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On Ternary Seminear Rings

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Abstract

The notion of Ternary seminear ring is introduced which is the generalization of seminear ring. The characterization of ternary seminear ring are derived. The concept of ternary subseminear ring, ideals of ternary seminear ring and ternary seminear ring homomorphism are defined and their properties are also obtained.

1 Introduction

Lehmer.D.H[5] introduced the theory of ternary algebraic systems in 1932. Later Dutta.T.K. and Kar.S[1] initiated an idea on ternary semiring. Especially Vandiver[7] the american mathematician in 1934 tried to prove semiring as fundamental and best algebraic structure whereas the study of seminear ring was found by Willey G. Van Hoorn and Van Rootselaar B.,[8] in 1967. Later different people yielded seminear ring and researched numerous interesting and elegant properties. Our main purpose of this paper is to introduce the notion of ternary seminear ring and derived many examples, especially one of the examples is about ternary seminear ring which is not a ternary semiring. We also discussed about ternary subseminear ring, ternary seminear ring homomorphism and Ideals of ternary seminear ring and derived their characters.

2 Preliminaries

Definition 2.1. *A nonempty set S together with a binary operation, called addition and a ternary multiplication, denoted by juxtaposition, is said to be a ternary semiring if S is an additive commutative semigroup satisfying the following conditions* $(i) abc \in S$ $(ii)(abc)de = a(bcd)e = ab(cde)$
$(iii)(a+b)cd = acd + bcd$ $(iv)a(b+c)d = abd + acd$ $(v)ab(c+d) = abc + abd$ *for all* $a, b, c, d, e \in S$.

Definition 2.2. *A nonempty set R together with two binary operations called addition and multiplication is called a seminear ring if it satisfies the following conditions:* $(i)(R, +)$ *is a semigroup* $(ii)(R, .)$ *is a semigroup (iii) Multiplication distributes over addition* $a.(b+c) = a.b + a.c$ *for all* $a, b, c \in R$.

Definition 2.3. *A seminear ring R is said to have an absorbing zero if (i) $a+0 = 0+a = a$ (ii)$a.0 = 0.a = 0$, hold for all $a \in R$.*

Definition 2.4. *Let R be a seminear ring. A nonempty subset A of R is called a subseminear ring if A itself is a seminear ring with respect to the same operations in R.*

Definition 2.5. *A nonempty subset I of a seminear ring R is called a left(right)ideal if, (i) for all $x, y \in I, x+y \in I$ and (ii)for all $x \in I$ and $a \in R, ax \in I (xa \in I)$. I is said to be an ideal of R it is both a left ideal and a right ideal of R.*

Definition 2.6. *Let R and S be two seminear rings. A mapping $\mu : R \to S$ is called a homomorphism if and only if (i) $\mu(a+b) = \mu(a)+\mu(b)$ (ii)$\mu(a.b) = \mu(a).\mu(b)$ for all $a, b \in R$.*

3 Ternary Seminear ring

In this section the concept of **ternary seminear ring** is introduced and its properties are obtained.

Definition 3.1. *A* ***ternary seminear ring*** *is a nonempty set T together with a binary operation called addition '+' and a ternary operation called ternary multiplication denoted by juxtaposition, such that* $(i)(T, +)$ *is a semigroup. (ii)T is a ternary semigroup under ternary multiplication.*
$(iii)xy(z+u) = xyz + xyu$ *for all* $x, y, z, u \in T$.

Example 3.2. *The set of all negative integers with zero forms a ternary seminear ring under usual binary addition and ternary multiplication.*

Example 3.3. ***A ternary seminear ring need not be a ternary semiring.*** *For, let* $T = \{(x, y)/x, y \in R$, *R is set of all real numbers*$\}$. *First we prove that* T *is a ternary seminear ring. Let* $(x_1, y_1), (x_2, y_2) \in T$. *We define* $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ *and* $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in T$ *and we define* $(x_1, y_1)(x_2, y_2)(x_3, y_3) = (x_1x_2x_3, y_1x_2x_3 + y_2x_3 + y_3)$.
(i) Clearly by definition of addition, + satisfies closure law. It can be easily seen that $[(x_1, y_1) + (x_2, y_2)] + (x_3 + y_3) = (x_1, y_1) + [(x_2, y_2) + (x_3, y_3)]$.
Therefore (T,+) is a semigroup.
(ii) Clearly by the definition of ternary multiplication defined above, closure law is satisfied under ternary multiplication.
Let $(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4), (x_5, y_5) \in T$

