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# Numerical Solution of Fractional Differential Equations by ADM

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**Abstract—** *Due to the role played by fractional differential equations in various sciences, many studies have been submitted to study them. In this paper, we focus on finding approximate solutions of fractional differential equations using Adomian Decomposition Method (ADM). In this paper, three numerical examples are solved. These examples showed the ease of finding approximate solutions of these equations using MADM.*

**Keywords** *:Fractional Equations, Adomian Method, Boundary Conditions*

# 1 Introduction

The fractional differential equations of various types play important roles and tools not only in mathematics but also in physics, engineering [1-3].
In recent years, Adomian decomposition method is applied to solving fractional differential equations. This method was made by George Adomian in 1980s [4-6]. The method efficiently works for initial value or boundary value problems, for linear or nonlinear, ordinary or partial differential equations[7,8].
The equation under study has been studied by researcher Yahya Hasan when $\alpha = 1$ [9], and will work on this when $0 < \alpha \leq 1$. The main objective of this paper is to solve fractional differential equations using Modified Adomian Method.

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# 2 Preliminaries

Before we start in the details of the equation solution under study using Adomian method, we will review some basic theories in the fractional calculus [10].

**Theorem 1.** Let $\alpha\in(0,1]$ and $f, g$ be $\alpha$- differentiable at a point $x > 0$. Then

(1) $D^{\alpha}f(x) = x^{1-\alpha}\frac{df}{dx}(x)$, where $f$ is differentiable.

(2) $D^{\alpha}(af + bg) = aD^{\alpha}(f) + bD^{\alpha}(g)$, for all $a, b\in R$.

(3) $D^{\alpha}(x^{p}) = px^{p-\alpha}$ for all $p \in R$.

(4) $D^{\alpha}(\lambda) = 0$, for all constant functions $f(x) = \lambda$.

(5) $D^{\alpha}(fg) = fD^{\alpha}(g) + gD^{\alpha}(f)$.

(6) $D^{\alpha}(\frac{f}{g}) = \frac{gD^{\alpha}f - gD^{\alpha}g}{g^{2}}$.

**Theorem 2.** (Conformable fractional derivative of known function)

(1) $D^{\alpha}(e^{cx}) = cx^{1-\alpha}e^{cx}$.

(2) $D^{\alpha}(sin(ax)) = ax^{1-\alpha}cos(ax), a \in R$.

(3) $D^{\alpha}(cos(ax)) = -ax^{1-\alpha}sin(ax), a \in R$.

(4) $D^{\alpha}(tan(ax)) = ax^{1-\alpha}sec^{2}(ax), a \in R$.

(5) $D^{\alpha}(cot(ax)) = -ax^{1-\alpha}csc^{2}(ax), a \in R$.

(6) $D^{\alpha}(sec(ax)) = ax^{1-\alpha}sec(ax)tan(at), a \in R$.

(7) $D^{\alpha}(csc(ax)) = -ax^{1-\alpha}csc(ax)cot(ax), a \in R$.

(8) $D^{\alpha}(\frac{1}{\alpha}x^{\alpha}) = 1$.

## 2.1 Conformable Fractional Integrals

Suppose that the function is continuous

Let $\alpha \in (0, \infty)$. Define $J^{\alpha}(x^{p}) = \frac{x^{p+\alpha}}{p+\alpha}$, for any $p \in R, \alpha \neq -p$.

If $f(x) = \sum_{n=0}^{k} a_n x^n$, then we define $J^{\alpha}(f) = \sum_{n=0}^{k} a_n J^{\alpha}(x^n) = \sum_{n=0}^{k} a_n \frac{x^{n+\alpha}}{n+\alpha}$.

Cleary, $J^{\alpha}$ is linear in its domain. Further, if $\alpha = 1$, then $J^{\alpha}$ the usual integrai.

