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Relation between Lah matrix and *k*-Fibonacci Matrix

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Abstract. - *The Lah matrix is represented by L_n, is a matrix where each entry is Lah number. Lah number is count the number of ways a set of n elements can be partitioned into k nonempty linearly ordered subsets. k-Fibonacci matrix, $F_n(k)$ is a matrix which all the entries are k-Fibonacci numbers. k-Fibonacci numbers are consist of the first term being 0, the second term being 1 and the next term depends on a natural number k. In this paper, a new matrix is defined namely A_n where it is not commutative to multiplicity of two matrices, so that another matrix B_n is defined such that $A_n \neq B_n$. The result is two forms of factorization from those matrices. In addition, the properties of the relation of Lah matrix and k-Fibonacci matrix is yielded as well.*

Keywords — *Lah numbers, Lah matrix, k-Fibonacci numbers, k-Fibonacci matrix.*

I. INTRODUCTION

Guo and Qi [5] stated that Lah numbers were introduced in 1955 and discovered by Ivo. Lah numbers are count the number of ways a set of n elements can be partitioned into k non-empty linearly ordered subsets. Lah numbers [9] is denoted by $L(n,k)$ for every n, k are elements of integers with initial value $L(0,0) = 1$. Lah numbers can also be presented in Lah matrix where all the entries are Lah numbers and denoted by L_n [9].

Reference [1] discussed about Stirling matrix and Lah matrix with the inverses and yields some interesting properties. Reference [5] gives six proofs of Lah numbers identity property by using Lah numbers generator function, Chu-Vandermonde addition formula, inverse formula, Gauss hypergeometric series, and first kind Stirling numbers generator function. In [9] discussed Lah numbers and Lindstorm theorem by giving a combinatorial interpretation from Lah numbers through planar network resulting in some properties. Moreover, *k*-Fibonacci numbers were introduced by Falcon [2] and denoted by $F_{n,k}$ where n, k are elements of integers.The *k*-Fibonacci numbers can be represented into *k*-Fibonacci matrix where all entries are *k*-Fibonacci numbers. It is denoted by $F_n(k)$. Reference [8] talks about the relation of *k*-Fibonacci sequence and its generalization. Wahyuni et al. [13] introduces some identities of *k*-Fibonacci sequence modulo ring. Reference [2] discusses the relation between *k*-Fibonacci matrix and Pascal matrix where two new matrices are yielded. [10] also talks about the relation of Bell polynomial matrix and *k*-Fibonacci matrix such that two new matrices as form of factorization of Bell polynomial matrix.

This article discusses Lah matrix and *k*-Fibonacci matrix, invers of *k*-Fibonacci matrix which is used to find new matrices, and relationship of Lah matrix and *k*-Fibonacci matrix. The second section of this article discusses about Lah matrix and *k*-Fibonacci matrix. The third section contains the main result which is the relationship of Lah matrix and *k*-Fibonacci matrix. Conclusion is given in the last section.

II. LAH MATRIX AND *K*-FIBONACCI MATRIX

In this section, we recall the notions of Lah and *k*-Fibonacci matrices and review some properties which we will need in the next section. Some definitions and theories related to Lah and *k*-Fibonacci matrices that have been discussed by several authors will also be presented.

Lah numbers are defined by Guo and Qi [5] as follows.

Definition 2.1. Lah numbers, $L(n,k)$ are defined as

$$L(n,k) = \binom{n-1}{k-1}\frac{n!}{k!}.$$

for n, k are natural numbers.

It can be concluded that

If $k = 1$ then $L(n,1) = n!$.

If $k = n$ then $L(n,n) = 1$.

If $n < k$ then $L(n,k) = 0$.

If $k = 0$ then $L(n,0) = 0$.

Martinjak [9] gives recursive formula for Lah numbers:

$$L(n+1,k) = L(n,k-1) + (n+k)L(n,k).$$

where n, k are integers and $n \geq k$ with intial value $L(0,0) = 1$.

Lah matrix is a lower triangle matrix with every entry is Lah number. Definition of Lah matrix according to Martinjak [9] is given below.

