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Alternative Proofs for the Length of Angle Bisectors Theorem on Triangle

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Abstract — *The purpose of the paper is to introduces alternative proofs for the length of Angle Bisectors Theorem on Triangle. In this paper the author proofs it with a simple way that uses law of sines, Pythagoras theorem, Ptolemy's theorem, and similarity.*

Keywords — *Angle bisector, Ptolemy's theorem, similar triangles*

I. INTRODUCTION

Given ΔABC with a, b, and c as side lengths. If d is the length of the angle bisector of the triangle drawn from the vertices A and p, q is the side facing the bisector, then $d^2 = bc\text{-}pq$. There are several ways to determine the formula for the length of angle bisector of a triangle, including those written by G.W.I.S. Amarasinghe [1] in 2012. Amarasinghe determines the formula for the length of angle bisector of the triangle using three methods. These three methods use similarity and the Pythagorean theorem. Then in 2013 Ovidiu T. Pop and Rodica D. Pop [8] determined the formula for the length of angle bisector of the triangle using Stewart's theorem. Another alternative in determining the length of angle bisector of the triangle is to use the chord concept, as described by Alec Scooper [12]. In his writing, Alec Scooper constructs the excircle of a triangle where the three sides of the triangle are the chord. Furthermore, Putu Darmayasa [13] used the law of cosines in determining the length formula for the angle bisector of the triangle. In this article, a new alternative in determining the length of the angle bisector of a triangle is discussed, namely using the law of sines, the Pythagorean theorem, Ptolemy's theorem, and similarity. The concepts and illustrations used are simple, so that it can be understood by student in senior high school.

II. LITERATURE REVIEW

In a circle, there is the term cyclic quadrilateral, which is a rectangle formed by the intersection of four chords on the circumference [2 and 3].

Definition1**.** Cyclic quadrilateral is a quadrilateral whose four vertices lie on the circumference of the circle.

One of the theorems dealing with cyclic quadrilaterals is Ptolemy's theorem. Ptolemy's theorem states the relationship between 4 sides and 2 diagonals in a cyclic quadrilateral [2, 3, 9, 10, and 11].

Theorem 1. If $ABCD$ is a cyclic quadrilateral, then the sum of the product of opposite sides equals the product of the diagonal, $(AB.CD) + (AD.BC) = AC.BD$.

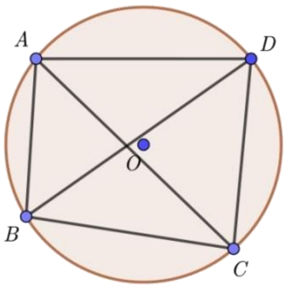

Figure 1. A Cyclic Quadrilateral

Proof. We can see in [2, 3, 9,10, and 11].

Besides circles, a shape that is often discussed in geometry is a triangle. In triangles there are special lines, one of which is a bisector, which is a line drawn from a vertex of the triangle so that it divides the angle into two equal parts. On triangle ABC a line is drawn from point A to the side of BC at point D so that it divides $\angle A$ into two equal parts, namely $\angle BAD = \angle CAD$, then AD is the bisector of triangle ABC [1 and 3].

Theorem 2. If the triangle ABC with side lengths a, b, and c and d is a bisector of angle A, then AD divides the side in front of it into two parts, p and q whose length ratio is equal to the ratio of the sides adjacent to the section, $p/q = c/b$.

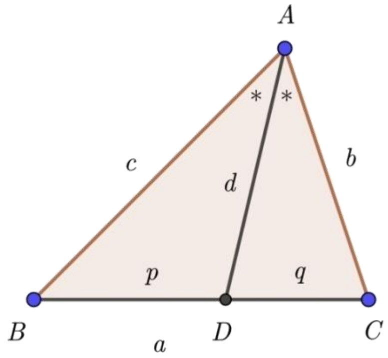

Figure 2. Triangle *ABC* and Bisector *AD*

Proof. We can see in [1 and 3].

