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# Harmonic Zagreb-K-Banhatti Index of a Graph

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***Abstract:*** *The Zagreb indices and K-Banhatti indices are closely related. In this paper, we introduce the reformulated harmonic index, harmonic Zagreb-K-Banhatti index of a graph. We establish some bounds for the harmonic index and reformulated harmonic index. We also obtain lower and upper bounds for the harmonic Zagreb-K-Banhatti index of a graph in terms of Zagreb and K-Banhatti indices.*

**Keywords:** *harmonic index, reformulated harmonic index, harmonic Zagreb-K-Banhatti index, graph.*

**Mathematics Subject Classification:** 05*C*05, 05*C*12, 05*C*35.

## I. Introduction

All graphs considered in this paper are finite, connected, undirected without loops and multiple edges. Let $G$=($V$, $E$) be a connected graph with $n$ vertices and $m$ edges. The degree $d_G(u)$ of a vertex $u$ is the number of vertices adjacent to $u$. The degree $d_G(u)$ of a vertex $u$ is the number of vertices adjacent to $u$. The degree of an edge $e = uv$ in $G$, is defined by $d_G(e) = d_G(u) + d_G(v) – 2$. The vertices and edges of $G$ are called the element of $G$. If $e = uv$ is an edge of $G$, then the vertex $u$ and edge $e$ are incident as are $u$ and $e$. For all further and terminology, we refer the reader to [1].

A molecular graph is a graph such that its vertices correspond to the atoms and edges to the bonds. Chemical Graph Theory is a branch of Mathematical Chemistry which has an important effect on the development of Chemical Sciences. A single number that can be computed from the molecular graph, and used to characterize some property of underlying molecule is said to be a topological index or a graph index. There are numerous topological indices [2] that have found some applications in Theoretical Chemistry, especially in QSPR/QSAR research see [3, 4, 5].

The first and second Zagreb indices were introduced by Gutman et al. in [6] and they are defined as

$$M_1(G) = \sum_{u\in V(G)} d_G(u)^2 = \sum_{uv\in E(G)} \left[d_G(u) + d_G(v)\right]$$

$$M_2(G) = \sum_{uv\in E(G)} d_G(u)d_G(v).$$

The Zagreb indices were studied extensively see [7, 8, 9].

The harmonic index [10] of a graph $G$ is defined as

$$H(G) = \sum_{uv\in E(G)} \frac{2}{d_G(u) + d_G(v)}.$$

This index was studied, for example, in [11, 12, 13, 14, 15, 16, 17].

In [18], Miličević et al. introduced the first and second reformulated Zagreb indices of a graph $G$ in terms of edge degrees instead of vertex degrees and they are defined as

$$EM_1(G) = \sum_{e\in E(G)} d_G(e)^2, \qquad EM_2(G) = \sum_{e\sim f} d_G(e)d_G(f).$$

where $e \sim f$ means that the edges $e$ and $f$ are adjacent.

We introduce the reformulated harmonic index of a graph $G$ and defined it as

$$HEM(G) = \sum_{e\sim f} \frac{2}{d_G(e) + d_G(f)}.$$

The reformulated Zagreb indices were studied, in [19, 20, 21, 22].

In [23], Kulli introduced the first and second $K$-Banhatti indices, intending to take into account the contributions of pairs of incident elements. The first and second $K$-Banhatti indices of a graph $G$ are defined as

$$B_1(G)=\sum_{ue}[d_G(u)+d_G(e)], \qquad B_2(G)=\sum_{ue}d_G(u)d_G(e)$$

where *ue* means that the vertex *u* and edge *e* are incident.

The *K*-Banhatti indices have been studied extensively. For their applications and mathematical properties, see [24, 25,26, 27, 28, 29, 30, 31, 32, 33].

