Logistic Inverse Exponential Distribution with Properties and Applications

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Abstract: *In this study, we have introduced a two-parameter univariate continuous distribution called Logistic inverse exponential distribution. Some mathematical and statistical properties of the distribution such as the shapes of the probability density, cumulative distribution and hazard rate functions, survival function, quantile function, the skewness, and kurtosis measures are derived and established. To estimate the model parameters, we have employed three well-known estimation methods namely maximum likelihood estimation (MLE), least-square estimation (LSE), and Cramer-Von-Mises estimation (CVME) methods. A real data set is considered to explore the applicability and capability of the proposed distribution also AIC, BIC, CAIC and HQIC are calculated to assess the validity of the Logistic inverse exponential distribution.*

I. INTRODUCTION

In most of the literature of probability distributions and applied statistics, it is observed that the study of reliability and survival analysis in various fields of applied statistics and life sciences, the probability distributions are often used. In modeling survival data, existing models do not always reveal a better fit. Hence most of the researchers are interested to generalizing standard distributions and investigating their flexibility and applicability. Usually, these new compounded models produces an improved fit as compared to usual classical survival models and are obtained by introducing one or more additional shape parameter(s) to the parent distribution.

The exponential distribution plays a vital role in analyses of life testing data in statistics and probability theory. It is the probability distribution of the time between events in a Poisson point process, i.e., a process in which events occur independently and continuously at a constant average rate. It is a specific case of the gamma distribution. It is the continuous analog of the geometric distribution, and it has the key property of being memoryless. In addition to being used for the analysis of Poisson point processes, it is found in various other contexts.

A compounded survival model that includes the different shapes like Increasing, decreasing, bathtub-shaped, and inverted Bathtub-Shaped failure rate in a single model would be constructive in survival analysis. Such a model would provide considerable flexibility and goodness of fit for fitting a broad variety of lifetime data sets. Such a survival model might also be taken to determine the distribution class from which the data is selected, by constructing confidence interval over its parameters. The proposed distribution introduced here satisfies these criteria.

The inverse exponential distribution was proposed as an alternative to the Exponential distribution because it does not have a constant failure rate and it does not also reveal the memoryless property. Details about the Inverse Exponential distribution are readily available in (Keller & Kamath, 1982) and (Prakash, 2012). The inverse exponential distribution has been compounded in recent times resulting into Kumaraswamy Inverse Exponential distribution (Oguntunde et al., 2017), (Chaudhary, et al., 2020) has presented Truncated Cauchy power–inverse exponential distribution, The Exponential Inverse Exponential distribution has introduced by (Oguntunde et al., 2017). Rao (2013) has used inverse exponential distribution for the estimation of reliability in
multi-component stress-strength. Basheer (2019) has introduced the Marshall–Olkin alpha power inverse exponential distribution.

The logistic distribution is a univariate continuous distribution and both its PDF and CDF functions have been used in many different areas such as logistic regression, logit models and neural networks. It has been used in the physical sciences, demography, sports modeling, and recently in finance. The logistic distribution has wider tails than a normal distribution so it is more consistent with the underlying data and provides better insight into the likelihood of extreme events.
If *X* follows the logistic random variable with shape parameter

$\lambda > 0$, its cumulative distribution function is given by

$$F(x;\lambda) = \frac{1}{1+e^{-\lambda x}}; \quad \lambda > 0, x \in \Re$$

and its corresponding PDF is

$$f(x;\lambda) = \frac{\lambda e^{-\lambda x}}{\left(1+e^{-\lambda x}\right)^2}; \quad \lambda > 0, x \in \Re$$

Tahir et al. (2016) has defined the *logistic-X family* as a new generating family of continuous distributions produced from a logistic random variable whose density function can be defined as being right-skewed, left-skewed, symmetrical and reversed-J shaped, and can have decreasing, increasing, bathtub and upside-down bathtub hazard rates shaped. Lan and Leemis, (2008) has introduced an approach to define the logistic compounded model and introduced the logistic–exponential survival distribution. This has several useful probabilistic properties for lifetime modeling. Unlike most distributions in the bathtub and upside down bathtub classes, the logistic–exponential distribution exibit closed-form density, hazard, cumulative hazard, and survival functions. The survival function of the logistic–exponential distribution is

$$S(x;\lambda) = \frac{1}{1+\left(e^{\lambda x} - 1\right)^{\alpha}}; \quad \alpha > 0, \lambda > 0, x \geq 0$$

Using the same approach used by (Lan & Leemis, 2008) we have defined the new distribution called logistic- inverse exponential (LIE) distribution. The main aim of this study is to present a more flexible distribution by adding just one extra parameter to the inverse exponential distribution to attain a better fit to the lifetime data sets. We have discussed some distributional properties and its applicability. The remaining sections of the proposed study are arranged as follows. In Section 2 we present the new logistic- inverse exponential (LIE) distribution and its various mathematical and statistical properties. We have make use of three well-known estimation methods to estimate the model parameters namely the maximum likelihood estimation (MLE), least-square estimation (LSE) and Cramer-Von-Mises estimation (CVME) methods. For the maximum likelihood (ML) estimate, we have constructed the asymptotic confidence intervals using the observed information matrix are presented in Section 3. In Section 4, a real data set has been analyzed to explore the applications and capability of the proposed distribution. In this section, we present the estimated value of the parameters and log-likelihood, AIC, BIC and CAIC criterion for ML, LSE, and CVME. Finally, in Section 5 we present some concluding remarks

