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Superperfect Numbers and Some Ideas

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Abstract: A positive integer n is called perfect number if $\sigma(n) = 2n$. In this paper we discuss few properties of perfect numbers and then extend the idea of perfect numbers to superperfect numbers. A positive integer n is said to be a superperfect number if $\sigma(\sigma(n)) = 2n$. Then we study few properties of these superperfect numbers.

Keywords: Divisor Function, Mersenne Primes, Multiplicative Functions, Perfect Numbers, Superperfect Numbers.

1 Introduction

A positive integer n is called perfect if it is equal to the sum of all its positive divisors,excluding itself. This is a well-known result in number theory. The history of perfect number dates back to 300 BC. For many centuries philosophers were more concerned with mystical and religious significance of perfect numbers.
Euclid proved that $2^{p-1}(2^p - 1)$ is an even perfect number whenever $2^p - 1$ is prime. Prime numbers of the form $2^p - 1$ are called Mersenne Primes. Only first four perfect numbers were known to ancient Greek Mathematics. Nichomachus in his *Introductio Arithmeticae* lists the first four perfect numbers $16, 28, 496, 8128$.
History of perfect numbers abounds with famous open problems. Where infinitude of perfect numbers is one of the most difficult open problems in Number Theory.Another famous open problems related with the perfect numbers is presence of odd perfect numbers.
In this project we would study a extension of the idea of perfect numbers. Which idea leads to the generalization of perfect numbers.
In 1969 D Suryanarayana [1] defined the idea superperfect numbers. The numbers with the property $\sigma(\sigma(n)) = 2n$. The first few superperfect numbers are 2, 4, 16 ,64, 4096, 65536.

2 Perfect Numbers and Some Properties

Definition 1: Let $\sigma(n)$ denote the sum of positive divisors of positive integer n. n is called perfect if $\sigma(n) = 2n$.

Theorem 1: An even number is perfect if and only if it has the form $2^{p-1}(2^p - 1)$. Where $2^p - 1$ is a prime. This theorem is known as Euclid-Euler Theorem. As stated earlier Euclid proved the sufficiency of the result. Later Euler proved the necessity condition.[2]

Theorem 2: An even perfect number n ends in the digit 6 or 8. [2]

It is still unknown whether any odd perfect numbers exists or not. Though Mathematicians throughout the centuries proved different results. Various properties have been found about the odd perfect numbers.Ratzan [4] studied many important properties of odd perfect numbers. Bege-Fogarasi [5] tried to generalize the form of perfect numbers, in their paper they have mentioned some interesting properties of odd perfect numbers.
If N is an odd perfect number, it must satisfy the following conditions.

1. $N > 10^{1500}$ [7]
2. N must be of the form $12m + 1$ or $36m + 9$.(Touchard [6])
3. $N = p^k m^2$ where p is a prime of the form $4k + 1$.(Euler [2])
4. N is not divisible by 105.[8]
5. N is divisible by at least seven distinct primes. [9]

These are some basic and common properties of odd perfect numbers.

3 Extending the Idea of Perfect Numbers

Definition 2: A positive integer n is called superperfect if $\sigma(\sigma(n)) = 2n$.

Theorem 3: An even number is perfect if and only if it has the form 2^{p-1}. Where $2^p - 1$ is a prime.p is an integer.
Proof:
Claim 1: $\sigma(n) \geq n$ for any positive integer n.
Every positive integer is divisible by 1 and itself. So for $n \geq 2$, we have $\sigma(n) \geq 1 + n$ and $\sigma(1) = 1$. So for any positive integer n, we have $\sigma(n) \geq n$, the equality holds for $n = 1$.
Claim 2: $\sigma(mn) \geq m\sigma(n)$ for any two positvie integers m, n
$\sigma(mn) = \sum_{d|mn} d = \sum_{d|n} md + C \geq m \sum_{d|n} d = m\sigma(n)$. Here $C \geq 0$.Hence equality holds when $m = 1$.

Now let $2^p - 1$ is a prime and let $2^{p-1} = n$
$\sigma(\sigma(n)) = \sigma(\sigma(2^{p-1})) = \sigma(2^p - 1)$
$= 2^p - 1 + 1$ [As $2^p - 1$ is prime]
$= 2^p = 2.2^{p-1} = 2n$
Hence n is superperfect.

Conversely, Let n be an even superperfect number.
So we write $n = 2^a b$
Here $a >= 1$ is an integer and b is an odd integer.
$\sigma(n) = \sigma(2^a b) > \sigma(2^a)c$ [Let $\sigma(b) = c$]
$= (2^{a+1} - 1)c$
So $\sigma(\sigma(n)) = \sigma((2^{a+1} - 1)c) \geq \sigma(2^{a+1} - 1)c \geq 2^{a+1}b = 2.2^a b = 2n$
As n is an superperfect number, we need to check the equality cases. i.e $\sigma((2^{a+1} - 1)c) = \sigma(2^{a+1} - 1)c$ and $\sigma(2^{a+1} - 1)c = 2^{a+1}b$.By claim 1 and 2, we conclude the first equality is true when $c = 1$ and the second equality is true when $b = 1$ and $\sigma(2^{a+1} - 1) = 2^{a+1}$. Thus $2^{a+1} - 1$ must be a prime. So $n = 2^a$ and $2^{a+1} - 1$ is a prime. Therefore every even superperfect number is of the form 2^{p-1}, Where $2^p - 1$ is a prime.