In [34], Kulli introduced the harmonic *K*-Banhatti index of a graph *G*, which is defined as

$$HB(G)=\sum_{ue}\frac{2}{d_G(u)+d_G(e)}.$$

Motivated by the work on Zagreb and *K*-Banhatti indices, Kulli et al. introduced the Zagreb-*K*-Banhatti index [35] of a graph *G* and defined as

$$MB(G)=\sum_{\substack{a \text{ is either adjacent}\\ \text{or incident to } b}}[d_G(a)+d_G(b)]$$

where *a* and *b* are elements of *G*.

Recently, some Zagreb-*K*-Banhatti indices were introduced and studied such as the second Zagreb-*K*-Banhatti index [36], hyper Zagreb-*K*-Banhatti indices [37], sum connectivity and product connectivity Zagreb-*K*-Banhatti indices [38].

We introduce the harmonic Zagreb-*K*-Banhatti index of a graph *G* and defined as

$$HMB(G)=\sum_{\substack{a \text{ is either adjacent}\\ \text{or incident to } b}}\frac{2}{d_G(a)+d_G(b)}$$

where *a* and *b* are elements of *G*.

In this paper, we obtain some bounds for the harmonic index and reformulated harmonic index. Also we provide lower and upper bounds for *HMB*(*G*) of a graph *G* in terms of other topological indices.

## II. PRELIMINARIES

We give some inequalities for harmonic index *H*(*G*) which will be needed in the subsequent considerations.

Ilić [13] and Xu [16] independently proved the following inequality.

**Theorem 1.** Let *G* be a graph with *n* vertices and $m \geq 1$ edges. Then

$$H(G)\geq\frac{2m^2}{M_1(G)}$$

with equality if and only if $d_G(u) + d_G(v)$ is constant for each edge *uv* in *G*.

**Theorem 2 [14].** Let *G* be a graph with *n* vertices and *m* edges. Then

$$H(G)\leq\frac{2m^2}{M_1(G)}\left(\sqrt{\frac{\Delta(G)}{\delta(G)}}+\sqrt{\frac{\delta(G)}{\Delta(G)}}\right)^2$$

with equality if *G* is regular.

**Theorem 3 [15].** Let *G* be a nontrivial connected graph with *n* vertices and *m* edges. Then

$$H(G)\leq\frac{2m\left(\Delta(G)+\delta(G)\right)-n\Delta(G)\delta(G)}{2\delta(G)^2}.$$

**Theorem 4 [16].** Let *G* be a nontrivial connected graph with *n* vertices and *m* edges. Then

$$H(G)\leq P(G)$$

with equality if and only if *G* is a $\frac{2m}{n}$ regular graph.

**Theorem 5 [17].** Let *G* be a connected graph with $n \geq 3$ vertices. Then

$$\sqrt{\frac{2}{n-1}}S(G) \le H(G) \le \frac{2}{\sqrt{3}}S(G).$$

The lower bound holds if and only if $G=K_n$ and the upper bound holds if only if $G=P_3$.

## III. BOUND FOR HARMONIC INDEX

We obtain the lower and upper bounds on $H(G)$ in terms of the number of pendant vertices and minimal nonpendant vertex degree $\delta_1(G)$.

**Theorem 6.** For any $(n, m)$-connected graph $G$ with $p$ pendant vertices and minimal nonpendant vertex degree $\delta_1(G)$,

$$H(G) \le \frac{2p}{\delta_1(G)+1} + \frac{m-p}{\delta_1(G)},$$

and $$H(G) \ge \frac{2p}{\Delta(G)+1} + \frac{m-p}{\Delta(G)}.$$

**Proof:** By definition, we have

$$H(G) = \sum_{uv\in E(G)} \frac{2}{d_G(u)+d_G(v)}$$

$$= \sum_{uv\in E(G):d_G(v)=1} \frac{2}{d_G(u)+1} + \sum_{uv\in E(G):d_G(u),d_G(v)\ne 1} \frac{2}{d_G(u)+d_G(v)}$$

Since $2\Delta(G) \ge d_G(u)+d_G(v) \ge 2\delta_1(G)$, this implies

$$\frac{1}{\Delta(G)} \le \frac{2}{d_G(u)+d_G(v)} \le \frac{1}{\delta_1(G)}$$

and $$\frac{1}{\Delta(G)} \le \frac{1}{d_G(u)} \le \frac{1}{\delta_1(G)}.$$

Hence the upper bound follows.
Similarly the lower bound of

$$H(G) \ge \frac{2p}{\Delta(G)+1} + \frac{m-p}{\Delta(G)}.$$

follows.