II. THE LOGISTIC-INVERSE EXPONENTIAL(LIE) DISTRIBUTION

The Inverse Exponential (IE) distribution has been introduced by (Keller & Kamath, 1982) and it has been studied and discussed as a lifetime model. If a random variable $Y \sim IE(\lambda)$ then the variable $U = \frac{1}{Y}$ will have an inverse exponential distribution and its CDF and PDF can be written as,

$$G(y) = e^{-\lambda/y}; \quad \lambda > 0, y > 0 \tag{2.1}$$

and

$$g(y) = \frac{\lambda}{y^2} e^{-\lambda/y}; \quad \lambda > 0, y > 0 \tag{2.2}$$

Let *X* be a positive random variable with a positive shape parameter α and a positive scale parameter λ then CDF of logistic-inverse exponential distribution can be defined as

$$F(x) = \frac{1}{1+\left[\exp\{\lambda/x\} - 1\right]^{\alpha}}; \quad (\alpha,\lambda) > 0, \ x > 0 \tag{2.3}$$

And its PDF is

$$f(x) = \frac{\alpha\lambda}{x^2} \frac{\exp\{\lambda/x\}\left[\exp\{\lambda/x\}-1\right]^{\alpha-1}}{\left\{1+\left[\exp\{\lambda/x\}-1\right]^{\alpha}\right\}^2} ; \ (\alpha,\lambda) > 0, \ x > 0 \tag{2.4}$$

This CDF function be similar to the log logistic CDF function with the second term of the denominator being changed in its base to an inverse exponential function, hence we named it Logistic inverse exponential distribution.

A. Reliability function
The reliability function of Logistic inverse exponential (LIE) distribution is

$$R(x) = 1 - F(x) = 1 - \frac{1}{1+\left[\exp\{\lambda/x\}-1\right]^{\alpha}} ; \ (\alpha,\lambda) > 0, \ x > 0 \tag{2.5}$$

B. Hazard function
The failure rate function of LIE distribution can be defined as,

$$h(x) = \frac{f(x)}{R(x)} = \frac{\alpha\lambda}{x^2} \frac{\exp\{\lambda/x\}}{\left[\exp\{\lambda/x\}-1\right]\left\{1+\left[\exp\{\lambda/x\}-1\right]^{\alpha}\right\}} ; \ (\alpha,\lambda) > 0, \ x > 0$$

(2.6)

In Figure 1, we have displayed the plots of the PDF and hazard rate function of LIE distribution for different values of α and λ.
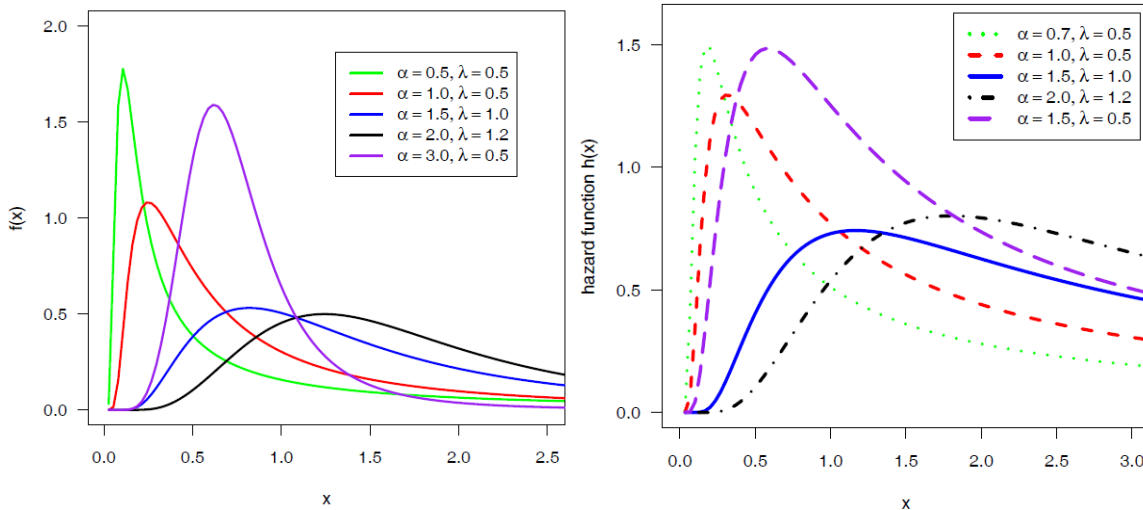

Figure 1. Plots of PDF (left panel) and hazard function (right panel) for different values of α and λ.