$$\begin{aligned}[(x_1, y_1)(x_2, y_2)(x_3, y_3)](x_4, y_4)(x_5, y_5) &= (x_1x_2x_3, y_1x_2x_3 + y_2x_3 + y_3)(x_4x_5, y_4x_5 + y_5)\\ &= (x_1x_2x_3x_4x_5, y_1x_2x_3x_4x_5 + y_2x_3x_4x_5 + y_3x_4x_5\\ &\quad + y_4x_5 + y_5)\end{aligned}$$

$$\begin{aligned}(x_1, y_1)[(x_2, y_2)(x_3, y_3)(x_4, y_4)](x_5, y_5) &= (x_1, y_1)(x_2x_3x_4, y_2x_3x_4 + y_3x_4 + y_4)(x_5, y_5)\\ &= (x_1x_2x_3x_4, y_1x_2x_3x_4 + y_2x_3x_4 + y_3x_4 + y_4)(x_5, y_5)\\ &= (x_1x_2x_3x_4x_5, y_1x_2x_3x_4x_5 + y_2x_3x_4x_5 + y_3x_4x_5\\ &\quad + y_4x_5 + y_5)\end{aligned}$$

$$\begin{aligned}(x_1, y_1)(x_2, y_2)[(x_3, y_3)(x_4, y_4)(x_5, y_5)] &= (x_1x_2, y_1x_2 + y_2)(x_3x_4x_5, y_3x_4x_5 + y_4x_5 + y_5)\\ &= (x_1x_2x_3x_4x_5, y_1x_2x_3x_4x_5 + y_2x_3x_4x_5 + y_3x_4x_5\\ &\quad + y_4x_5 + y_5)\end{aligned}$$

Thus associative law is satisfied under ternary multiplication. Therefore T *is a ternary semigroup under ternary multiplication.*

(iii)Let $(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4) \in T$

$$\begin{aligned}(x_1, y_1)(x_2, y_2)[(x_3, y_3) + (x_4, y_4)] &= (x_1x_2, y_1x_2 + y_2)(x_3 + x_4, y_3 + y_4)\\ &= (x_1x_2x_3 + x_1x_2x_4, y_1x_2x_3 + y_1x_2x_4\\ &\quad + y_2x_3 + y_2x_4 + y_3 + y_4)\\ [(x_1, y_1)(x_2, y_2)(x_3, y_3)] + [(x_1, y_1)&\\ (x_2, y_2)(x_4, y_4)] &= (x_1x_2x_3, y_1x_2x_3 + y_2x_3 + y_3)+\\ &\quad (x_1x_2x_4, y_1x_2x_4 + y_2x_4 + y_4)\\ &= (x_1x_2x_3 + x_1x_2x_4, y_1x_2x_3 + y_1x_2x_4\\ &\quad + y_2x_3 + y_2x_4 + y_3 + y_4)\end{aligned}$$

Therefore $(x_1, y_1)(x_2, y_2)[(x_3, y_3)+(x_4, y_4)] = [(x_1, y_1)(x_2, y_2)(x_3, y_3)]+[(x_1, y_1)(x_2, y_2)(x_4, y_4)]$. *Thus Left distributive law is satisfied. Therefore from (i),(ii) and (iii) T is a ternary seminear ring. Now let us prove that T is not a ternary semiring. For, let* $(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4) \in T$

$$\begin{aligned}[(x_1, y_1) + (x_2, y_2)](x_3, y_3)(x_4, y_4) &= (x_1 + x_2, y_1 + y_2)(x_3x_4, y_3x_4 + y_4)\\ &= (x_1x_3x_4 + x_2x_3x_4, y_1x_3x_4 + y_2x_3x_4 + y_3x_4 + y_4)\\ [(x_1, y_1)(x_3, y_3)(x_4, y_4)] + [(x_2, y_2)&\\ (x_3, y_3)(x_4, y_4)] &= (x_1x_3x_4, y_1x_3x_4 + y_3x_4 + y_4)+\\ &\quad (x_2x_3x_4 + y_2x_3x_4 + y_3x_4 + y_4)\\ &= (x_1x_3x_4 + x_2x_3x_4, y_1x_3x_4 + y_2x_3x_4 + y_3x_4\\ &\quad + y_4 + y_3x_4 + y_4)\end{aligned}$$

Therefore $[(x_1, y_1)+(x_2, y_2)](x_3, y_3)(x_4, y_4) \neq [(x_1, y_1)(x_3, y_3)(x_4, y_4)]+[(x_2, y_2)(x_3, y_3)(x_4, y_4)]$. *Thus right distributive law is not satisfied.*
Similarly lateral distributive law is also not satisfied. Therefore only one distributive law holds is an example of ternary seminear ring which is not a ternary semiring.

Definition 3.4. *A ternary seminear ring T is said to have an absorbing zero if there exists an element* $0 \in T$ *such that* $i)x + 0 = 0 + x = x$ *for all* $x \in T$. $ii)xy0 = x0y = 0xy = 0$ *for all* $x, y \in T$.