## IV. BOUNDS FOR REFORMULATED HARMONIC INDEX

**Theorem 7.** For any $(n, m)$-connected graph $G$,

$$\frac{M_1(G)-2m}{4(\Delta(G)-1)} \le HEB(G) \le \frac{M_1(G)-2m}{4(\delta(G)-1)}$$

with equality if and only if $G$ is regular.

**Proof:** We have $\delta(G) \le d_G(u) \ge \Delta(G)$ for any vertex $u$ of $G$.

Thus $4(\delta(G)-1) \le d_G(e)+d_G(f) \ge 4(\Delta(G)-1)$ for every adjacent edges $e$ and $f$ in $G$. Therefore

$$\frac{2}{4(\Delta(G)-1)} \le \frac{2}{d_G(e)+d_G(f)} \le \frac{2}{4(\delta(G)-1)}$$ for every adjacent edges $e$ and $f$ in G.

Thus from definition of reformulated harmonic index, we have

$$\frac{M_1(G)-2m}{4(\Delta(G)-1)} \le HEB(G) \le \frac{M_1(G)-2m}{4(\delta(G)-1)},$$

where the number of pairs of edges which have a common end point is

$$\sum_{i=1}^{n} \binom{d_G(v_i)}{2} = \frac{1}{2}\left(M_1(G)-2m\right).$$

Obviously, in the above inequality, equality will hold when $G$ is regular.

We give an upper bound for $HEB(G)$.

**Theorem 8.** Let $G$ be a connected graph with $n \geq 3$ vertices and $m$ edges.
Then

$$HEB(G) \leq \frac{1}{2} M_1(G) - 2m$$

with equality if and only if $G = P_3$.

**Proof:** Since $d_G(e), d_G(f) \geq 1$, $d_G(e) + d_G(f) \geq 2$ for every adjacent edges $e$ and $f$ in $G$.

By summing inequalities $\frac{2}{d_G(e) + d_G(f)} \leq 1$, we obtain $HEB(G) \leq \frac{1}{2} M_1(G) - m.$

Second part is obvious.

## V. BOUNDS FOR HARMONIC *K*-BANHATTI INDEX

**Theorem 9.** For any $(n, m)$-connected graph $G$ with $\delta(G) \geq 2$ and $n \geq 3$ vertices. Consider the harmonic index of $G$ is

$$H(G) = \sum_{uv \in E(G)} \frac{2}{d_G(u) + d_G(v)}$$

with equality if and only if $G$ is $C_n$.

**Proof:** Let $G$ be a connected graph with $\delta(G) \geq 2$ and $n \geq 3$ vertices. Consider the harmonic index of $G$ is

$$HB(G) = \sum_{uv \in E(G)} \frac{2}{d_G(u) + d_G(v)}$$

and the harmonic Banhatti index of $G$ is

$$HB(G) = \sum_{ue} \frac{2}{d_G(u) + d_G(e)}$$

Since $\delta(G) \geq 2$, $d_G(e) \geq d_G(u)$, and $d_G(e) \geq d_G(v)$ for every edge $e = uv$ of G.

Thus $\qquad d_G(u) + d_G(e) \geq d_G(u) + d_G(v)$

Therefore $\qquad HB(G) = \sum_{ue} \left[ \frac{2}{d_G(u) + d_G(e)} + \frac{2}{d_G(v) + d_G(e)} \right]$

$$\leq \sum_{uv \in E(G)} \frac{4}{d_G(u) + d_G(v)}$$

Hence $HB(G) \leq 2H(G)$.