C. Quantile function:

Quantile function of Logistic Inverse Exponential distribution can be expressed as

$$Q(p) = \lambda\left[\ln\left\{\left(\frac{1-p}{p}\right)^{1/\alpha}+1\right\}\right]^{-1} ; 0 < p < 1 \tag{2.7}$$

D. Skewness and Kurtosis:
The Skewness and Kurtosis based on quantile function are,
Bowley's coefficient of skewness is

$$\mathfrak{I}_{Sk} = \frac{Q(0.75) + Q(0.25) - 2Q(0.5)}{Q(0.75) - Q(0.25)}, \tag{2.8}$$

Coefficient of kurtosis based on octiles which was defined by (Moors, 1988) is

$$K_u(M) = \frac{Q(0.875) - Q(0.625) + Q(0.375) - Q(0.125)}{Q(3/4) - Q(1/4)}, \tag{2.9}$$

III. METHODS OF ESTIMATION

In this section, we have presented some well-known estimation methods for estimating parameters of the proposed model, which are as follows

A. Maximum Likelihood Estimates

For the estimation of the parameter, the maximum likelihood method is the most commonly used method (Casella & Berger, 1990). Let, $x_1, x_2, \ldots x_n$ is a random sample from $LIE(\alpha, \lambda)$ and the likelihood function, $L(\alpha, \lambda)$ is given by,

$$L(\vartheta; x_1, x_2 \ldots x_n) = f(x_1, x_2, \ldots x_n / \vartheta) = \prod_{i=1}^{n} f(x_i / \vartheta)$$

$$L(\alpha, \lambda) = \alpha\lambda \prod_{i=1}^{n} \frac{\exp\{\lambda / x_i\} \left[\exp\{\lambda / x_i\} - 1\right]^{\alpha-1}}{x_i^2 \left\{1 + \left[\exp\{\lambda / x_i\} - 1\right]^{\alpha}\right\}^2} ; \ (\alpha, \lambda) > 0, \ x > 0$$

Now log-likelihood density is

$$\ell(\alpha, \lambda \mid \underline{x}) = n\ln(\alpha\lambda) - 2\sum_{i=1}^{n}\ln(x_i) + \lambda\sum_{i=1}^{n}\frac{1}{x_i} + (\alpha-1)\sum_{i=1}^{n}\ln\left\{\exp(\lambda / x_i) - 1\right\} - 2\sum_{i=1}^{n}\ln\left\{1 + \left[\exp\{\lambda / x_i\} - 1\right]^{\alpha}\right\} \tag{3.1.1}$$

Differentiating (3.1.1) with respect to α and λ we get,

$$\frac{\partial l}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\ln\left\{A(x_i)\right\} - 2\sum_{i=1}^{n}\frac{\left[A(x_i)\right]^{\alpha}\ln\left[A(x_i)\right]}{1 + \left[A(x_i)\right]^{\alpha}}$$

(3.1.2)

$$\frac{\partial l}{\partial \lambda} = \frac{n}{\lambda} + \sum_{i=1}^{n}\frac{1}{x_i} + (\alpha-1)\sum_{i=1}^{n}\frac{e^{\lambda/x_i}}{x_i A(x_i)} - 2\alpha\sum_{i=1}^{n}\frac{e^{\lambda/x_i}}{x_i}\frac{\left[A(x_i)\right]^{\alpha-1}}{1 + \left[A(x_i)\right]^{\alpha}}$$

(3.1.3)

where $A(x_i) = \exp(\lambda / x_i) - 1$

Equating (3.1.2) and (3.1.3) to zero and solving simultaneously for α and λ, we get the maximum likelihood estimate $\hat{\alpha}$ *and* $\hat{\lambda}$ of the parameters α and λ. By using computer software like R, Matlab, Mathematica etc for maximization of (3.1.1) we can obtain the estimated value of α and λ. For the confidence interval estimation of α and λ and testing of the hypothesis, we have to calculate the observed information matrix. The observed information matrix for α and λ can be obtained as,

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

Where

$$B_{11} = \frac{\partial^2 l}{\partial \alpha^2} = -\frac{n}{\alpha^2} - 2\sum_{i=1}^{n} \frac{[\ln\{A(x_i)\}]^2 [A(x_i)]^{\alpha}}{\{1+[A(x_i)]^{\alpha}\}^2}$$

$$B_{22} = \frac{\partial^2 l}{\partial \lambda^2} = -\frac{n}{\lambda^2} - (\alpha-1)\sum_{i=1}^{n} \frac{e^{\lambda/x_i}}{x_i^2 [A(x_i)]^2} - 2\alpha\sum_{i=1}^{n} \frac{e^{\lambda/x_i}}{x_i^2} \frac{[A(x_i)]^{\alpha-2}\left[\alpha e^{\lambda/x_i} - [A(x_i)]^{\alpha} - 1\right]}{\{1+[A(x_i)]^{\alpha}\}^2}$$

$$B_{12} = \frac{\partial^2 l}{\partial \alpha \partial \lambda} = -\frac{n}{\lambda^2} - (\alpha-1)\sum_{i=1}^{n} \frac{e^{\lambda/x_i}}{x_i [A(x_i)]^2} - 2\sum_{i=1}^{n} \frac{e^{\lambda/x_i}}{x_i} \frac{[A(x_i)]^{\alpha-1}\left[\alpha \ln\{A(x_i)\} + [A(x_i)]^{\alpha} + 1\right]}{\{1+[A(x_i)]^{\alpha}\}^2}$$

$$B_{21} = \frac{\partial^2 l}{\partial \alpha \partial \lambda} = -\frac{n}{\lambda^2} - (\alpha-1)\sum_{i=1}^{n} \frac{e^{\lambda/x_i}}{x_i [A(x_i)]^2} - 2\sum_{i=1}^{n} \frac{e^{\lambda/x_i}}{x_i} \frac{[A(x_i)]^{\alpha-1}\left[\alpha \ln\{A(x_i)\} + [A(x_i)]^{\alpha} + 1\right]}{\{1+[A(x_i)]^{\alpha}\}^2}$$