Definition 2.2. Let $L(i,j)$ be Lah numbers. The $n \times n$ Lah matrix is denoted by $L_n = [l_{i,j}]$ where $i,j = 1,2,\dots,n$ and defined as

$$l_{i,j} = \begin{cases} L(i,j) & \text{if} \quad i \geq j, \\ 0, & \text{otherwise.} \end{cases}$$

The general form of Lah matrix is

$$L_n = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ l_{2,1} & 1 & 0 & 0 & 0 & 0 \\ l_{3,1} & l_{3,2} & 1 & 0 & 0 & 0 \\ l_{4,1} & l_{4,2} & l_{4,3} & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ l_{n,1} & l_{n,n-1} & l_{n,n-2} & l_{n,n-3} & \cdots & 1 \end{bmatrix}.$$

From the general form above, it can be seen that the main diagonal is 1 and the determinant (det) of L_n is obtained from the multiplication of the diagonal entries and so det (L_n) = 1. Since det (L_n) $\neq 0$ then L_n has inverse. Engbers et al. [1] defines matrix inverse of Lah matrix as follows.

Definition 2.3. Let $L(i,j)$ be Lah numbers and $L_n = [l_{i,j}]$ be $n \times n$ Lah matrix. The inverse of Lah matrix is $L_n^{-1} = [(-1)^{i-j} l_{i,j}]$ where $i,j = 1,2,\dots,n$.

The following is the general form of Lah matrix:

$$L_n^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -l_{2,1} & 1 & 0 & 0 & 0 & 0 \\ l_{3,1} & -l_{3,2} & 1 & 0 & 0 & 0 \\ -l_{4,1} & l_{4,2} & -l_{4,3} & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ l_{n,1} & -l_{n,n-1} & l_{n,n-2} & -l_{n,n-3} & \cdots & 1 \end{bmatrix}.$$

The notion of k-Fibonacci numbers defined by Falcon [3] is given below.

Definition 2.4. For any integer number $k \geq 1$, the k^{th} Fibonacci sequence, say $\{F_{k,n}\}_{n \in N}$ is defined recurrently by $F_{k,0} = 0$, $F_{k,1} = 1$, dan $F_{k,n+1} = k\, F_{k,n} + F_{k,n-1}$ for any $n \geq 1$.

The k-Fibonacci numbers can also be represented in a lower triangular matrix where each entry is k-Fibonacci number. This definition is given in [2].

Definition 2.5. Let $F_{k,n}$ be n^{th} k-Fibonacci number, the $n \times n$ k-Fibonacci matrix as the unipotent lower triangular matrix $F_n(k) = [f_{i,j}]_{i,j=1,\dots,n}$ defined with entries. That is

$$f_{i,j}(k) = \begin{cases} F_{k,i-j+1} & \text{if } i \geq j, \\ 0, & \text{otherwise } i < j. \end{cases}$$

Generally, k-Fibonacci matrix is written in this form

$$F_n(k) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ F_{k,2} & 1 & 0 & 0 & 0 & 0 \\ F_{k,3} & F_{k,2} & 1 & 0 & 0 & 0 \\ F_{k,4} & F_{k,3} & F_{k,2} & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ F_{k,n} & F_{k,n-1} & F_{k,n-2} & F_{k,n-3} & \cdots & 1 \end{bmatrix}.$$

From the above general formula of *k*-Fibonacci matrix it can be seen that the main diagonal is 1 and determinant (det) is yielded from the multiplication of the entries of the diagonal and so that det ($F_n(k)$) = 1. Since det $F_n(k)) \neq 0$, $F_n(k)$ is has inverse. Falcon [2] defines the inverse matrix of the *k*-Fibonacci matrix as follows.

Definition 2.6. Let $F^{-1}_n(k)$ be inverse matrix of the *k*-Fibonacci matrix, the $n \times n$ inverse *k*-Fibonacci matrix as lower triangular matrix $F^{-1}_n(k) = [f'_{i,j}(k)]_{i,j=1,\dots,n}$ where

$$f'_{i,j}(k) = \begin{cases} 1 & if\ j = i, \\ -k & if\ j = i-1, \\ -1 & if\ j = i-2, \\ 0 & otherwise, \end{cases}$$

Since $F_n(k)$ has inverse, then $F_n(k)\, F_n^{-1}(k) = I_n = F_n^{-1}(k) F_n(k)$. Therefore *k*-Fibonacci $F_n(k)$ is an *invertible* matrix. Generally *k*-Fibonacci matrix can be written by

$$F_n^{-1}(k) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -k & 1 & 0 & 0 & 0 & 0 \\ -1 & -k & 1 & 0 & 0 & 0 \\ 0 & -1 & -k & 1 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & -1 & -k & 1 \end{bmatrix}.$$

From the general form above, it can be concluded that each entry of the inverse of the *k*-Fibonacci matrix is $1, -k, -1,$ and 0. In other words, the entry of $F_n^{-1}(k)$ for any n is fixed.