# 3 Adomian Decomposition Method

In this work, we consider the fractional differential equation:

$$D_x^{3\alpha} y = f(x, y), \;\; 0 < \alpha \leq 1, \tag{1}$$

with boundary conditions

$$y(a) = A, D_x^{\alpha} y(0) = B, D_x^{\alpha} y(b) = C, \tag{2}$$

or

$$y(0) = A, D_x^{\alpha} y(0) = B, y(a) = C, \tag{3}$$

or

$$y(0) = A, y(a) = B, D_x^{2\alpha} y(b) = C, \tag{4}$$

where $f(x, y)$ is given function, and $A, B, C, a, b$ are constants.
We write equation (1) in the standard operator form

$$L_{\alpha} = f(x, y). \tag{5}$$

Where $L_{\alpha}$ are given as

$$L_{\alpha}(.) = x^{-\alpha} \frac{d^{\alpha}}{dx^{\alpha}} x^{2\alpha} \frac{d^{\alpha}}{dx^{\alpha}} x^{-\alpha} \frac{d^{\alpha}}{dx^{\alpha}} (.), \tag{6}$$

$$L_{\alpha}(.) = x^{-\alpha} \frac{d^{2\alpha}}{dx^{2\alpha}} x^{3\alpha} \frac{d^{\alpha}}{dx^{\alpha}} x^{-2\alpha} (.), \tag{7}$$

$$L_\alpha(.) = \frac{d^\alpha}{dx^\alpha}x^{-\alpha}\frac{d^\alpha}{dx^\alpha}x^{2\alpha}\frac{d^\alpha}{dx^\alpha}x^{-\alpha}(.). \tag{8}$$

And $L_\alpha^{-1}$ are set respectively as

$$L_\alpha^{-1}(.) = \int_a^x x^\alpha \int_b^x x^{-2\alpha} \int_0^x x^\alpha (.) dx^\alpha dx^\alpha dx^\alpha, \tag{9}$$

$$L_\alpha^{-1}(.) = x^{2\alpha} \int_a^x x^{-3\alpha} \int_0^x \int_0^x x^\alpha (.) dx^\alpha dx^\alpha dx^\alpha, \tag{10}$$

$$L_\alpha^{-1}(.) = x^\alpha \int_a^x x^{-2\alpha} \int_0^x x^\alpha \int_b^x (.) dx^\alpha dx^\alpha dx^\alpha. \tag{11}$$

Applying (9),(10),(11) on (5) and using (2),(3),(4) respectively, we get

$$y(x) = A - \frac{a^\alpha B}{\alpha} - \frac{Ca^{2\alpha}}{2\alpha b^\alpha} + \frac{Ba^{2\alpha}}{2\alpha b^\alpha} + \frac{B}{\alpha}x^\alpha + \frac{C-B}{2\alpha b^\alpha}x^{2\alpha} + L_\alpha^{-1} f(x,y), \tag{12}$$

$$y(x) = A + \frac{B}{\alpha}x^\alpha + (\frac{C}{a^{2\alpha}} - \frac{B}{\alpha a^\alpha} - \frac{A}{a^{2\alpha}})x^{2\alpha} + L_\alpha^{-1} f(x,y), \tag{13}$$

$$y(x) = A + (\frac{B}{a^\alpha} - \frac{a^\alpha C}{2\alpha^2} - \frac{A}{a^\alpha})x^\alpha + \frac{C}{2\alpha^2}x^{2\alpha} + L_\alpha^{-1} f(x,y). \tag{14}$$

The general solution of the given equation is decomposed into the sum

$$y(x) = \sum_{n=0}^{\infty} y_n(x). \tag{15}$$

The non-linear part can be decomposed into the infinite polynomial series obtained by

$$f(x,y) = \sum_{n=0}^{\infty} A_n, \tag{16}$$

where the elements $y_n(x)$ of the solution $y(x)$ will be determined repeatable. Specific algorithms were seen [4,6] to formulate Adomian polynomials. The following algorithm:

$$A_0 = G(y_0),$$

$$A_1 = y_1 G'(y_0),$$

$$A_2 = y_2 G'(y_0) + \frac{1}{2!} y_1^2 G''(y_0),$$

$$A_3 = y_3 G'(y_0) + y_1 y_2 G''(y_0) + \frac{1}{3!} y_1^3 G'''(y_0), \tag{17}$$

....