Let $\Theta = (\alpha, \lambda)$ denote the parameter space and the corresponding MLE of Θ as $\hat{\Theta} = (\hat{\alpha}, \hat{\lambda})$, then $\left(\hat{\Theta} - \Theta\right) \rightarrow N_2\left[0, \left(B(\Theta)\right)^{-1}\right]$ where $B(\Theta)$ is the Fisher's information matrix. Using the Newton-Raphson algorithm to maximize the likelihood creates the observed information matrix and hence the variance-covariance matrix is obtained as,

$$\left[B(\Theta)\right]^{-1} = \begin{pmatrix} \text{var}(\hat{\alpha}) & \text{cov}(\hat{\alpha}, \hat{\lambda}) \\ \text{cov}(\hat{\alpha}, \hat{\lambda}) & \text{var}(\hat{\lambda}) \end{pmatrix} \tag{3.1.4}$$

Hence from the asymptotic normality of MLEs, approximate 100(1-α) % confidence intervals for α and λ can be constructed as,

$$\hat{\alpha} \pm z_{\alpha/2} \sqrt{\text{var}(\hat{\alpha})} \text{ and } \hat{\lambda} \pm z_{\alpha/2} \sqrt{\text{var}(\hat{\lambda})} \tag{3.1.5}$$

where $z_{\alpha/2}$ is the upper percentile of standard normal variate.

B. Method of Least-Square Estimation (LSE)
The weighted square estimators and ordinary least square estimators are proposed by Swain et al. (1988) to estimate the parameters of Beta distributions. Here we have applied the same procedure for the LIE distribution. The least-square estimators of the unknown parameters α and λ of LIE distribution can be obtained by minimizing

$$D(X; \alpha, \lambda) = \sum_{i=1}^{n} \left[G(X_i) - \frac{i}{n+1}\right]^2 \tag{3.2.1}$$

with respect to unknown parameters α and λ.

Consider $G(X_i)$ denotes the distribution function of the ordered random variables $X_{(1)} < X_{(2)} < \dots < X_{(n)}$ where $\{X_1, X_2, \dots, X_n\}$ is a random sample of size n from a distribution function G(.). The least-square estimators of α and λ say $\hat{\alpha}$ and $\hat{\lambda}$ respectively, can be obtained by minimizing

$$D(X;\alpha,\lambda) = \sum_{i=1}^{n} \left[\frac{1}{1+\left[\exp\{\lambda/x_i\}-1\right]^{\alpha}} - \frac{i}{n+1} \right]^2 ; \; (\alpha,\lambda) > 0, \; x > 0$$

(3.2.2)

with respect to α and λ.

Differentiating (3.2.2) with respect to α and λ we get,

$$\frac{\partial D}{\partial \alpha} = -2\sum_{i=1}^{n} \left[\frac{1}{1+\left[\exp\{\lambda/x_i\}-1\right]^{\alpha}} - \frac{i}{n+1} \right] \frac{\left[\exp\{\lambda/x_i\}-1\right]^{\alpha} \ln\left[\exp(\lambda/x_i)-1\right]}{\left\{1+\left[\exp\{\lambda/x_i\}-1\right]^{\alpha}\right\}^2}$$

$$\frac{\partial D}{\partial \lambda} = -2\alpha\sum_{i=1}^{n} \left[\frac{1}{1+\left[\exp\{\lambda/x_i\}-1\right]^{\alpha}} - \frac{i}{n+1} \right] \frac{\left[\exp\{\lambda/x_i\}-1\right]^{\alpha-1} \exp(\lambda/x_i)}{x_i\left\{1+\left[\exp\{\lambda/x_i\}-1\right]^{\alpha}\right\}^2}$$

The weighted least square estimators can be obtained by minimizing

$$D(X;\alpha,\lambda) = \sum_{i=1}^{n} w_i \left[G(X_{(i)}) - \frac{i}{n+1} \right]^2$$

with respect to α and λ. The weights w_i are $\quad w_i = \dfrac{1}{Var(X_{(i)})} = \dfrac{(n+1)^2(n+2)}{i(n-i+1)}$

Hence, the weighted least square estimators of α and λ respectively can be obtained by minimizing,

$$D(X;\alpha,\lambda) = \sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[\frac{1}{1+\left[\exp\{\lambda/x_i\}-1\right]^{\alpha}} - \frac{i}{n+1} \right]^2$$

(3.2.3)

with respect to α and λ.

