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A New Complex Inversion Formula For A Laplace Function, In Solving Some Complicated Problems Using The Laplace Transform Method, Methods Based On Complex Variable Theory May Come In Handy For Finding The Inverse Transform

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Abstract:

With the remarkable advances made in various branches of science, engineering and technology, today, more than ever before, the study of partial differential equations has become essential. For, to have an in-depth understanding of subjects like fluid dynamics and heat transfer, aerodynamics, elasticity, waves, and electromagnetics, the knowledge of finding solutions to partial differential equation is absolutely necessary. In this article, Laplace transform method is self-contained since the subject matter has been developed from the basic definition.

Introduction:

In this paper, we present a new complex inversion formula for a Laplace function. In solving some complicated problems using the Laplace transform method. Methods based on complex variable theory may come in handy for finding the inverse transform. Also it can be noted that the Laplace transform of f(t) is expressed as in integral. Similarly, the inverse Laplace transform of F(s) can be expressed as in integral which is known as inverse integral. This integral can be evaluated by using contour integration methods. The complex inversion formula is stated below.

Theorem:

Let f(t) and f'(t) be continuous functions on t≥0 and f(t)=0 for t<0. In addition, if f (t) is $0(e^{t\gamma_0})$ and F(s) =L [f (t); s] Then L^{-1}[F(s); t]=f (t) $=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}F(s)ds$, t>0 and γ is a positive constant.

Proof:

Let g(t) and g'(t) be continuous functions and if $\int_{-\infty}^{\infty} g(t)dt$ converges absolutely and uniformly then g(t) may be represented by the Fourier integral formula

$$g(t)= \frac{1}{2\pi}\int_{-\infty}^{\infty} g(v)[\int_{-\infty}^{\infty} \cos\omega(t-v)d\omega]dv$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}[\int_{-\infty}^{\infty} g(v)\cos\omega(t-v)dv]d\omega \qquad \text{.......................} (1)$$

Since sin ω (t-v) is an odd function of ω,

We have

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}[\int_{-\infty}^{\infty} g(v)\sin\omega(t-v)dv]d\omega = 0$$

Combining this expression with equation (1) we get

$$g(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}[\int_{-\infty}^{\infty}g(v)e^{i\omega(t-v)}dv]d\omega$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\omega t}[\int_{-\infty}^{\infty}g(v)e^{-i\omega v}dv]d\omega \qquad \text{......................} (2)$$

In addition we assume that g(t) is of exponential order =0($e^{t\gamma_0}$).

Now we consider the function

$$g(t) = \begin{cases} e^{-\gamma t} & t\geq 0 \\ 0 & t<0 \end{cases}$$

Where γ is a real number greater thanγ_0. Thus, g(t) satisfies all the conditions required by the Fourier integral theorem and, therefore we have from equation (2).

For t≥0 the relation

$$e^{-\gamma t}f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\omega t}[\int_{-\infty}^{\infty}e^{\gamma v}f(v)e^{-i\omega v}dv]\,d\omega$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\omega t}[\int_{-\infty}^{\infty}e^{-(\gamma+i\omega)v}f(v)dv]\,d\omega$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\omega t}F(\gamma+i\omega)d\omega \quad \text{[definition of Laplace]}$$

Let $\gamma+i\omega$=s, so that$d\omega=ds/i$. It follows that $e^{-\gamma t}$f (t)=$\frac{1}{2\pi}\int_{-\gamma+i\infty}^{\gamma+i\infty}e^{t(s-\gamma)}$F(s) ds

Therefore,

$$f(t)=\frac{1}{2\pi}\int_{-\gamma+i\infty}^{\gamma+i\infty}e^{st}F(s)\,ds, \quad t\geq 0 \qquad \text{…………….... (3)}$$

Hence proved

Definition 1:

A continuous function can be formally defined as a function $f: X\to Y$ where the preimage of every open set in Y is open in X. More concretely, a function $f(x)$ in a single variable x is said to be continuous at point x_0 if

*$f(x_0)$is defined, so that x_0 is in the domain of f.

*$\lim_{x\to x_0} f(x)$ exists for x in the domain of f.

*$\lim_{x\to x_0} f(x)=f(x_0)$.