In order to prove our next results (upper bounds) of $HB(G)$ in terms of Randić index[] is defined as $P(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d_G(u) d_G(v)}}$, the modified second Zagreb index [] is defined as $M_2^*(G) = \sum_{uv \in E(G)} d_G(u) d_G(v)$ and the first Zagreb index $M_1(G)$ of a graph $G$.

**Theorem 10.** For any $(n, m)$-connected graph $G$ with $\delta(G) \geq 2$ and $n \geq 3$ vertices,

(i) $\quad HB(G) \leq (m+1) P(G)$.

(ii) $\quad HB(G) \leq (m+1) M_2^*(G)$.

(iii) $\quad HB(G) \leq M_1(G)$.

**Proof:** Let $G$ be a connected graph with $\delta(G) \geq 2$ and $n \geq 3$ vertices. Consider

$$\sum_{ue}\frac{2}{d_G(u)+d_G(e)}\le\frac{1}{2}\sum\frac{d_G(u)+d_G(e)}{d_G(u)d_G(e)}.$$

Since $\frac{a+b}{2}\ge\frac{2ab}{a+b}$ or $\frac{2}{a+b}\ge\frac{a+b}{2ab}$ for any two positive integers.

$$\sum_{ue}\frac{2}{d_G(u)+d_G(e)}\le\frac{1}{2}\sum_{ue}\frac{d_G(u)+d_G(e)}{d_G(u)d_G(e)}$$

$$\le\frac{1}{2}\sum_{ue}\left(\frac{1}{d_G(e)+d_G(u)}\right)$$

$$\le\frac{1}{2}\sum_{uv\in E(G)}\left[\left(\frac{1}{d_G(e)+d_G(u)}\right)+\left(\frac{1}{d_G(e)+d_G(v)}\right)\right].$$

Since $\delta(G)\ge 2,$ we have $d_G(u)\le d_G(e)$ and hence $\frac{1}{d_G(e)}\le\frac{1}{d_G(u)}.$

Hence

$$\sum_{ue}\frac{2}{d_G(u)+d_G(e)}\le\frac{1}{2}\sum_{ue}2\left(\frac{1}{d_G(u)}+\frac{1}{d_G(v)}\right)$$

$$\le\sum_{uv\in E(G)}\frac{d_G(u)+d_G(v)}{d_G(u)+d_G(v)}. \qquad (1)$$

(i) Since $d_G(u)+d_G(v)\le m+1$ for any edge *e*=*uv* of *G*, by inequality (1) we have

$$\sum_{ue}\frac{2}{d_G(u)+d_G(e)}\le(m+1)\sum_{ue}\frac{1}{d_G(u)d_G(v)} \qquad (2)$$

Since $\sum_{uv\in E(G)}\frac{1}{d_G(u)d_G(v)}\le\sum_{uv\in E(G)}\frac{1}{\sqrt{d_G(u)d_G(v)}},$

$$\sum_{ue}\frac{2}{d_G(u)d_G(e)}\le(m+1)\sum_{uv\in E(G)}\frac{1}{\sqrt{d_G(u)d_G(v)}}.$$

Thus $HB(G)\le(m+1)P(G).$

(ii) By inequality (2), we have

$HB(G)\le(m+1)M_2^*(G).$

(iii) Since *G* is connected with *n* ≥ 3 vertices, we have $d_G(u)d_G(v)\ge 1.$

Thus

$$\frac{d_G(u)+d_G(v)}{d_G(u)d_G(v)}\le d_G(u)+d_G(v)$$

Inequality (1) gives

$$\sum_{ue}\frac{2}{d_G(u)+d_G(e)}\le\sum_{uv\in E(G)}\frac{d_G(u)+d_G(v)}{d_G(u)d_G(v)}$$

$$\le(m+1)\sum_{uv\in E(G)}\left[d_G(u)+d_G(v)\right]=M_1(G).$$

Thus $HB(G)\le M_1(G).$

We obtain a lower bound for *HB*(*G*).