C. Method of Cramer-Von-Mises estimation (CVME)

The CVME estimators of α and λ are obtained by minimizing the function

$$W(X;\alpha,\lambda) = \frac{1}{12n} + \sum_{i=1}^{n} \left[G(x_{i:n}|\alpha,\lambda) - \frac{2i-1}{2n} \right]^2$$

$$= \frac{1}{12n} + \sum_{i=1}^{n} \left[\frac{1}{1+\left[\exp\{\lambda/x_i\}-1\right]^{\alpha}} - \frac{2i-1}{2n} \right]^2$$

(3.3.1)

Differentiating (3.3.1) with respect to α and λ we get,

$$\frac{\partial W}{\partial \alpha} = -2\sum_{i=1}^{n} \left[\frac{1}{1+\left[\exp\{\lambda/x_i\}-1\right]^{\alpha}} - \frac{2i-1}{2n} \right] \frac{\left[\exp\{\lambda/x_i\}-1\right]^{\alpha} \ln\left[\exp(\lambda/x_i)-1\right]}{\left\{1+\left[\exp\{\lambda/x_i\}-1\right]^{\alpha}\right\}^2}$$

$$\frac{\partial W}{\partial \lambda} = -2\alpha \sum_{k=1}^{n} \left[\frac{1}{1+\left[\exp\{\lambda/x_i\}-1\right]^{\alpha}} - \frac{2i-1}{2n} \right] \frac{\left[\exp\{\lambda/x_i\}-1\right]^{\alpha-1} \exp(\lambda/x_i)}{x_i \left\{1+\left[\exp\{\lambda/x_i\}-1\right]^{\alpha}\right\}^{2}}$$

Solving $\frac{\partial W}{\partial \alpha} = 0 \; and \; \frac{\partial W}{\partial \lambda} = 0$ simultaneously we get the CVM estimators.

IV. ILLUSTRATION WITH A REAL DATASET

The data given below represents the fatigue life of 6061-T6 aluminum coupons oscillated at 18 cycles per seconds (cps), cut parallel to the direction of rolling which consists of 101 observations with maximum stress per cycle 31,000 psi. Birnbaum and Saunders (1969) initially studied this data set.

70, 90, 96, 97, 99, 100, 103, 104, 104, 105, 107, 108, 108, 108, 109, 109, 112, 112, 113, 114, 114, 114, 116, 119, 120, 120, 120, 121, 121, 123, 124, 124, 124, 124, 124, 128, 128, 129, 129, 130, 130, 130, 131, 131, 131, 131, 131, 132, 132, 132, 133, 134, 134, 134, 134, 134, 136, 136, 137, 138, 138, 138, 139, 139, 141, 141, 142, 142, 142, 142, 142, 142, 144, 144, 145, 146, 148, 148, 149, 151, 151, 152, 155, 156, 157, 157, 157, 157, 158, 159, 162, 163, 163, 164, 166, 166, 168, 170, 174, 196, 212

The contour plot and fitted CDF with empirical distribution function (EDF) are presented in Figure 2, Kumar & Ligges (2011).
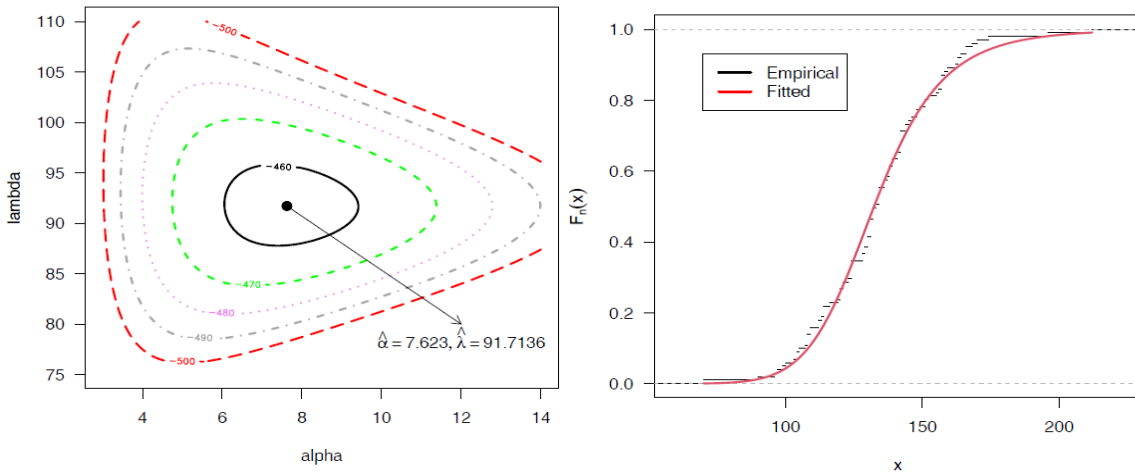

Figure 2. Contour plot (left panel) and the fitted CDF with empirical distribution function (right panel) of LIE distribution.

The MLEs are calculated directly by using optim() function (Ming, 2019) in R software (R Core Team, 2020) and (Rizzo, 2008) by maximizing the likelihood function (3.1.1). We have obtained $\hat{\alpha} = 7.6230$ and $\hat{\lambda} = 91.7136$ and the corresponding Log-Likelihood value is -456.4885. In Table 1 we have demonstrated the MLE's with their standard errors (SE) and 95% confidence interval for α and λ.