Definition 2:

Laplace transform is the integral transform of the given derivative function with real variable t to convert into complex function with variable s. For t ≥ 0, let f(t) be given and assume the function satisfies certain conditions to be stated later on. The Laplace transform of f(t), that it is denoted by **f(t) or F(s)** is defined by the equation

$$F(s)=\int_0^{\infty}f(t)e^{-st}dt$$

Definition 3:

The integral $\int_A f(x)dx$ of a real or complex-valued function is said to converge absolutely if $\int_A |f(x)|\,dx<\infty$.

Definition 4:

If a function $f(X)$ satisfies the Dirichlet condition on every finite interval and if the integral $\int_{-\infty}^{\infty}|f(x)|dx$ converges then, $f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}g(v)[\int_{-\infty}^{\infty}\cos\omega(t-v)d\omega]dv$.

Example 1:

Find the Laplace inverse of$\frac{1}{s^2+1}$

Solution:

From the formula

$$L^{-1}[\text{F(s); t}]=\text{f (t)} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}F(s)ds$$

$$L^{-1}\left[\frac{1}{s^2+1};\text{t}\right] = \frac{1}{2\pi}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{st}ds}{s^2+1}$$

$$= \text{sum of residues of} \frac{e^{st}}{s^2+1}$$

It has two poles s=i and s = $-i$

$$= \text{sum of the residues of} \frac{e^{st}}{(s+i)(s-i)}$$

$$=R_1+R_2 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

R_1 is a simple pole at s = i, Formula for simple pole is $f(z) = \lim_{s\to s_0}(s-s_0)f(z)$

Appling the formula

$$R_1 = \lim_{s\to i}(s-i)\frac{e^{st}}{(s+i)(s-i)}$$

$$= \lim_{s\to i}\frac{e^{st}}{(s+i)}$$

$$= \frac{e^{it}}{(i+i)}$$

$$= \frac{e^{it}}{2i}$$

$$R_2 = \lim_{s\to -i}(s+i)\frac{e^{st}}{(s+i)(s-i)}$$

$$= \lim_{s\to -i}\frac{e^{st}}{(s-i)}$$

$$= \frac{e^{-it}}{(-i-i)}$$

$$= \frac{e^{-it}}{-2i}$$

Substitute the values of R_1 and R_2 in equation (1)

Laplace inverse of $\frac{1}{s^2+1} = \frac{e^{it}}{2i} + \frac{e^{-it}}{-2i}$

$$= \frac{e^{it}-e^{-it}}{2i}$$

Hence the Laplace inverse of $\frac{1}{s^2+1}$ is Sin t

Verification through normal Laplace inverse formula

$$L^{-1}\left[\frac{1}{(s^2+1)}\right]$$

First we solve the partial fraction $\frac{1}{s^2+1}$

The denominator s^2+1can be factorized into linear factors.

$$s^2+1 = (s+i)(s-i)$$

We assume $\frac{1}{(s-i)(s+i)} = \frac{A}{(s-i)} + \frac{B}{(s+i)}$…………..(1)where A and B are constants.

$$\Rightarrow \frac{1}{(s-i)(s+i)} = \frac{A(s+i)+B(s-i)}{(s-i)(s+i)}$$

$$\Rightarrow \qquad 1 = A(s+i) + B(s-i) \ldots\ldots\ldots\ldots\ldots(2)$$

To find A, put s = i in equation (2)

$$\Rightarrow \qquad 1 = A(i+i) + B(i-i)$$

$$\Rightarrow \qquad 1 = A(2i) + B(0)$$

$$\Rightarrow \qquad A = \frac{1}{2i}$$

To find B, put s = $-i$ in equation (2)

$$\Rightarrow \qquad 1 = A(-i+i) + B(-i-i)$$

$$\Rightarrow \qquad 1 = A(0) + B(-2i)$$

$$\Rightarrow \qquad B = \frac{1}{-2i}.$$

Substitute the values of A and B in equation (1)

$$\frac{1}{(s-i)(s+i)} = \frac{1}{2i(s-i)} - \frac{1}{2i(s+i)}$$

Appling Laplace inverse

$$L^{-1}\left[\frac{1}{(s-i)(s+i)}\right] = L^{-1}\left[\frac{1}{2i(s-i)} - \frac{1}{2i(s+i)}\right]$$