III. RELATION BETWEEN LAH MATRIX AND *K*-FIBONACCI MATRIX

Relationship of Lah matrix and *k*-Fibonacci matrix is obtained from multiplication of two matrices, which is multiplication of inverse of *k*-Fibonacci matrix and Lah matrix and the other way around. It starts from multiplying 2×2 matrices, follows by 3×3 and eventually a new matrix is obtained. To construct the general form of the new matrix, the properties of this new matrix should be investigated. Therefore, a larger size of matrices are needed. To make the calculation easier, software Maple 13 is used. In the first factorization, a new matrix A_n, for any *n* integer, is obtained from multiplying inverse of *k*-Fibonacci matrix and Lah matrix. In the second factorization, a new matrix B_n is yielded from multiplying Lah matrix and *k*-Fibonacci inverse matrix.

A. First Factorization of Lah matrix and k-Fibonacci Matrix

To get the relationship of Lah matrix and *k*-Fibonacci matrix, multiply two 2×2 matrices which are inverse of *k*-Fibonacci matrix $F_2^{-1}(k)$ and L_2 Lah matrix and obtain

$$F_2^{-1}(k)\, L_2 = \begin{bmatrix} 1 & 0 \\ -k & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2-k & 1 \end{bmatrix} = A_2.$$

For $n = 3$, by multiplying inverse of *k*-Fibonacci $F_3^{-1}(k)$ matrix and Lah matrix L_3 it is obtained

$$F_3^{-1}(k)\, L_3 = \begin{bmatrix} 1 & 0 & 0 \\ -k & 1 & 0 \\ -1 & -k & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 6 & 6 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2-k & 1 & 0 \\ 5-2k & 6-k & 1 \end{bmatrix} = A_3.$$

Then, for $n > 3$, multiplication of inverse of *k*-Fibonacci matrix and Lah matrix is done by Maple 13. By investigating each entry of A_4 for $i = j$ entry the entries of the diagonal is 1 and for $i > j$ the following construction is given :

i. In the first row, $a_{1,1} = 1$, $a_{1,j} = 0$ for $j \geq 2$.
ii. In the second row, $a_{2,1} = 2 - k$, $a_{2,2} = 1$, $a_{2,j} = 0$ for $j \geq 3$.
iii. In the third row $a_{3,1} = 5 - 2k$, $a_{3,2} = 6 - k$, $a_{3,3} = 1$, $a_{3,j} = 0$ for $j \geq 3$.
iv. $a_{i,j} = 1$ for any $i = j$ and for $i < j$, $a_{i,j} = 0$.

The entry of A_4 matrix is listed in the following Table 3.1

By investigating the entries of A_4 for $i \geq j$, the value of entries of A_4 can be derived as listed in the table below.

Table 3.1: Elements of A_n

Entries of A_n	Value of the entries of A_n
$a_{1,1}$	$(1)\ l_{1,1} = L(1,1)$
$a_{2,2}$	$(1)\ l_{2,2} = L(2,2)$
$a_{3,3}$	$(1)\ l_{3,3} = L(3,3)$
$a_{4,4}$	$(1)\ l_{4,4} = L(4,4)$
$a_{2,1}$	$(1)\ l_{2,1} + (-k)\ l_{1,1} = L(2,1) - kL(1,1)$
$a_{3,2}$	$(1) l_{3,2} + (-k)\ l_{2,2} = L(3,2) - kL(2,2)$
$a_{4,3}$	$(1) l_{4,3} + (-k)\ l_{3,3} = L(4,3) - kL(3,3)$
$a_{3,1}$	$(1) l_{3,1} + (-k)\ l_{2,1} + (-1)\ l_{1,1} = L(3,1) - kL(2,1) - L(1,1)$
$a_{4,2}$	$(1) l_{4,2} + (-k)\ l_{3,2} + (-1)\ l_{2,2} = L(4,2) - kL(3,2) - L(2,2)$
$a_{4,1}$	$(1) l_{4,1} + (-k)\ l_{3,1} + (-1)\ l_{2,1} + (0)\ (1) l_{1,1} = L(4,1) - kL(3,1) - L(2,1)$
$\vdots$	$\vdots$
$a_{i,j}$	$L(i,j) - k\ L(i-1,j) - L(i-2,j)$

Hence, multiplication of inverse of k-Fibonacci matrix and Lah matrix yields a new matrix A_n and the definition generally is given as follows.