III. RESULT

To prove the length of angle bisector theorem, the author uses simple concepts, namely the law of sines, Pythagorean theorem, Ptolemy's theorem, and similarity. Ideas for using the law of sines and similarity are given in [4, 5, 6, and 7]. For any triangle *ABC* with side lengths *a*, *b*, and *c*. *AD* is angle bisector in *A*. The length of *AD* is given in the following theorem.

Theorem 3. On any triangle *ABC* with side lengths *a*, *b*, and *c*, a bisector of angle *A* is drawn with the side length *d*. If *p* and *q* are the lengths of the sides in front of the angle bisector *A*, then the bisector of the triangle is $d^2 = bc - pq$.

Proof. The prove will be provided in 3 methods, they are

Method 1. Using the Law of Sines

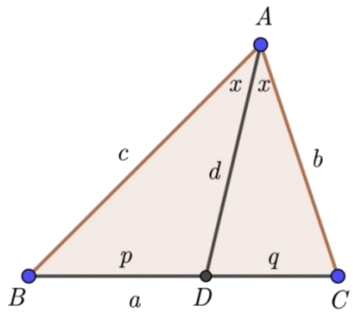

Figure 3. Triangle *ABC* and Angle *A*

Consider Figure 3. Using the law of sines of triangles *ABC* and *ABD* are obtained $\sin B = b/a.\sin 2x$ and $\sin B = d/p.\sin x$ so $b/a.\sin 2x = d/p.\sin x$. By performing algebraic operations it is obtained $\sin 2x/\sin x = ad/bp$. Because $\sin 2x=2 \sin x.\cos x$, then $(2\sin x.\cos x) /\sin x = ad/bp$, so that it is obtained

$$\cos x = ad/2bp \qquad ...(1)$$

Then in the same way, the triangle *ABC* and *ADC* are obtained

$$\cos x = ad/2cq \qquad ...(2)$$

By multiplying equations (1) and (2) we get

$$\cos^2 x = (a^2.d^2) / (4bp.cq) \qquad ...(3)$$

At ΔABC the law of cosines applies $\cos 2x = (b^2+c^2-a^2)/2bc$. Because $\cos 2x = 2\cos^2 x - 1$, then $2\cos^2 x – 1 = (b^2+c^2-a^2)/2bc$, so that

$$\cos^2 x = [(b+c)^2 - a^2] / 4bc \qquad ...(4)$$

Substituting equation (3) to equation (4) is obtained

$$(a^2.d^2) / (4bp.cq)= [(b+c)^2 - a^2] / 4bc$$

$$a^2.d^2 = bp.bq + cp.cq + 2bc.pq - a^2 pq \qquad ...(5)$$

Using Theorem 2, equation (5) becomes

$$a^2.d^2 = cq.bq + bp.cq + 2bc.pq - a^2pq$$

$$a^2.d^2 = bc\,(p+q)^2 - a^2\,pq$$

Since $p + q = a$, we get

$$a^2.d^2 = bca^2 - a^2\,pq$$
$$d^2 = bc - pq \qquad \ldots \blacksquare$$

Method 2. Using the Pythagorean Theorem and Similarity

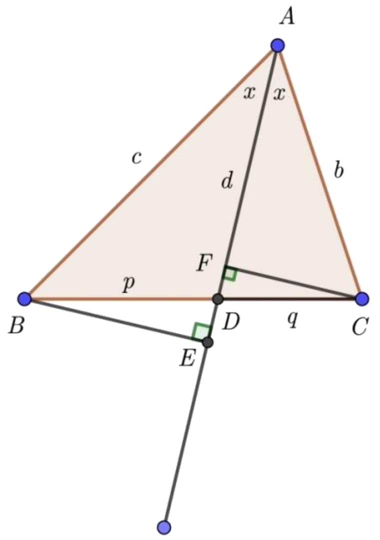

Figure 4. Altitudes of Triangles

In Figure 4, *BE* is the altitude of the *BED* and *BEA* triangles, so that the *BED* and *BEA* triangles apply the Pythagorean theorem $BE^2 = p^2 - DE^2 = c^2 - AE^2$, so we get $p^2 - DE^2 = c^2 - AE^2$. Since $AE = d + DE$, then