$$= L^{-1}\left[\frac{1}{2i(s-i)}\right] - L^{-1}\left[\frac{1}{2i(s+i)}\right]$$

$$= \frac{1}{2i} L^{-1}\left[\frac{1}{(s-i)}\right] - \frac{1}{2i} L^{-1}\left[\frac{1}{(s+i)}\right]$$

$$= \frac{1}{2i}\left\{L^{-1}\left[\frac{1}{(s-i)}\right] - L^{-1}\left[\frac{1}{(s+i)}\right]\right\}$$

$$= \frac{1}{2i}\{e^{it} - e^{-it}\}$$

$$= \sin t$$

Hence proved.

Example 2:

Find the Laplace inverse of $\frac{1}{(s+1)(s-2)^2}$.

Solution:

From the formula

$$L^{-1}[F(s); t] = f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}F(s)ds$$

$$L^{-1}\left[\frac{1}{(s+1)(s-2)^2}; t\right] = \frac{1}{2\pi}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{st}ds}{(s+1)(s-2)^2}$$

$$= \text{sum of residues of } \frac{e^{st}}{(s+1)(s-2)^2}$$

It has two poles at s = -1 and s = 2.

$$= R_1 + R_2 \qquad \ldots\ldots\ldots\ldots\ldots\ldots..(3)$$

R_1 is a simple pole at s = -1

Formula for simple pole is $f(z) = \lim_{z\to z_0}(z-z_0)f(z)$

Here the function is s so z = s, $z_0 = -1$, f(z) = $\frac{e^{st}}{(s+1)(s-2)^2}$.

Applying the formula,

$$R_1 = \lim_{s\to -1}(s+1)\frac{e^{st}}{(s+1)(s-2)^2}$$

$$= \lim_{s\to -1}\frac{e^{st}}{(s-2)^2}$$

$$= \frac{e^{-t}}{(-1-2)^2}$$

$$= \frac{e^{-t}}{(-3)^2}$$

$$= \frac{e^{-t}}{9}$$

R_2 is a double pole at s = 2

Formula for double pole for power m is $\frac{1}{(m-1)!}\lim_{s\to s_0}\frac{d^{m-1}}{dz^{m-1}}[(s-s_0)^m f(z)]$

$$R_2 = \frac{1}{(2-1)!}\lim_{s\to 2}\frac{d^{2-1}}{ds^{2-1}}[(s-2)^2\frac{e^{st}}{(s+1)(s-2)^2}]$$

$$= \frac{1}{1!}\lim_{s\to 2}\frac{d}{ds}[\frac{e^{st}}{(s+1)}]$$

$$= \lim_{s\to 2}(\frac{(s+1)e^{st}(t)-e^{st}(1)}{(s+1)^2})$$

$$= \lim_{s\to 2}\left[\frac{(s+1)t.e^{st}-e^{st}}{(s+1)^2}\right]$$

$$= \lim_{s\to 2}\left[\frac{e^{st}[(s+1)t-1]}{(s+1)^2}\right]$$

$$= \left[\frac{e^{2t}[(2+1)t-1]}{(2+1)^2}\right]$$

$$= \frac{e^{2t}(3t-1)}{3^2}$$

$$= \frac{3t.e^{2t}}{9} - \frac{e^{2t}}{9}$$

$$= \frac{te^{2t}}{3} - \frac{e^{2t}}{9}$$

Substitute the values of R_1 and R_2 in equation (1)

Laplace inverse of $\frac{1}{(s+1)(s-2)^2} = \frac{e^{-t}}{9} + \frac{te^{2t}}{3} - \frac{e^{2t}}{9}.$

CONCLUSION :

In this paper, we present a new complex inversion formula for a Laplace function. In solving some complicated problems using the Laplace transform method. Methods based on complex variable theory may come in handy for finding the inverse transform. Also it can be noted that the Laplace transform of f(t) is expressed as in integral. Similarly, the inverse Laplace transform of F(s) can be expressed as in integral which is known as inverse integral. With the remarkable advances made in various branches of science, engineering and technology, today, more than ever before, the study of partial differential equations has become essential.

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