Definition 3.1. For every natural number n, it is defined an $(n+1) \times (n+1)$ matrix $A_n = [a_{i,j}]$ with $i, j = 0,1,2, \dots, n$ as follows

$$a_{i,j} = L(i,j) - k\ L(i-1,j) - L(i-2,j).$$

From the definition above, $a_{i,j} = 1$ for every $i = j$, $a_{i,j} = 0$ when $i < j$. For every $i > j$, apply $a_{i,j} = L(i,j) - k\ L(i-1,j) - L(i-2,j)$.

From definitions of Lah matrix, k-Fibonacci matrix, and A_n matrix, the following theorem is derived.

Theorem 3.2. Lah matrix defined in Definition 2.2 can be defined as multiplication of k-Fibonacci $F_n(k)$ defined in Definition 2.5 with A_n and given as $L_n = F_n(k)\ A_n$.

Proof. Since k-Fibonacci matrix $F_n(k)$ has inverse, it will be proven that

$$A_n = F_n^{-1}(k)\ L_n.$$

Suppose $F_n^{-1}(k)$ is inverse of k-Fibonacci matrix so that the main diagonal is 1. L_n is Lah matrix and so the main diagonal is also 1. Multiplication of $F_n^{-1}(k)$ with L_n yields a new matrix with main diagonal 1. If $i = j$ then $a_{i,j} = 1$ and if $i < j$ then $a_{i,j} = 0$, and for every $i > 2$ then

$$a_{i,j} = f'_{i,i}(k)\ l_{i,j} + f'_{i,i-1}(k)\ l_{i-1,j} + f'_{i,i-2}(k)\ l_{i-2,j} + f'_{i,i-3}(k)\ l_{i-3,j} + \dots + f'_{i,n}(k)\ l_{n,j},$$
$$= \sum_{r=1}^{n} f'_{i,r}(k)\ l_{r,j}.$$

It can be concluded that $F_n^{-1}(k)\ L_n = A_n$. Hence $L_n = F_n(k)\ A_n$. ∎

Theorem 3.2 gives the relation of Lah numbers and k-Fibonacci numbers defined in Theorem 3.3.

Theorem 3.3. Suppose $L(i,j)$ is Lah number defined in Definition 2.1 and $F_{k,n}$ is the k-Fibonacci numbers defined in Definition 2.5. Then for $i \geq j+2$,

$$L(i,j) = F_{k,i-j+1} + (j^2 + j - k)\ F_{k,i-j} + \sum_{r=j+2}^{i} (L(r,j) - k\ L(r-1,j) - L(r-2,j)) F_{k,i-r+1}.$$

For $i < j+2$ it is obtained that $L(i,j) = F_{k,i-j+1} + (j^2 + j - k)\ F_{k,i-j}$ and for $i < j$ then $L(i,j) = 0$ for i to be natural numbers.

Proof.

From Definition 3.2

$a_{j,j} = L(j,j) = 1,$

$$\begin{aligned} a_{j+1,j} &= L(j+1,j) - kL(j,j) - L(j-1,j) \\ &= j(j+1) - k(1) - 0 \\ &= j^2 + j - k, \end{aligned}$$

for $r \geq j+2$
$a_{t,j} = L(r,j) - kL(r-1,j) - L(r-2,j)$.

Definition 2.5 and Theorem 3.2 gives the following

$$\begin{aligned} L(i,j) &= l_{i,j} \\ &= \sum_{r=j}^{i} f_{i,r}(k)\ a_{r,j} \\ &= \sum_{r=j}^{i} F_{k,i-r+1}\ a_{r,j} \\ &= F_{k,i-j+1}\, a_{j,j} + F_{k,i-(j+1)+1}\, a_{j+1,j} + \sum_{r=j+2}^{i} F_{k,i-r+1}\ a_{r,j} \\ &= F_{k,i-j+1} + F_{k,i-j}(j^2+j-k) + \sum_{r=j+2}^{i} F_{k,i-r+1}\ (L(r,j) - kL(r-1,j) - L(r-2,j)) \end{aligned}$$

$$L(i,j) = F_{k,i-j+1} + (j^2+j-k)\ F_{k,i-j} + \textstyle\sum_{r=j+2}^{i}(L(r,j) - kL(r-1,j) - L(r-2,j))\ F_{k,i-r+1}. \qquad \blacksquare$$