**Theorem 11.** For any (*n*, *m*)-connected graph *G* with δ(*G*) ≥ 2 and *n* ≥ 3 vertices,

$2M_2^*(G)\le HB(G).$

Further, equality holds if and only if $G = C_n$.

**Proof:** Let $G$ be a connected graph with $\delta(G) \geq 2$ and $n \geq 3$ vertices. Then

$$d_G(u)d_G(e) \geq d_G(u) + d_G(e)$$

$$\frac{2}{d_G(u)d_G(e)} \leq \frac{2}{d_G(u) + d_G(e)}$$

$$\sum_{ue} \frac{2}{d_G(u)d_G(e)} \leq \sum_{ue} \frac{2}{d_G(u) + d_G(e)}$$

Thus $\quad 2M_2^*(G) \leq HB(G).$

We obtain the lower and upper bounds for $HB(G)$ in terms of the number of pedant vertices and minimal nonpendant vertex degree $\delta_1(G)$.

**Theorem 12.** For any $(n, m)$-connected graph $G$ with $p$ pendant vertices and minimal nonpendant vertex degree $\delta_1(G)$,

$$HB(G) \leq 2p\left[\frac{2\delta_1(G)-1}{\delta_1(u)\left(2\delta_1(G)-1\right)}\right] + \frac{4(m-p)}{3\delta_1(G)-2}$$

$$HB(G) \geq 2p\left[\frac{3\Delta(G)-1}{\Delta(u)\left(2\Delta(G)-1\right)}\right] + \frac{4(m-p)}{3\Delta(G)-2}$$

**Proof:** By definition, we have

$$HB(G) = \sum_{e=uv \in E(G)} \left[\frac{2}{d_G(u)+d_G(e)} + \frac{2}{d_G(v)+d_G(e)}\right]$$

$$= \sum_{e=uv \in E(G): d_G(u)=1} \left[\frac{2}{d_G(v)} + \frac{2}{2d_G(v)-1}\right]$$

$$+ \sum_{e=uv \in E(G): d_G(u), d_G(v) \neq 1} \left[\frac{2}{d_G(u)+d_G(e)} + \frac{2}{d_G(v)+d_G(e)}\right]$$

$$= \sum_{e=uv \in E(G): d_G(u)=1} \frac{2\left(3d_G(v)-1\right)}{d_G(v)\left(2d_G(v)-1\right)}$$

$$+ \sum_{e=uv \in E(G): d_G(u), d_G(v) \neq 1} \left[\frac{2}{d_G(u)+d_G(e)} + \frac{2}{d_G(v)+d_G(e)}\right]$$

We have

$$2\Delta(G) - 2 \geq d_G(u) + d_G(e) \geq 3\delta_1(G) - 2.$$

Therefore

$$\frac{2}{3\Delta(G)-2} \leq \frac{2}{d_G(u)+d_G(e)} \leq \frac{2}{3\delta_1(G)-2}$$

and $\quad \dfrac{1}{\Delta(G)} \leq \dfrac{1}{d_G(u)} \leq \dfrac{1}{\delta_1(G)}.$

Thus

$$HB(G) \leq 2p\left[\frac{3\delta_1(G)-1}{\delta_1(G)\left(2\delta_1(G)-1\right)}\right] + \frac{4(m-p)}{3\delta_1(G)-2}.$$

Similarly, the lower bound of $HB(G)$ follows.

Now we obtain lower and upper bounds on $HB(G)$ in terms of $\delta(G)$, $\Delta(G)$ and $m$.

