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OBSERVING REPRESENTATIONS OF THE LIE GROUP SL$(3,\mathbb{R})$ FROM VANTAGE POINT OF SYMMETRIES OF SECOND ORDER LINEAR O.D.E's

Uchechukwu Opara

Veritas University (VUNA), Abuja, Nigeria

Abstract- ***An in-depth view of the reducibility properties of second order linear O.D.E's also offers a unique setting to draw up links between representations of the admitted symmetry group*** $SL(3,\mathbb{R})$***. These links are exposed in this article, with further elements of symmetry methods for differential equations and abstract group algebra also elucidated.***

Keywords - Kummer-Liouville point transform, Lie group representations and isomorphisms, Projective special linear group, Semi-invariants of second order linear O.D.E's .

1 Introduction

The foremost matter to be rued from the onset is the need for knowledge of at least one explicit solution to the generic Ordinary Differential Equation (O.D.E) in view here, which is

$$y''(x) + a_1(x)y'(x) + a_0(x)y = 0 \qquad (1),$$

to explicitly determine almost all admitted symmetries. This is again the only deterrent encountered in harnessing benefits of Sophus Lie's reducibility theorem of (1) to the form $[\ddot{z}(t) = 0]$ as referenced in [1]. To be able to realize the proof of Sophus Lie's theorem, we engage the tool of the Kummer-Liouville transform, which is the most general point transform that preserves order and linearity of second order linear O.D.E's [4]. The Kummer-Liouville transform is given by

$$y = v(x)z\ ,\ dt = u(x)dx \qquad (KL); \quad u, v \in C^2(I),\ uv \neq 0\ \forall x \in I$$

which rearranges (1) to be of the form

$$\ddot{z}(t) + b_1(t)\dot{z}(t) + b_0(t)z = 0 \qquad (2); \qquad b_1(t) \in C^1(J)\ ,\ b_0(t) \in C(J)$$

where I and J are compact sub-intervals of the real number line with non-empty interiors. The functions u and v are respectively recognized as the kernel and multiplier of the Kummer-Liouville transform.

Consider that equation (1) is well-posed and let λ be a non-trivial solution to this generic equation. It has been shown in [1] that a Kummer-Liouville (KL) kernel of $[u = |\lambda^{-2}exp(-\int a_1 dx)|]$ would lead to reduction to the normal form $[\ddot{z}(t) = 0]$ after a double point transform of (1), regardless of choice of the KL multiplier. As an improvement on this development, the additional choice of multiplier $[v = \lambda]$ in the first KL transform leads to the desired result $[\ddot{z}(t) = 0]$ after a single point transform. This transform to the most simplified second order autonomous O.D.E form shall constitute the crux of the subsequent 'Results' section of this article. As a noteworthy remark, any kernel to be useful for reduction to an autonomous form of (1) is the reciprocal of some solution to the third order linear O.D.E:

$$A'''(x) + 4p.A'(x) + 2p'(x)A = 0\ ,$$

whereby $[p(x) = a_0 - \frac{1}{2}a_1'(x) - a_1^2]$ is identified as the semi-invariant of (1). In other words, $p(x)$ is the expression obtained as the coefficient of y_* in the conversion of (1) to its normal form $[y_*''(x)+p(x)y_* = 0]$ via the KL transform functional specifications:

$$u(x) \equiv 1\ ;\ v(x) = exp\left(-\frac{1}{2}\int a_1(x)dx\right)\ ;\ z = y_*\ .$$

The above salient remark is clearly suggested in the explicit initial computation of infinitesimal symmetries admitted by (1) in the article [1].

2 Results

As a brief background review, it is important to observe that (1) admits a Lie symmetry group of dimension eight. The criteria for admittance of a symmetry group by any given O.D.E are given succinctly in the prolongation theorem stated in Olver ([3], p.100). Let y_1 and y_2 be linearly independent solutions to equation (1). As given below, observe the eight linearly independent one-parameter infinitesimal symmetries admitted by (1) from ([2], p.53), also derived explicitly in the same order in [1].

$$\begin{aligned}
&\mathbf{v}_1 = y\tfrac{\partial}{\partial y}\\
&\mathbf{v}_2 = y_1\tfrac{\partial}{\partial y}\\
&\mathbf{v}_3 = y_2\tfrac{\partial}{\partial y}\\
&\mathbf{v}_4 = exp[\textstyle\int a_1dx](y_1y\tfrac{\partial}{\partial x} + y_1'y^2\tfrac{\partial}{\partial y})\\
&\mathbf{v}_5 = exp[\textstyle\int a_1dx](y_2y\tfrac{\partial}{\partial x} + y_2'y^2\tfrac{\partial}{\partial y})\\
&\mathbf{v}_6 = exp[\textstyle\int a_1dx](y_1^2\tfrac{\partial}{\partial x} + y_1y_1'y\tfrac{\partial}{\partial y})\\
&\mathbf{v}_7 = exp[\textstyle\int a_1dx](2y_1y_2\tfrac{\partial}{\partial x} + (y_1'y_2 + y_1y_2')y\tfrac{\partial}{\partial y})\\
&\mathbf{v}_8 = exp[\textstyle\int a_1dx](y_2^2\tfrac{\partial}{\partial x} + y_2y_2'y\tfrac{\partial}{\partial y})
\end{aligned}$$