B. Second Factorization for Lah matrix and k-Fibonacci Matrix

To find the relation of Lah matrix and k-Fibonacci matrix from second factorization, multiplication of two matrices is needed. It is done by multiplying Lah matrix L_n and $F_n^{-1}(k)$ inverse of k-Fibonacci matrix and eventually yielding a new matrix called B_n. It starts from multiplying 2×2 matrices such that the following is obtained.

$$L_2F_2^{-1}(k) = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -k & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2-k & 1 \end{bmatrix} = B_2.$$

Then proceed with multiplication of 3×3 matrices to get

$$L_3\ F_3^{-1}(k) = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 6 & 6 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -k & 1 & 0 \\ -1 & -k & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2-k & 1 & 0 \\ 5-6k & 6-k & 1 \end{bmatrix} = B_3.$$

Then, for $n > 3$ is done by Maple 13 to get the general form. By observing each entry of B_4 it can be seen that for $i = j$ the entry in the main diagonal is 1 and for $i > j$ a simple construction is obtained as follows:

i. In the first row $b_{1,1} = 1$, $b_{1,j} = 0$for $j \geq 2$.
ii. In the second row $b_{2,1} = 2-k$, $b_{2,2} = 1$, $b_{2,j} = 0$ for $j \geq 3$.
iii. In the third row $b_{3,1} = 5-6k$, $b_{3,2} = 6-k$, $b_{3,3} = 1$, $b_{3,j} = 0$ for $j \geq 3$.
iv. The entry is $b_{i,j} = 1$ for every $i = j$ and for $i < j$ the entry is $b_{i,j} = 0$.

By observing entries of B_4 for $i \geq j$, all entries of B_4 is listed in the following Table 3.2

Table 3.2: Elements of B_n

Entries B_n	The value of entries of B_n
$b_{1,1}$	$(1)\ l_{1,1} = L(1,1)$
$b_{2,2}$	$(1)\ l_{2,2} = L(2,2)$
$b_{3,3}$	$(1)\ l_{3,3} = L(3,3)$
$b_{4,4}$	$(1)\ l_{4,4} = L(4,4)$
$b_{2,1}$	$(1)\ l_{2,1} + (-k)\ l_{2,2} = L(2,1) - kL(2,2)$
$b_{3,2}$	$(1)l_{3,2} + (-k)\ l_{3,3} = L(3,2) - kL(3,3)$
$b_{4,3}$	$(1)l_{4,3} + (-k)\ l_{4,4} = L(4,3) - kL(4,4)$
$b_{3,1}$	$(1)l_{3,1} + (-k)\ l_{3,2} + (-1)\ l_{3,3} = L(3,1) - kL(3,2) - L(3,3)$
$b_{4,2}$	$(1)l_{4,2} + (-k)\ l_{4,3} + (-1)\ l_{4,4} = L(4,2) - kL(4,3) - L(4,4)$
$b_{4,1}$	$(1)l_{4,1} + (-k)\ l_{4,2} + (-1)\ l_{4,3} + (0)\ (1)l_{4,4} = L(4,1) - kL(4,2) - L(4,3)$
$\vdots$	$\vdots$
$b_{i,j}$	$L(i,j) - k\ L(i,j+1) - L(i,j+2)$

So, Lah matrix and inverse of k-Fibonacci matrix multiplication gives a new matrix, which is B_n where the general definition is given by the following.

Definition 3.4. For every n being natural number, the $n \times n$ matrix B_n with entry $B_n = [b_{i,j}]$ for every $i, j = 1,2, \dots, n$, is defined as follows

$$b_{i,j} = L(i,j) - k\, L(i,j+1) - L(i,j+2).$$

It is easy to see that $b_{i,j} = 1$ for every $i = j$, $b_{i,j} = 0$ for every $i < j$. Moreover, for $i > j$ then

$$b_{i,j} = L(i,j) - k\, L(i,j+1) - L(i,j+2).$$

The following theorem is constructed from defining B_n matrix, Lah matrix, and k-Fibonacci matrix.