**Theorem 13.** For any $(n, m)$-connected graph $G$ with $\delta(G) \geq 2$ and $n \geq 3$ vertices,

$$\frac{4m}{3\Delta(G)-2} \leq HB(G) \leq \frac{4m}{3\delta(G)-2}.$$

**Proof:** Let $G$ be a connected graph with $\delta(G) \geq 2$ and $n \geq 3$ vertices. Then

$$\frac{2}{3\Delta(G)-2} \le \frac{2}{d_G(u)+d_G(e)} \le \frac{2}{3\delta(G)-2}.$$

Therefore $\sum_{ue}\frac{2}{3\Delta(G)-2} \le \sum_{ue}\frac{2}{d_G(u)+d_G(e)} \le \sum_{ue}\frac{2}{3\delta(G)-2}.$

Thus

$$\frac{4m}{3\Delta(G)-2} \le HB(G) \le +\frac{4m}{3\delta(G)-2}.$$

## VI. BOUND ON HARMONIC ZAGREB-*K*-BANHATTI, ZAGREB, *K*-BANHATTI-TYPE INDICES

**Theorem 14.** Let $G$ be a graph with $n$ vertices and $m$ edges. Then

$$HMB(G) = H(G) + HEM(G) + HB(G).$$

**Proof:** Let $G$ be a graph with $n$ vertices and $m$ edges. Then

$$HMB(G) = \sum_{\substack{a \text{ is either adjacent} \\ \text{or incident to } b}} \frac{2}{d_G(a)+d_G(b)}$$

$$= \sum_{ab\in E(G)} \frac{2}{d_G(a)+d_G(b)} + \sum_{ef\in E(G), e\sim f} \frac{2}{d_G(e)+d_G(f)} + \sum_{a(ab)} \frac{2}{d_G(a)+d_G(b)}$$

$$= H(G) + HEM(G) + HB(G).$$

**Theorem 15.** For any ($n$, $m$)-connected graph $G$ with $n \ge 3$ vertices and $m$ edges,

$$HMB(G) \le 3H(G) + HEM(G).$$

**Proof:** From Theorem 14, we have

$$HMB(G) = H(G) + HEM(G) + HB(G).$$

Using Theorem 9, we obtain

$$HMB(G) \le 3H(G) + HEM(G).$$

**Theorem 16.** For any ($n$, $m$)-connected graph $G$ with $n \ge 3$ vertices,

$$HMB(G) \le \frac{3m^2}{2M_1(G)}\left(\sqrt{\frac{\Delta(G)}{\delta(G)}} + \sqrt{\frac{\delta(G)}{\Delta(G)}}\right)^2 + HEM(G).$$

**Proof:** From Theorem 15, we have

$$HMB(G) \le 3H(G) + HEM(G).$$

Using Theorem 2, we obtain the desired result.

**Theorem 17.** For any ($n$, $m$)-connected graph $G$ with $n \ge 3$ vertices

$$HMB(G) \le \frac{6m(\Delta(G)+\delta(G)) - 3n\Delta(G)\delta(G)}{2\delta(G)^2} + HEM(G).$$

**Proof:** From Theorem 15, we have

$$HMB(G) \le 3H(G) + HEM(G).$$

Then from Theorem 3, we get the desired result.

**Theorem 18.** Let $G$ be any ($n$, $m$)-connected graph with $n \ge 3$ vertices. Then

$$HMB(G) \le (m+2)P(G) + HEM(G).$$

**Proof:** From Theorem 14, we have

$$HMB(G) = H(G) + HB(G) + HEM(G).$$

Using Theorem 4, we get

$$HMB(G) \le P(G) + HB(G) + HEM(G).$$

Then from Theorem 10(1), we obtain

$$HMB(G) \le P(G) + (m+1)P(G) + HEM(G).$$

Thus $HMB(G) \le (m+2)P(G) + HEM(G).$

**Theorem 19.** For any (*n*, *m*)-connected graph *G* with δ(*G*) ≥ 2 and *n* ≥ 3 vertices,

$$HMB(G) \le \frac{m^2}{2M_1(G)}\left(\sqrt{\frac{\Delta(G)}{\delta(G)}} + \sqrt{\frac{\delta(G)}{\Delta(G)}}\right)^2 + M_1(G) + HEM(G).$$