Table 1
MLEs, SE and 95% confidence interval

Parameter	MLE	SE	95% ACI
alpha	7.6230	0.6118	(6.4239, 8.8221)
lambda	91.7136	1.5853	(88.6064, 94.8208)

Hence the Hessian variance-covariance matrix is obtained as,

$$\left[B(\Theta)\right]^{-1} = \begin{pmatrix} var(\hat{\alpha}) & cov(\hat{\alpha},\hat{\lambda}) \\ cov(\hat{\alpha},\hat{\lambda}) & var(\hat{\lambda}) \end{pmatrix} = \begin{pmatrix} 0.3743 & 0.0000 \\ 0.0000 & 2.5132 \end{pmatrix}$$

We have displayed the graph of the profile log-likelihood function of α and λ in Figure 3 and observed that the MLEs are unique.
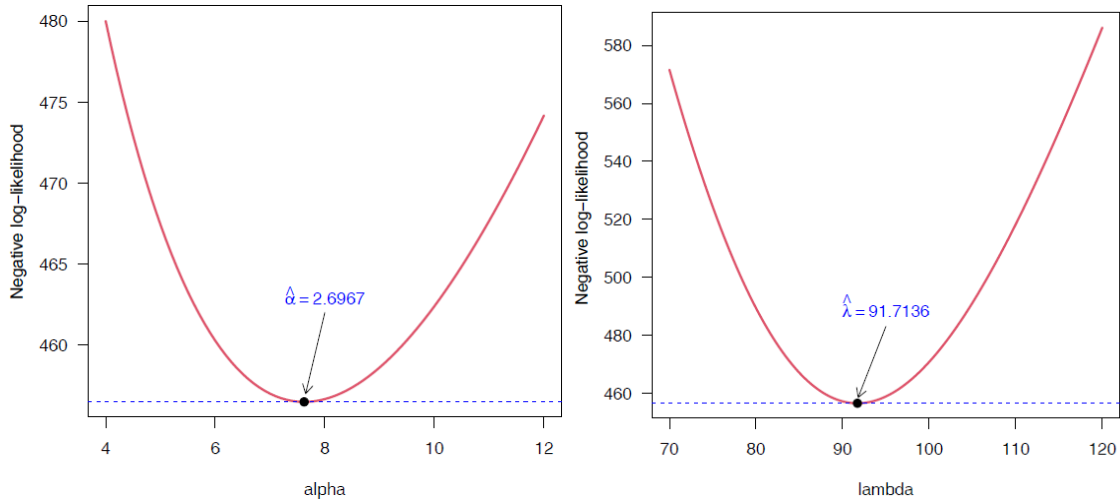
Figure 3. Graph of profile log-likelihood function of α and λ.

By using MLE method we estimate the parameter of each of these distributions. For the goodness of fit purpose we use negative log-likelihood (-LL), Bayesian information criterion (BIC), Akaike information criterion (AIC), Corrected Akaike Information criterion (CAIC) and Hannan-Quinn information criterion (HQIC), statistic to select the best model among selected models. The expressions to calculate AIC, BIC, CAIC and HQIC are listed below:

a) $AIC = -2l(\hat{\theta}) + 2k$

b) $BIC = -2l(\hat{\theta}) + k\log(n)$

c) $CAIC = AIC + \dfrac{2k(k+1)}{n-k-1}$

d) $HQIC = -2l(\hat{\theta}) + 2k\log\left[\log(n)\right]$

where the number of parameters is denoted by k and n denotes sample size in the model under consideration. Further, in order to evaluate the fits of the LHC distribution with some selected distributions we have taken the Kolmogorov-Simnorov (KS), the Anderson-Darling (W) and the Cramer-Von Mises (A^2) statistic. These statistics are widely used to compare non-nested models and to illustrate how closely a specific CDF fits the empirical distribution of a given data set. These statistics are calculated as

$$KS = \max_{1 \le i \le n}\left(d_i - \frac{i-1}{n}, \frac{i}{n} - d_i\right)$$

$$W = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\ln d_i + \ln\left(1 - d_{n+1-i}\right)\right]$$

$$A^2 = \frac{1}{12n} + \sum_{i=1}^{n}\left[\frac{(2i-1)}{2n} - d_i\right]^2$$

where $d_i = CDF\left(x_i\right)$; the x_i's being the ordered observations.

In Table 2 we have displayed the estimated value of the parameters of Logistic inverse exponential distribution using MLE, LSE and CVME method and their corresponding negative log-likelihood, AIC, BIC CAIC and HQIC information criteria.