$$p^2 - DE^2 = c^2 - (d+DE)^2$$
$$2d.DE = c^2 - p^2 - d^2 \qquad \ldots (6)$$

Furthermore, in the *CFD* and *CFA* triangles the Pythagorean theorem applies $CF^2 = q^2 - DF^2 = b^2 - AF^2$, so $q^2 - DF^2 = b^2 - AF^2$. Since $AF = d - DF$, then

$$q^2 - DF^2 = b^2 - (d - DF)^2$$
$$2d.DF = q^2 - b^2 + d^2 \qquad \ldots (7)$$

By dividing equation (6) by equation (7), we get

$$DE / DF = (c^2 - p^2 - d^2) / (q^2 - b^2 + d^2) \qquad \ldots (8)$$

Next, notice that $\Delta BED \sim \Delta CFD$ so it applies $p/q = DE / DF$.

Substitution $p/q = DE / DF$ into the equation (8), obtained

$$p/q = (c^2 - p^2 - d^2) / (q^2 - b^2 + d^2)$$
$$pq^2 - pb^2 + pd^2 = qc^2 - qp^2 - qd^2$$
$$d^2\,(p+q) = cq.c - qp^2 - pq^2 + bp.b \qquad \ldots (9)$$

Applying Theorem 2 to equation (9) is obtained

$$d^2\,(p+q) = bp.c - qp^2 - pq^2 + bp.b$$
$$d^2\,(p+q) = bc(p+q) - pq(p+q)$$
$$d^2 = bc - pq \qquad \ldots \blacksquare$$

Method 3. Using Ptolemy's Theorem and Similarity

In triangle *ABC*, make the excircle of the triangle centered on *O*. Extend the dividing line *AD* so that it intersects the circle *O* at point *E*. Connect point *B* to *E* and points *C* and *E* as in Figure 5 below.

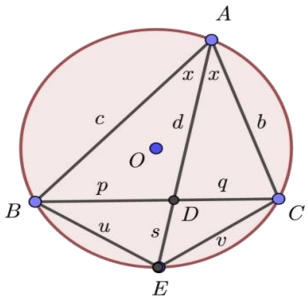

Figure 5. Cyclic Quadrilateral *ABEC*

Look at the *BED* and *ACD* triangles. Based on the similarity theorem $\Delta BED \sim ACD$, and apply

$$p/d = u/b \Leftrightarrow u = bp/d \quad \ldots (10)$$

Next look at the *BAD* and *ECD* triangles. $\Delta BAD \sim ECD$ so it applies $c/v = d/q = p/s$.

$$v = cq/d \quad \ldots (11)$$

and

$$s = pq/d \quad \ldots (12)$$

To the cyclic quadrilateral *ABEC* applies Ptolemy's theorem

$$(u.b) + (c.v) = (p+q)\,(d+s) \quad \ldots (13)$$

Substitution of equations (10), (11), and (12) into equation (13) is obtained

$$[(bp/d)\, b]+[c(cq/d)] = (p+q)\times[d + (pq/d)]$$

$$(bp.b + c.cq)/\, d = (p+q)d + (p^2q+pq^2)/d$$

$$bp.b + c.cq = (p+q)\, d^2 + pq(p+q) \quad \ldots (14)$$

Applying Theorem 2 to equation (14) is obtained

$$cq.b + c.bp = (p+q)\, d^2 + pq(p+q)$$

$$bc(p+q) = (p+q)\, d^2 + pq(p+q)$$

$$d^2 = bc - pq \quad \ldots \blacksquare$$

IV. CONCLUSION

The bisector of the triangle is an important material in mathematics learning, especially in the field of geometry. Various methods can be used to determine the length of the angle bisector of the triangle. One method that can be used by students in high school is the method of the law of sines, Pythagorean theorem, Ptolemy's theorem, and similarity where these four materials have been studied by students in school learning.

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