**Proof:** From Theorem 14, we have

$$HMB(G) = H(G) + HB(G) + HEM(G).$$

Using Theorem 2, we get

$$HMB(G) \le \frac{m^2}{2M_1(G)}\left(\sqrt{\frac{\Delta(G)}{\delta(G)}} + \sqrt{\frac{\delta(G)}{\Delta(G)}}\right)^2 + HB(G) + HEM(G).$$

Then from Theorem 10(iii), we obtain

$$HMB(G) \le \frac{m^2}{2M_1(G)}\left(\sqrt{\frac{\Delta(G)}{\delta(G)}} + \sqrt{\frac{\delta(G)}{\Delta(G)}}\right)^2 + M_1(G) + HEM(G).$$

**Theorem 20.** For any (*n*, *m*)-connected graph *G* with δ(*G*) ≥ 2 and *n* ≥ 3 vertices,

$$HMB(G) \le \frac{2m(\Delta(G)+\delta(G)) - \Delta(G)\delta(G)n}{2\delta(G)^2} + (m+1)M_2^*(G) + HEM(G).$$

**Proof:** From Theorem 14, we have

$$HMB(G) = H(G) + HB(G) + HEM(G).$$

Then from Theorem 10(ii), we obtain

$$HMB(G) \le H(G) + M_2^*(G) + HEM(G).$$

Using Theorem 3, we get

$$HMB(G) \le \frac{2m(\Delta(G)+\delta(G)) - \Delta(G)\delta(G)n}{2\delta(G)^2} + (m+1)M_2^*(G) + HEM(G).$$

**Theorem 21.** For any (*n*, *m*)-connected graph *G* with δ(*G*) ≥ 2 and *n* ≥ 3 vertices,

$$HMB(G) \ge \frac{2m^2}{M_1(G)} + 2M_2^*(G) + HEM(G).$$

**Proof:** From Theorem 14, we have

$$HMB(G) = H(G) + HB(G) + HEM(G).$$

Then from Theorem 1, we get

$$HMB(G) \ge \frac{2m^2}{M_1(G)} + HB(G) + HEM(G).$$

Them from Theorem 11, we obtain

$$HMB(G) \ge \frac{2m^2}{M_1(G)} + 2M_2^*(G) + HEM(G).$$

**Theorem 22.** For any (*n*, *m*)-connected graph *G* with δ(*G*) ≥ 2 and *n* ≥ 3 vertices,

$$HMB(G) \le \frac{3m^2}{2M_1(G)}\left(\sqrt{\frac{\Delta(G)}{\delta(G)}} + \sqrt{\frac{\delta(G)}{\Delta(G)}}\right)^2 + \frac{M_1(G) - 2m}{4(\delta(G)-1)}.$$

**Proof:** From Theorem 16, we have

$$HMB(G) \le \frac{3m^2}{2M_1(G)}\left(\sqrt{\frac{\Delta(G)}{\delta(G)}} + \sqrt{\frac{\delta(G)}{\Delta(G)}}\right)^2 + HEB(G).$$

Then from Theorem 7, we obtain the desired result.

**Theorem 23.** For any (*n*, *m*)-connected graph *G* with δ(*G*) ≥ 2 and *n* ≥ 3 vertices,

$$HMB(G) \le \frac{6m(\Delta(G)+\delta(G)) - 3n\Delta(G)\delta(G)}{2\delta(G)^2} + \frac{M_1(G)-2m}{4(\delta(G)-1)}.$$

**Proof:** From Theorem 17, we have

$$HMB(G) \le \frac{6m(\Delta(G)+\delta(G)) - 3n\Delta(G)\delta(G)}{2\delta(G)^2} + HEB(G).$$

Then from Theorem 7, we get the desired result.

