$, then $(p_1, \ldots, p_n) \in ker \prod_{i=1}^{n} ker\,\psi_i$. Therefore, it is proved that $ker\,\psi = \prod_{i=1}^{n} ker\,\psi_i$. ■

(iii) Let $(r_1, \ldots, r_n) \in \psi(\prod_{i=1}^{n} P_i)$, then there is $(p_1, \ldots, p_n) \in \prod_{i=1}^{n} P_i$ so that

$$(r_1, \ldots, r_n) = \psi(p_1, \ldots, p_n) = (\psi_1(p_1), \ldots, \psi_n(p_n)).$$

There is $p_i \in P_i$ so that $r_i = \psi(p_i) \in \psi(P_i)$ each $i \in I_n$, then $(r_1, \ldots, r_n) \in \prod_{i=1}^{n} \psi_i(P_i)$. Therefore, it is proved that $\psi(\prod_{i=1}^{n} P_i) = \prod_{i=1}^{n} \psi_i(P_i)$. ■

Theorem 3.7. Let $\psi_1: P_1 \to Q_1$, $\psi_2: P_2 \to Q_2$, and $\psi: P_1 \times P_2 \to Q_1 \times Q_2$ by $(p_1, p_2) \mapsto \big(\psi_1(p_1), \psi_2(p_2)\big)$, then

(i) ψ is a *BP*-monomorphisms if and only if ψ_1 and ψ_2 are *BP*-monomorphisms,

(ii) ψ is a *BP*-epimorphism if and only if ψ_1 and ψ_2 are *BP*-epimorphisms.

Proof.

(i) Let ψ is a *BP*-monomorphism. If $\psi_1(p_1) = \psi_1(r_1)$ for each $p_1, r_1 \in P_1$ and $\psi_2(p_2) = \psi_2(r_2)$ for each $p_2, r_2 \in P_2$, then

$$\psi(p_1, p_2) = \big(\psi_1(p_1), \psi_2(p_2)\big)$$
$$= \big(\psi_1(r_1), \psi_2(r_2)\big)$$
$$\psi(p_1, p_2) = \psi(r_1, r_2),$$

Since ψ is a *BP*-monomorphism, then ψ is one-one function, such that $(p_1, p_2) = (r_1, r_2)$ implies $p_1 = r_1$ and $p_2 = r_2$. Thus, it is proved that ψ_1 and ψ_2 are one-one functions, such that ψ_1 and ψ_2 are *BP*-monomorphisms. Conversely, let ψ_1 and ψ_2 is *BP*-monomorphisms. If $\psi(p_1, p_2) = \psi(r_1, r_2)$ for each $(p_1, p_2), (r_1, r_2) \in P_1 \times P_2$, then

$$\big(\psi_1(p_1), \psi_2(p_2)\big) = \psi(p_1, p_2)$$
$$= \psi(r_1, r_2)$$
$$\big(\psi_1(p_1), \psi_2(p_2)\big) = \big(\psi_1(r_1), \psi_2(r_2)\big),$$

Since ψ_1 and ψ_2 is *BP*-monomorphisms, then ψ_1 and ψ_2 are one-one functions, such that $p_1 = r_1$ and $p_2 = r_2$ yields $(p_1, p_2) = (r_1, r_2)$. Thus, it is proved that ψ is a one-one function, such that ψ is a *BP*-monomorphism.

(ii) Let ψ is a *BP*-epimorphism. Let $q_1 \in Q_1$ and $q_2 \in Q_2$, then $(q_1, q_2) \in Q_1 \times Q_2$. Since ψ is a function onto, then there is $(p_1, p_2) \in P_1 \times P_2$ that $(q_1, q_2) = \psi(p_1, p_2) = \big(\psi_1(p_1), \psi_2(p_2)\big)$, which results $q_1 = \psi_1(p_1)$ and $q_2 = \psi_2(p_2)$. Thus, it is proved that ψ_1 and ψ_2 are functions onto such that ψ_1 and ψ_2 are *BP*-epimorphisms. Conversely, Let ψ_1 and ψ_2 are *BP*-epimorphisms. Let $(q_1, q_2) \in Q_1 \times Q_2$ then $q_1 \in Q_1$ and $q_2 \in Q_2$. Since ψ_1 and ψ_2 is a onto function, then there are $p_1 \in P_1$ and $p_2 \in P_2$ so that $q_1 = \psi_1(p_1)$ and $q_2 = \psi_2(p_2)$ which results $(q_1, q_2) = \big(\psi_1(p_1), \psi_2(p_2)\big) = \psi(p_1, p_2)$. Therefore, it shows that ψ is a onto function, its mean ψ is also a BP-epimorphism. ■

Based on Theorem 3.7, the following corollary 3.8 result is obtained.

Corollary 3.8. Let $\psi_i: P_i \to Q_i : i \in I_n$ and $\psi: \prod_{i=1}^{n} P_i \to \prod_{i=1}^{n} Q_i$ by $(p_1, \ldots, p_n) \mapsto (\psi_1(p_1), \ldots, \psi_n(p_n))$, then

(i) ψ is a *BP*-monomorphism if and only if ψ_i is a *BP*-monomorphism,

(ii) ψ is a *BP*-epimorphism if and only if ψ_i is a *BP*-epimorphism.