The component $y(x)$ can be given by using Adomian decomposition method as follows

$$y_0 = A - \frac{a^\alpha B}{\alpha} - \frac{Ca^{2\alpha}}{2\alpha b^\alpha} + \frac{Ba^{2\alpha}}{2\alpha b^\alpha} + \frac{B}{\alpha}x^\alpha + \frac{C-B}{2\alpha b^\alpha}x^{2\alpha},$$

$$y_{(n+1)} = L_\alpha^{-1}A_n,\ n \geq 0, \tag{18}$$

$$y_0 = A + \frac{B}{\alpha}x^\alpha + (\frac{C}{a^{2\alpha}} - \frac{B}{\alpha a^\alpha} - \frac{A}{a^{2\alpha}})x^{2\alpha},$$

$$y_{(n+1)} = L_\alpha^{-1}A_n,\ n \geq 0, \tag{19}$$

$$y_0 = A + (\frac{B}{a^\alpha} - \frac{a^\alpha C}{2\alpha^2} - \frac{A}{a^\alpha})x^\alpha + \frac{C}{2\alpha^2}x^{2\alpha},$$

$$y_{(n+1)} = L_\alpha^{-1}A_n,\ n \geq 0. \tag{20}$$

From equation (15) we can get the series solution of equation (1).

# 4 Experiment of the method

In this part, we will give three examples that show the quality of the method.
**Example 1.** Consider the following problems:

$$D_x^{3\alpha}y = -3\alpha^3 e^{x^\alpha} + \alpha^3 y, \tag{21}$$

with boundray condition

$$y(0) = 1, D_x^\alpha y(0) = 0, y(1) = 0,$$

where $y(x) = (1 - x^\alpha)e^{x^\alpha}$is the exact solution.
Eq.(21) in an operator form (7) becomes

$$L_\alpha = -3\alpha^3 e^{x^\alpha} + \alpha^3 y. \tag{22}$$

Applying eq.(10 ) on eq.(22) we get

$$y = 1 - x^{2\alpha} - 3\alpha^3 L_\alpha^{-1}e^{x^\alpha} + L_\alpha^{-1}y.$$

$$y_0 = 4 + 3x^\alpha + (3e - 7)x^{2\alpha} - 3e^{x^\alpha},$$

$$y_{n+1} = \alpha^3 L_\alpha^{-1} y_n, \quad n \geq 0.$$

Therefore

$$y_1 = 3-3\,e^{\alpha x}+3\,x^{\alpha}+\left(\frac{267}{40}+\frac{59\,e}{20}\right)\,x^{2\,\alpha}+\frac{2\,x^{3\,\alpha}}{3}+\frac{x^{4\,\alpha}}{8}+\left(-\frac{7}{60}+\frac{e}{20}\right)\,x^{5\,\alpha}+...,$$

$$y_2 = 3-3\,e^{\alpha x}+3\,x^{\alpha}+\left(-\frac{33421}{5040}+\frac{267-118\,e}{2400}+\frac{7-3\,e}{20160}+3\,e\right)\,x^{2\,\alpha}+\frac{x^{3\,\alpha}}{2}+\frac{x^{4\,\alpha}}{8}$$
$$+\frac{(-267+118\,e)\,x^{5\,\alpha}}{2400}+\frac{x^{6\,\alpha}}{180}+\frac{x^{7\,\alpha}}{1680}+\frac{(-7+3\,e)\,x^{8\,\alpha}}{20160}+...,$$