Under constraints $b_1 = b_0 = 0$ achieved with the identified single KL transform to $[\ddot{z}(t) = 0]$ in focus, then we see in [1] that (1) is expressible in the following form with the aid of the relevant KL kernel (u) and multiplier (v) as identified previously:

$$y'' + y'\overbrace{\left(\frac{-2v'}{v} - \frac{u'}{u}\right)}^{a_1(x)} + y\overbrace{\left(\frac{2(v')^2}{v^2} - \frac{v''}{v} + \frac{v'u'}{vu}\right)}^{a_0(x)} = 0 \qquad (3).$$

Upon computing the infinitesimal symmetries of (3) in this realized scenario, we have the following results.

$$\begin{aligned}
&\chi_1 = \frac{1}{u}\frac{\partial}{\partial x} + \frac{v'}{uv}y\frac{\partial}{\partial y} && \chi_2 = v\frac{\partial}{\partial y}\\
&\chi_3 = \chi_1\int udx && \chi_4 = \chi_2\int udx\\
&\chi_5 = \frac{y}{v}\chi_1 && \chi_6 = \frac{y}{v}\chi_2\\
&\chi_7 = \left(\int udx\right)^2\chi_1 + \left(\frac{y}{v}\int udx\right)\chi_2 && \chi_8 = \left(\frac{y}{v}\int udx\right)\chi_1 + \left(\frac{y}{v}\right)^2\chi_2.
\end{aligned}$$

We ought to recall quite importantly from Berkovich in [4] that the first infinitesimal symmetry (χ_1) listed in the above set of one-parameter symmetries admitted by (3) is the symmetry invariably required for conversion to an autonomous form. The precise correspondence between the explicitly above stated representations of Hydon and Berkovich is immediately verified by way of the functional substitutions:

$$u = y_1^{-2}exp\left(-\int a_1dx\right),\ v = y_1 \text{ and } \int udx = \frac{y_2}{y_1}.$$

The final functional substitution for $\int udx$ given above may be justified by way of the fundamental theorem of calculus. Observe that:

$$\frac{d}{dx}\left[\frac{y_2}{y_1}\right] = \frac{y_2'y_1 - y_1'y_2}{y_1^2} = \frac{W(x)}{y_1^2} = \frac{exp(-\int a_1 dx)}{y_1^2} = u \ ,$$

whereby $W(x)$ denotes the Wronskian of (1). With these functional substitutions, we are able to realize easily that: $\chi_1 = \mathbf{v}_6, \ \chi_2 = \mathbf{v}_2, \ \chi_3 = \frac{1}{2}(\mathbf{v}_7 - \mathbf{v}_1), \ \chi_4 = \mathbf{v}_3, \ \chi_5 = \mathbf{v}_4, \ \chi_6 = \mathbf{v}_1, \ \chi_7 = \mathbf{v}_8, \ \chi_8 = \mathbf{v}_5$.

Finally, with functional substitutions $t = \displaystyle\int udx$ and $z = \dfrac{y}{v}$ in the case in focus, we determine the vector field representations:

$$\frac{\partial}{\partial x} = u\frac{\partial}{\partial t} - \frac{v'z}{v}\frac{\partial}{\partial z} \quad \text{and} \quad \frac{\partial}{\partial y} = \frac{1}{v}\frac{\partial}{\partial z} \ .$$

Making these substitutions in Berkovich's symmetry representation $(\chi_1 - \chi_8)$ then turns out to yield:

$$\begin{aligned}
\chi_1 &= \frac{\partial}{\partial t} & \chi_2 &= \frac{\partial}{\partial z} \\
\chi_3 &= t\frac{\partial}{\partial t} & \chi_4 &= t\frac{\partial}{\partial z} \\
\chi_5 &= z\frac{\partial}{\partial t} & \chi_6 &= z\frac{\partial}{\partial z} \\
\chi_7 &= t^2\frac{\partial}{\partial t} + tz\frac{\partial}{\partial z} & \chi_8 &= tz\frac{\partial}{\partial t} + z^2\frac{\partial}{\partial z}.
\end{aligned}$$

These are the generators of the projective special linear group $PSL(3, \mathbb{R})$ as identified explicitly in ([5], p.31). We reckon that the projective special linear group $PSL(3, \mathbb{R})$ and projective general linear group $PGL(3, \mathbb{R})$ coincide, due to the existence of a unique real third root of unity. The immediate correspondence between the three aforementioned representations of the same Lie group as determined respectively by Hydon, Berkovich and Stephani was made possible by identification of the requisite KL transform kernel and multiplier to be used for realization of Sophus Lie's reducibility theorem in focus here.

3 Discussion

The projective special linear group $PSL(3,\mathbb{R})$ is identified as the quotient group of the special linear group $SL(3,\mathbb{R})$ by its center, so that

$$PSL(3,\mathbb{R}) = SL(3,\mathbb{R})/ZSL(3,\mathbb{R}) \ .$$

Since the center of $SL(3,\mathbb{R})$ is

$$ZSL(3,\mathbb{R}) = \{\lambda I_3 : \lambda^3 = 1\} \ ,$$

then we see that $PSL(3,\mathbb{R})$ is isomorphic to $SL(3,\mathbb{R})$ because the center $ZSL(3,\mathbb{R})$ is a singleton set comprising the identity matrix alone.

As a noteworthy conclusive remark, we re-emphasize the necessity of the infinitesimal symmetry χ_1 in transformation of (1) to any autonomous form. In the new KL coordinate system (t,z) here, χ_1 is merely the translation $\frac{\partial}{\partial t}$. To interprete this computational observation in literal terms, a linear O.D.E is autonomous if and only if it is invariant under the translation of its independent variable from the origin.

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