Proof.

(i) Let ψ is a *BP*-monomorphism. If $\psi_i(p_i) = \psi_i(r_i)$ for each $p_i, r_i \in P_i, i \in I_n$, then

$$\begin{aligned} \psi(p_1, \dots, p_n) &= (\psi_1(p_1), \dots, \psi_n(p_n)) \\ &= (\psi_1(r_1), \dots, \psi_n(r_n)) \\ \psi(p_1, \dots, p_n) &= \psi(r_1, \dots, r_n). \end{aligned}$$

Since ψ is *BP*-monomorphism, then ψ_1 and ψ_2 is a one-one function, such that $(p_1, \dots, p_n) = (r_1, \dots, r_n)$ yields $p_i = r_i$ for each $i \in I_n$. Thus it is proved that ψ is a one-one function, such that ψ_i is a *BP*-monomorphism. Coversely, Let ψ_i is a *BP*-monomorphism. If $\psi(p_1, \dots, p_n) = \psi(r_1, \dots, r_n)$ for each $(p_1, \dots, p_n), (r_1, \dots, r_n) \in \prod_{i=1}^{n} P_i$, then

$$\begin{aligned} (\psi_1(p_1), \dots, \psi_n(p_n)) &= \psi(p_1, \dots, p_n) \\ &= \psi(r_1, \dots, r_n) \\ (\psi_1(p_1), \dots, \psi_n(p_n)) &= (\psi_1(r_1), \dots, \psi_n(r_n)). \end{aligned}$$

Since ψ_i is a *BP*-monomorphism, then ψ_i is one-one function, such that $p_i = r_i$ for each $i \in I_n$ which results $(p_1, \dots, p_n) = (r_1, \dots, r_n)$. Thus, it is proved that ψ is a one-one function. Hence, ψ_i is a *BP*-monomorphism. ■

(ii) Let ψ is a *BP*-epimorphism and let $q_i \in Q_i$ for each $\in I_n$, then $(q_1, \dots, q_n) \in \prod_{i=1}^{n} Q_i$. Since ψ is a onto function, then $(p_1, \dots, p_n) \in \prod_{i=1}^{n} P_i$ each $p_i \in P_i$ each $i \in I_n$, so $(q_1, \dots, q_n) = \psi(p_1, \dots, p_n) = (\psi_1(p_1), \dots, \psi_n(p_n))$, resulting in $q_i = \psi_i(p_i)$ each $i \in I_n$. Thus, it is proven that ψ_i is a onto function, such that ψ is a *BP*-epimorphism. Conversely, let ψ_i for each $i \in I_n$ be a *BP*-epimorphism and $(q_1, \dots, q_n) \in \prod_{i=1}^{n} Q_i$, then $q_i \in Q_i$ for each $i \in I_n$. Since ψ_i is a function onto, then there is $p_i \in P_i$ for each $i \in I_n$, then $q_i = \psi_i(p_i)$ for each $i \in I_n$ implies $(q_1, \dots, q_n) = (\psi_1(p_1), \dots, \psi_n(p_n)) = \psi(p_1, \dots, p_n)$. Therefore, it is proved that ψ is a onto function. Hence, ψ is also a *BP*-epimorphism. ■

IV. CONCLUSION

In this paper, the notion of direct product of *BP*-algebra is equivalent to *B*-algebra. We obtain some of their properties being similar. Then, the notion of the direct product of *BP*-algebra applied to finite family *BP*-algebra, finite family 0-commutative *BP*-algebra, and finite family *BP*-homomorphism.

REFERENCES

[1] S. S. Ahn and J. S. Han, *On BP-algebras*, Hacettepe Journal of Mathematics and Statistics, 42 (2013), 551-557.

[2] K. Iseki, *On BCI-algebras*, Mathematics Seminar Notes, 8 (1980), 125-130.

[3] C. B. Kim and H. S. Kim, *On BG-algebras*, Demonstratio Mathematica, 41 (2003), 497-505.

[4] C. B. Kim and H. S. Kim, *On BN-algebras*, Kyungpook Math, 53 (2013), 175-184.

[5] H. S. Kim and H. G. Park, *On 0-commutative B-algebras*, Scientiae Mathematicae Japonicae Online, (2005), 31-36.

[6] J. A. V. Lingcong and J. C. Endam, *Direct Product of B-algebras*, International Journal of Algebra, 10 (2016), 33-40.

[7] J. Neggers and H. S. Kim, *On B-algebras*, Matematicki Vesnik, 54 (2002), 21-29.

[8] C. Teves and Joemar C. Endam, *Direct Product of Bf-algebras*, International Journal of Algebra, 10 (2016), 125-132.

[9] S. Widianto, S. Gemawati, and Kartini, *Direct Product in BG-algebras*, International Journal of Algebra, 13 (2019), 239 - 247.

[10] S. A. N. Zadeh, A. Radfar, and A. B. Saied, *On BP-algebras and QS-algebras*, The Journal of Mathematics and Computer Science, 5 (2012), 17-21.