Remark. *Throughout this paper T will always stand for the ternary seminear ring will always mean that ternary seminear ring with an absorbing zero.*

Definition 3.5. *Let T be a ternary seminear ring. A nonempty subset R of T is said to be a ternary subseminear ring of T if R itself is a ternary seminear ring under the same operations of T.*

Theorem 3.6. *Let R be a subset of a ternary seminear ring T and* $0 \in R$*. Then R is a ternary subseminear ring of T if and only if* $R + R \subset R$ *and* $RRR \subset R$*.*

Proof. Let T be a ternary seminear ring, R be a subset of T and $0 \in R$. If R is a ternary subseminear ring of T, then R is a subgroup of T under addition. Now let $x + y \in R + R$. Then $x, y \in R$ which implies that $x + y \in R$ and hence $R+R \subset R$. Let $xyz \in RRR$. Then $x, y, z \in$ R. Since R is closed under ternary multiplication, $xyz \in R$ and hence $RRR \subset R$
Conversely, assume that $R + R \subset R$ and $RRR \subset R$. Let $x, y, z \in R$. Then $x + y \in R + R$ and $xyz \in RRR$. Therefore $x + y \in R$ and $xyz \in R$, by assumption. Since R contains 0, R is a ternary subseminear ring of T. □

Theorem 3.7. *The intersection of two ternary subseminear rings of a ternary seminear ring T is a ternary subseminear ring of T.*

Proof. Let T be a ternary seminear ring and let R_1 and R_2 be two ternary subseminear rings of T. Since $0 \in R_1$ and $0 \in R_2$, $0 \in R_1 \cap R_2$. Thus $R_1 \cap R_2 \neq \emptyset$. Let $x, y \in R_1 \cap R_2$. Then $x, y \in R_1$ and $x, y \in R_2$.Since R_1 and R_2 are ternary subseminear rings of T, $x + y \in R_1$ and $x + y \in R_2$. Therefore $x+y \in R_1 \cap R_2$. Similarly we can prove that $xyz \in R_1 \cap R_2$, where $x, y, z \in R_1 \cap R_2$. Hence $R_1 \cap R_2$ is a ternary subseminear ring of T. □

Definition 3.8. *Let T be a ternary seminear ring (T,+,.). A non empty subset I of T is said to be a left(lateral and right) ideal of T if it holds the following conditions i)* $i + j \in I$ *for all* $i, j \in I$ *ii)* t_1t_2i*(respectively* $t_1it_2, it_1t_2) \in I$ *for all* $t_1, t_2 \in T$ *and* $i \in I$*. If I is a left, a lateral and a right ideal of T then I is said to be an ideal of T.*

Definition 3.9. *The ternary seminear ring T is a Ternary seminear ring with unity if sometimes it has a multiplicative identity as* $1 \in T$ *such that* $t_1t_21 = t_11t_2 = 1t_1t_2 = 1$ *for all* $t_1, t_2 \in T$*.*

Theorem 3.10. *If T is a ternary seminear ring with unity and I is an ideal of T such that* $1 \in I$*, then I=T.*

Proof. Let T be a ternary seminear ring and I be an ideal of T such that $1 \in I$ which implies $I \subseteq T$. Let $t_1 \in T$ and let $t_2 = 1 \in T$. Where I is an ideal of T then $t_1 t_2 i \in I$ which implies $t_1 1 \in I$ then $t_1 \in I$. Hence $T \subseteq I$. Therefore I=T. □

Theorem 3.11. *The nonempty intersection of two ideals of a ternary seminear ring T is an ideal of T.*

Proof. Let I_1 and I_2 be two ideals of a ternary seminear ring T. Then $0 \in I_1$ and $0 \in I_2$ where 0 is the zero element of the ternary seminear ring T.
$0 \in I_1 \cap I_2$ which implies $I_1 \cap I_2 \neq \emptyset$. Let $i, j \in I_1 \cap I_2$ and $t_1, t_2 \in T$. $i, j \in I_1 \cap I_2$ which implies $i, j \in I_1$ and $i, j \in I_2$.Now,$i, j \in I_1$, $t_1, t_2 \in T$ which implies $i + j \in I_1$ and $it_1t_2, t_1it_2, t_1t_2i \in I_1$. Since I_1 is an ideal of T. Also $i, j \in I_2$, $t_1, t_2 \in T$ which implies $i + j \in I_2$ and $it_1t_2, t_1it_2, t_1t_2i \in I_2$. Since I_2 is an ideal of T. Therefore $i + j \in I_1$ and $i + j \in I_2$ which implies $i + j \in I_1 \cap I_2$. $it_1t_2, t_1it_2, t_1t_2i \in I_1$ and $it_1t_2, t_1it_2, t_1t_2i \in I_2$ which implies $it_1t_2, t_1it_2, t_1t_2i \in I_1 \cap I_2$. Therefore $I_1 \cap I_2$ is an ideal of T. □