Theorem 4: If n is an odd superperfect number. Then n is a perfect square.
Proof:
n is an odd superperfect number. We write the prime factorization of n, let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k} = \prod_{i=1}^{k} p_i^{\alpha_i}$.
As σ is a multiplicative function. We write $\sigma(n) = \prod_{i=1}^{k} \sigma(p_i^{\alpha_i})$
Or, $\sigma(n) = \prod_{i=1}^{k} (1 + p_i + p_i^2 + \cdots + p_i^{\alpha_i})$

$= \prod_{j=1}^{k} q_j^{\beta_j}$ [Be the prime factorization of $\sigma(n)$]

Hence $\sigma(\sigma(n)) = \sigma(\prod_{j=1}^{l} q_j^{\beta_j}) = \prod_{j=1}^{l} \sigma(q_j^{\beta_j}) = \prod_{j=1}^{l} (1 + q_j + q_j^2 + \cdots + q_j^{\beta_j})$

As n is superperfect. We know $\sigma(\sigma(n)) = 2n$

Therefore $\prod_{j=1}^{l} (1 + q_j + q_j^2 + \cdots + q_j^{\beta_j}) = 2n = 2 \prod_{i=1}^{k} p_i^{\alpha_i}$

Now, $\frac{\sigma(\sigma(n))}{\sigma(n)} = \frac{\prod_{j=1}^{l} (1+q_j+q_j^2+\cdots+q_j^{\beta_j})}{\prod_{j=1}^{l} q_j^{\beta_j}} = \prod_{j=1}^{l} (1 + \frac{1}{q_j} + \frac{1}{q_j^2} + \cdots + \frac{1}{q_j^{\beta_j}})$
Again, $\frac{\sigma(n)}{n} = \frac{\prod_{i=1}^{k} (1+p_i+p_i^2+\cdots+p_i^{\alpha_i})}{\prod_{i=1}^{k} p_i^{\alpha_i}} = \prod_{i=1}^{k} (1 + \frac{1}{p_i} + \frac{1}{p_i^2} + \cdots + \frac{1}{p_i^{\alpha_i}})$
We can write $\frac{\sigma(\sigma(n))}{n} = \frac{\sigma(\sigma(n))}{\sigma(n)} . \frac{\sigma(n)}{n}$

Or, $\prod_{j=1}^{l} (1 + \frac{1}{q_j} + \frac{1}{q_j^2} + \cdots + \frac{1}{q_j^{\beta_j}}) \prod_{i=1}^{k} (1 + \frac{1}{p_i} + \frac{1}{p_i^2} + \cdots + \frac{1}{p_i^{\alpha_i}}) = 2$

If n is not a perfect square.
Then $\tau(n)$ i.e number of divisors of n is even. Hence $\sigma(n)$ is even. Without the loss of generality let us assume

$q_1 = 2$.
From our previous results we get $p_1 \leq 1 + 2 + \cdots + 2^{\beta_1} = 2^{\beta_1+1} - 1$
Again we get,
$2 \geq \frac{2^{\beta_1+1}-1}{2^{\beta_1}} \cdot \frac{p_1+1}{p_1} \prod_{j=2}^{l}(1 + \frac{1}{q_j} + \frac{1}{q_j^2} + \cdots + \frac{1}{q_j^{\beta_j}}) \prod_{i=2}^{k}(1 + \frac{1}{p_i} + \frac{1}{p_i^2} + \cdots + \frac{1}{p_i^{\alpha_i}}) \geq \frac{2^{\beta_1+1}-1}{2^{\beta_1}} \cdot \frac{2^{\beta_1+1}-1+1}{2^{\beta_1+1}-1} = 2$
Thus we can say $k = l = 1$, $\alpha_1 = 1$ and $p_1 = 2^{\beta_1+1} - 1$
Hence $n = 2^{\beta_1+1} - 1$
So $\sigma(n) = 2^{\beta_1+1}$
Or, $\sigma(\sigma(n) = 2^{\beta_1+1+1} - 1 = 2^{\beta_1+2} - 1$. But as n is superperfect $\sigma(\sigma(n) = 2n = 2p_1$. So we get $2^{\beta_1+2} - 1 = 2p_1$. Which is clearly a contradiction. So we can finally conclude n is a perfect square.

The idea of this proof was originally given by Kanold [3] in 1969.

Corollary 1: All superperfect numbers greater than 2 are perfect squares.
Proof: For odd superperfect numbers it follows from **Theorem 4**. For even superperfect numbers by **Theorem 3** we can say it is of the form 2^{p-1} when $2^p - 1$ is prime. For $2^p - 1$ to be a prime we must have p prime. If $p = 2$, $2^2 - 1 = 3$ is a prime and $2^{p-1} = 2^{2-1} = 2$. Now if p is an odd prime, $p - 1$ is even and 2^{p-1} is a perfect square. Therefore all superperfect numbers greater than 2 are perfect squares.

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