Table 2
Estimated parameters, log-likelihood, AIC, BIC, CAIC and HQIC

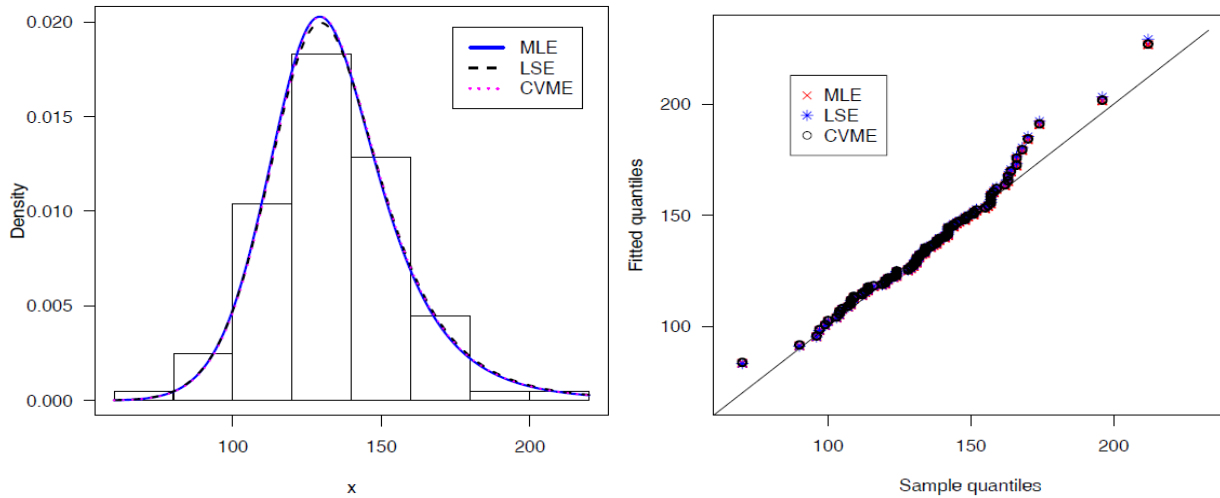| Method of Estimation	$\hat{\alpha}$	$\hat{\lambda}$	-LL	AIC	BIC	CAIC	HQIC
MLE	7.6230	91.7136	456.4885	916.9769	922.2071	917.0994	919.0943
LSE	7.5222	91.9984	456.5182	917.0364	922.2667	917.1589	919.1538
CVME	7.6485	92.0105	456.5093	917.0185	922.2488	917.1410	919.1359

Figure 4. The Histogram and the density function of fitted distributions (left panel) and Q-Q plot of estimation methods MLE, LSE and CVME(right panel).

Table 3
The KS, AD and CVM statistic with p-value

Method of Estimation	KS(p-value)	AD(p-value)	CVM(p-value)
MLE	0.0672(0.7511)	0.0608(0.8102)	0.4369(0.8112)
LSE	0.0598(0.8629)	0.0576(0.8302)	0.4523(0.7953)
CVME	0.0586(0.8791)	0.0569(0.8346)	0.4561(0.7915)

In order to illustrate the goodness of fit of the Lindley inverse exponential distribution, we have taken some well known distribution for comparison purpose which are listed blew,

A. Burr Type X distribution

The probability density function of Burr Type X distribution (Burr, 1942)

$$f_{BurrX}\left(x;\ \alpha,\ \lambda\right)\ =\ 2\alpha\lambda^{2}xe^{-(\lambda x)^{2}}\left[1\ -\ e^{-(\lambda x)^{2}}\right]^{\alpha-1};\ x\ >\ 0,\ \alpha\ >\ 0,\ \lambda\ >\ 0.$$

B. Generalized Exponential (GE) distribution

The probability density function of generalized exponential distribution (Gupta & Kundu, 1999)

$$f_{GE}\left(x;\alpha,\lambda\right)=\alpha\ \lambda\ e^{-\lambda\ x}\left\{1-e^{-\lambda\ x}\right\}^{\alpha-1};\left(\alpha,\lambda\right)>0,x>0.$$

C. Chen distribution
The probability density function of Chen distribution (Chen, 2000) is

$$f_{CN}\left(x; \lambda, \beta\right) = \lambda \ \beta \ x^{\beta-1} \ e^{x^\beta} \ exp\left\{\lambda\left(1 - e^{x^\beta}\right)\right\} \quad ;\left(\lambda, \beta\right) > 0, x > 0.$$

D. Exponential power (EP) distribution

The probability density function Exponential power (EP) distribution (Smith & Bain, 1975) is

$$f_{EP}(x) = \alpha \ \lambda^\alpha \ x^{\alpha-1} \ e^{(\lambda x)^\alpha} \ exp\left\{1 - e^{(\lambda x)^\alpha}\right\} \quad ;(\alpha, \lambda) > 0, \quad x \geq 0$$

where λ and α are the scale and shape parameters, respectively.

In Figure 5 we have presented the P-P plot (empirical distribution function against theoretical distribution function) and Q-Q plot (empirical quantile against theoretical quantile).
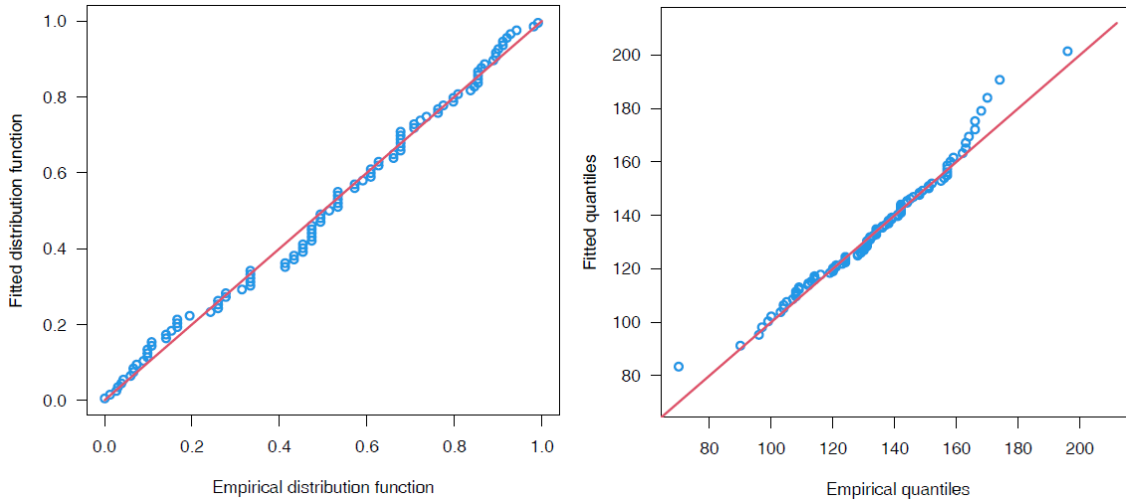