Remark. *From Ideal definition, we have every ideal is a ternary subseminear ring and every ternary subseminear ring is a ternary seminear ring. Then every ideal is a ternary seminear ring.*

Theorem 3.12. *Let I be a ternary subsemigroup of a ternary seminear ring T. Then I is a left(lateral or right)ideal of T if and only if $TTI \subseteq I$ (respectively $TIT \subseteq I$ and $ITT \subseteq I$).*

Proof. Let I be a ternary subsemigroup of a ternary seminear ring T. Let I be a left ideal of T. Then we have, $i + j \in I + I \subseteq I$ where $i, j \in I$. $t_1t_2i \in TTI \subseteq I$ where $t_1, t_2 \in T$ and $i \in I$. similarly, if I is a right ideal of T, then we have $ITT \subseteq I$ and if I is a lateral ideal of T then we have $TIT \subseteq I$. Conversely, assume that $TTI \subseteq I$ then by definition 3.8 it is obviously that I is a left(lateral or right) ideal of T. □

Definition 3.13. *Let T and U be two ternary seminear rings and $\phi : T \to U$ be a mapping. ϕ is said to be a Homomorphism of T into U if (i) $\phi(x+y) = \phi(x) + \phi(y)$ and (ii) $\phi(xyz) = \phi(x)\phi(y)\phi(z)$ for all x,y,z $\in$ T.*

Definition 3.14. *Let T and U be two ternary seminear rings and $\phi : T \to U$ be a homomorphism*

- *If $T = U$ then ϕ is said to be an Endomorphism.*

- *If ϕ is also onto then it is called an Epimorphism.*
- *If ϕ is also one to one then it is called Monomorphism.*
- *If homomorphism ϕ is epimorphism as well as monomorphism then it is said to be an Isomorphism.*
- *A isomorphism ϕ is said to be an automorphism if T=U.*

Definition 3.15. *Let T and U be two ternary seminear rings. If $\phi : T \to U$ is an isomorphism then T is said to be isomorphic to U and it is denoted as $T \cong U$.*

Theorem 3.16. *Let T and U be two ternary seminear rings and let $\phi : T \to U$ be a homomorphism.*
(i) The image of ϕ is a ternary subseminear ring of U.
(ii) The kernel of ϕ is a ternary subseminear ring of T. Furthermore, if $\alpha \in ker\phi$, then $t_1t_2\alpha, t_1\alpha t_2, \alpha t_1t_2 \in ker\phi$ for every $t_1, t_2 \in T$ that is $ker\phi$ is closed under multiplication by elements from T.

Proof. Let T and U be two ternary seminear rings and let $\phi : T \to U$ be a homomorphism. (i) If $u_1, u_2, u_3 \in im\phi$ then $u_1 = \phi(t_1)$, $u_2 = \phi(t_2)$ and $u_3 = \phi(t_3)$ for some $t_1, t_2, t_3 \in T$. Now

$$\begin{aligned}\phi(t_1 + t_2) &= \phi(t_1) + \phi(t_2)\\ &= u_1 + u_2\end{aligned}$$

where $t_1 + t_2 \in T$. Therefore $u_1 + u_2 \in im\phi$. Now

$$\begin{aligned}\phi(t_1t_2t_3) &= \phi(t_1)\phi(t_2)\phi(t_3)\\ &= u_1u_2u_3\end{aligned}$$

Since t_1, t_2 and $t_3 \in T$, $u_1u_2u_3 \in im\phi$. Thus the image of ϕ is closed under addition and ternary multiplication. Hence the image of ϕ is a ternary subseminear ring of U.
(ii) If $\alpha, \beta \in ker\phi$ then $\phi(\alpha) = \phi(\beta) = 0$. Hence $\phi(\alpha + \beta) = 0$ and $\phi(\alpha\beta\gamma) = 0$. Therefore $ker\phi$ is closed under addition and ternary multiplication, Thus it is a ternary subseminear ring of T.For any $t_1, t_2 \in T$, we have

$$\begin{aligned}\phi(t_1t_2\alpha) &= \phi(t_1)\phi(t_2)\phi(\alpha)\\ &= \phi(t_1)\phi(t_2)0\\ &= 0\end{aligned}$$

Similarly we can prove that $\phi(t_1\alpha t_2) = 0$ and $\phi(\alpha t_1 t_2) = 0$. So, we get $t_1t_2\alpha, t_1\alpha t_2, \alpha t_1 t_2 \in ker\phi$. $\square$

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