Theorem 3.5 Lah matrix defined in Definition 2.2 can be stated as multiplication of k-Fibonacci $F_n(k)$ in Definition 2.5 and B_n matrix in Definition 3.4 for n and k to be natural numbers such that $L_n = B_n\, F_n(k)$.

Proof. Since k-Fibonacci matrix $F_n(k)$ has inverse it will be proven that

$$B_n = L_n\, F_n^{-1}(k).$$

Suppose L_n is Lah matrix, then the main diagonal of Lah matrix is 1. $F_n^{-1}(k)$ is inverse of k-Fibonacci matrix, then the main diagonal of inverse of k-Fibonacci matrix is also 1. Multiplication of Lah matrix L_n and inverse of k-Fibonacci matrix $F_n^{-1}(k)$ resulting in a new matrix with main diagonal 1. Then, if $i = j$ then $b_{i,j} = 1$ and if $i < j$ then $b_{i,j} = 0$, and for $i > 2$ then

$$\begin{aligned} b_{i,j} &= l_{i,j}\, f'_{j,j}(k) + l_{i,j+1}\, f'_{j+1,j}(k) + l_{i,j+2}\, f'_{j+2,j}(k) + l_{i,j+3}\, f'_{j+3,j}(k) + \cdots + l_{i,n}\, f'_{n,j}(k), \\ &= \sum_{r=1}^{n} l_{i,r}\, f'_{r,j}(k). \end{aligned}$$

It can be concluded that $L_n\, F_n^{-1}(k) = B_n$. Hence, $L_n = B_n\, F_n(k)$. ■

From Theorem 3.5, it can derived the following properties

Theorem 3.6. Let $L(i,j)$ to be Lah numbers defined in Definition 2.1 and $F_{k,n}$ to be k-Fibonacci numbers defined in Definition 2.5 then for $i \geq j + 2$

$$L(i,j) = F_{k,i-j+1} + (i^2 - i - k)\, F_{k,i-j} + \sum_{r=j}^{i-2} (L(i,r) - k\, L(i,r+1) - L(i,r+2)) F_{k,r-j+1}.$$

$L(i,j) = F_{k,i-j+1} + (i^2 - i - k)\, F_{k,i-j}$ for $i < j + 2$ and for $i < j$ then $L(i,j) = 0$ for natural numbers i.

Proof. From Definition 3.4,

$b_{i,i} = L(i,i) = 1,$

$$\begin{aligned} b_{i,i-1} &= L(i,i-1) - kL(i,i) - L(i,i+1) \\ &= i(i-1) - k(1) - 0 \\ &= i^2 - i - k, \end{aligned}$$

For $j \leq r \leq i - 2$ then

$b_{i,t} = L(i,r) - kL(i,r+1) - L(i,r+2).$

Then, from Definition 2.5 and Theorem 3.5

$$\begin{aligned} L(i,j) &= l_{i,j} \\ &= \sum_{r=j}^{i} b_{i,r}\, f_{r,j}(k) \\ &= \sum_{r=j}^{i} b_{i,r}\, F_{k,r-j+1} \\ &= b_{i,i}\, F_{k,i-j+1} + b_{i,i-1}\, F_{k,i-1-j+1} + \sum_{r=j}^{i-2} b_{i,r}\, F_{k,r-j+1} \\ &= F_{k,i-j+1} + (i^2 - i - k)\, F_{k,i-j} + \sum_{r=j}^{i-2} \left(L_{i,r} - kL_{i,r+1} - L_{i,r+2}\right) F_{k,r-j+1} \end{aligned}$$

$$L(i,j) = F_{k,i-j+1} + (i^2 - i - k)F_{k,i-j} + \sum_{r=j}^{i-2}\big(L(i,r) - kL(i,r+1) - L(i,r+2)\big)\, F_{k,r-j+1}. \qquad \blacksquare$$

IV. CONCLUSION

This article discusses about the relation of Lah matrix and k-Fibonacci matrix. From this relation, derived two kinds of Lah matrix factorization. The first factorization gives a new matrix that is obtained from multiplication of inverse of k-Fibonacci matrix and Lah matrix. The second factorization gives a new matrix that is yielded from multiplication of Lah matrix and inverse of k-Fibonacci matrix. These new matrices are difference. In addition, some properties that also states the relation of Lah numbers and k-Fibonacci numbers are also obtained.

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