Figure 5. The P-P plot (left panel) and Q-Q plot (right panel) of LIE distribution

For the judgment of potentiality of the proposed model we have presented the value of Akaike information criterion (AIC), Bayesian information criterion (BIC), Corrected Akaike information criterion (CAIC) and Hannan-Quinn information criterion (HQIC) which are presented in Table 4.

Table 4
Log-likelihood (LL), AIC, BIC, CAIC and HQIC

Model	-LL	AIC	BIC	CAIC	HQIC
LIE	456.4885	916.9769	922.2071	917.0994	919.0943
BurrX	457.3766	918.7532	923.9835	918.8757	920.8706
GenExp	463.7324	931.4648	936.6951	931.5873	933.5822
Chen	467.0598	938.1196	943.3499	938.2421	940.2370
ExpPower	476.7897	957.5794	962.8096	957.6994	959.6967

The Histogram and the density function of fitted distributions and Empirical distribution function with estimated distribution function of LIE and some selected distributions are presented in Figure 6.
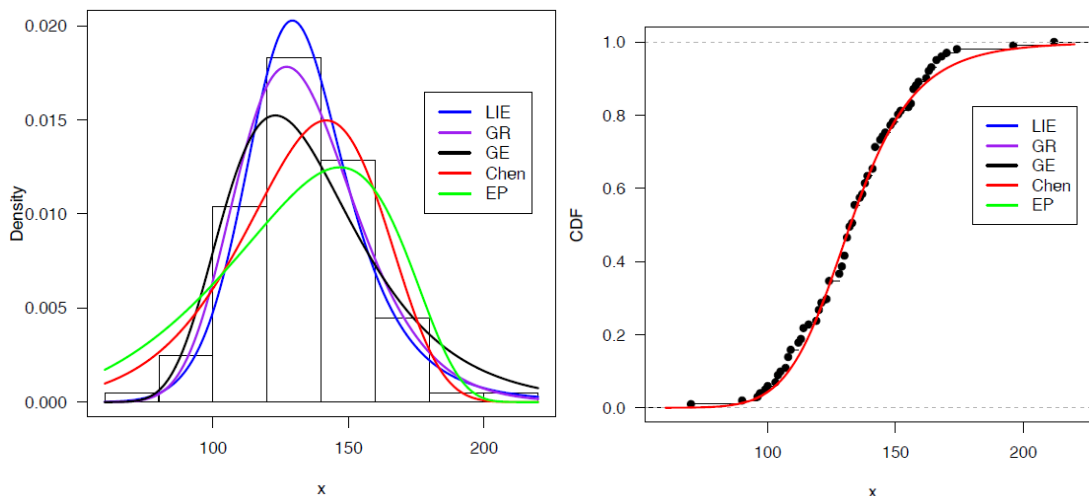
Figure 6. The Histogram and the density function of fitted distributions (left panel) and Empirical distribution function with estimated distribution function (right panel).

To compare the goodness-of-fit of the LIE distribution with other competing distributions we have presented the value of Kolmogorov-Simnorov (KS), the Anderson-Darling (AD) and the Cramer-Von Mises (CVM) statistics in Table 5. It is observed that the LIE distribution has the minimum value of the test statistic and higher *p*-value thus we conclude that the LIE distribution gets quite better fit and more consistent and reliable results from others taken for comparison.

Table 5

The goodness-of-fit statistics and their corresponding p-value

Model	KS(p-value)	AD(p-value)	CVM(p-value)
LIE	0.0672(0.7511)	0.0608(0.8102)	0.4369(0.8112)
BurrX	0.0901(0.3850)	0.1050(0.5620)	0.6033(0.6445)
GenExp	0.1066(0.2014)	0.3112(0.1257)	2.0724(0.0840)
Chen	0.1102(0.1718)	0.2960(0.1386)	2.0769(0.0835)
ExpPower	0.1378(0.0433)	0.6942(0.0130)	4.5057(0.0050)

VI. CONCLUSIONS

In this study, we have introduced a two-parameter univariate continuous Logistic inverse exponential (LIE) distribution. Some statistical and distributional properties of the LIE distribution are presented such as the shapes of the cumulative distribution function, probability density function and hazard rate function, survival function, hazard function quantile function the skewness, and kurtosis measures are derived and established and found that the proposed model is flexible and inverted bathtub shaped hazard function. the model parameters are estimated by using three well-known estimation methods namely maximum likelihood estimation (MLE), least-square estimation (LSE), and Cramer-Von-Mises estimation (CVME) methods and we concluded that the MLEs are quite better than LSE, and CVM. A real data set is considered to explore the applicability and suitability of the proposed distribution and found that the proposed model is quite better than other lifetime model taken into consideration.

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