**Theorem 24.** For any (*n*, *m*)-connected graph *G* with δ(*G*) ≥ 2 and *n* ≥ 3 vertices,

$$\sqrt{\frac{2}{n-1}}S(G) + \frac{M_1(G)-2m}{4(\Delta(G)-1)} + HB(G) \le HMB(G) \le \sqrt{\frac{2}{3}}S(G) + \frac{M_1(G)-2m}{4(\delta(G)-1)} + HB(G).$$

**Proof:** From Theorem 14, we have

$$HMB(G) = H(G) + HEM(G) + HB(G).$$

Then from Theorem 5, we get

$$\sqrt{\frac{2}{n-1}}S(G) + HEM(G) + HB(G) \le HMB(G) \le \sqrt{\frac{2}{3}}S(G) + HEM(G) + HB(G).$$

Using Theorem 7, we obtain

$$\sqrt{\frac{2}{n-1}}S(G) + \frac{M_1(G)-2m}{4(\Delta(G)-1)} + HB(G) \le HMB(G) \le \sqrt{\frac{2}{3}}S(G) + \frac{M_1(G)-2m}{4(\delta(G)-1)} + HB(G).$$

**Theorem 25.** For any (*n*, *m*)-connected graph *G* with *p* pendant vertices and minimal nonpendant vertex $\delta_1(G)$.

$$2p\frac{1}{\delta_1(G)-1} + \frac{3\Delta(G)-1}{\Delta(G)(2\Delta(G)-1)}\Bigg] + (m-p)\left[\frac{1}{\delta_1(G)} + \frac{4}{3\Delta(G)-2}\right] \le HMB(G) \le$$

$$2p\left[\frac{1}{\Delta(G)-1} + \frac{3\delta_1(G)-1}{\delta_1(G)(2\delta_1(G)-1)}\right] + (m-p)\left[\frac{1}{\Delta(G)} + \frac{4}{3\delta_1(G)-2}\right].$$

**Proof:** From Theorem 14, we have

$$HMB(G) = H(G) + HB(G) + HEM(G).$$

Using Theorem 6, we get

$$\frac{2p}{\delta_1(G)-1} + \frac{m-p}{\delta_1(G)} + HB(G) + HEM(G) \le HMB(G) \le \frac{2p}{\Delta(G)-1} + \frac{m-p}{\Delta(G)} + HB(G) + HEM(G).$$

Then from Theorem 12, we obtain

$$\frac{2p}{\delta_1(G)-1} + \frac{m-p}{\delta_1(G)} + 2p\left[\frac{3\Delta(G)-1}{\Delta(G)(2\Delta(G)-1)}\right] + \frac{4(m-p)}{3\Delta(G)-2} + HEM(G) \le HMB(G) \le$$

$$\frac{2p}{\Delta(G)-1} + \frac{m-p}{\Delta(G)} + 2p\left[\frac{3\delta_1(G)-1}{\delta_1(G)(2\delta_1(G)-1)}\right] + \frac{4(m-p)}{3\delta_1(G)-2} + HEM(G).$$

Thus

$$2p\frac{1}{\delta_1(G)-1} + \frac{3\Delta(G)-1}{\Delta(G)(2\Delta(G)-1)}\Bigg] + (m-p)\left[\frac{1}{\delta_1(G)} + \frac{4}{3\Delta(G)-2}\right] + HEM(G) \le HMB(G) \le$$

$$2p\left[\frac{1}{\Delta(G)-1} + \frac{3\delta_1(G)-1}{\delta_1(G)(2\delta_1(G)-1)}\right] + (m-p)\left[\frac{1}{\Delta(G)} + \frac{4}{3\delta_1(G)-2}\right] + HEM(G).$$

## CONCLUSION

In this study, we have introduced the reformulated harmonic index, harmonic Zagreb-*K*-Banhatti index of a graph. We have obtained some bounds for the harmonic index and reformulated harmonic index of a graph. Also we have provided some lower and upper bounds for *HMB*(*G*) of a graph *G* in terms of other topological indices.

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