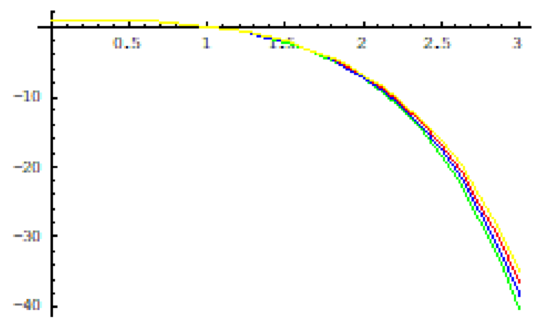
$$y(x) = y_0+y_1+y_2 =$$

$$10-9\,e^{\alpha x}+9\,x^{\alpha}+\left(-\frac{35059}{5040}+\frac{267-118\,e}{2400}+\frac{7-3\,e}{20160}+\frac{59\,e}{20}\right)\,x^{2\,\alpha}$$

$$+\frac{7\,x^{3\,\alpha}}{6}+\frac{x^{4\,\alpha}}{4}+\left(-\frac{7}{60}+\frac{e}{20}+\frac{-267+118\,e}{2400}\right)\,x^{5\,\alpha}+\frac{x^{6\,\alpha}}{180}+\frac{x^{7\,\alpha}}{1680}+\frac{(-7+3\,e)\,x^{8\,\alpha}}{20160}+....$$

Table 1: Approximate Solution of Example 1 for different values of $\alpha$ and absolute error at $\alpha = 1$

| | Approximate solutions by ADM | | | | Exact | Error |
|---|---|---|---|---|---|---|
| x | $\alpha = 1$ | $\alpha = 0.75$ | $\alpha = 0.50$ | $\alpha = 0.25$ | $\alpha = 1$ | $y_{Exact} - y_{ADM}$ |
| 0.0 | 1.000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.0000 |
| 0.1 | 0.994654 | 0.982183 | 0.938098 | 0.767997 | .994654 | 0.0000 |
| 0.3 | 0.94490 | 0.891869 | 0.782132 | 0.544819 | 0.944901 | 0.000002 |
| 0.5 | 0.824364 | 0.734709 | 0.594026 | 0.368879 | 0.824361 | 0.000003 |
| 0.7 | 0.60413 | 0.5045548 | 0.377097 | 0.212933 | 0.604126 | 0.000004 |
| 0.9 | 0.245963 | 0.191424 | 0.132517 | 0.0688526 | 0.24596 | 0.000003 |




| | | | | |
|---|---|---|---|---|
| —— Exact at($\alpha = 1$) | —— y($\alpha = 0.99$) | —— y($\alpha = 0.98$) | —— y($\alpha = 0.97$) | ——y($\alpha = 1$) |

Figure 1: Approximate solution and Exact solution at ($\alpha = 1$) of $y(x)$ of Example 1 at different values of $\alpha$

**Example 2.** Consider the following problems:

$$D_x^{3\alpha} y = \alpha^3 e^{-x^\alpha} y^2, \tag{23}$$

with boundray condition

$$y(0) = 1, D_x^\alpha y(0) = \alpha, D_x^\alpha y(1) = \alpha e,$$

where $y(x) = e^{x^\alpha}$is the exact solution.
Eq.(23) in an operator form (6) becomes

$$L_\alpha = \alpha^3 e^{-x^\alpha} y^2. \tag{24}$$

Applying eq.( 9) on eq.(24) we get

$$y = 1 + x^\alpha + 0.859141\, x^{2\,\alpha} + \alpha^3 L_\alpha^{-1} e^{-x^\alpha} y^2.$$

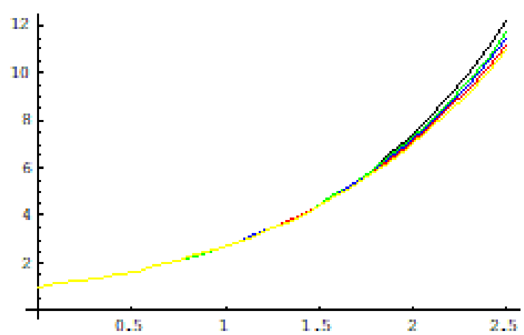
$$y_0 = 1 + x^\alpha + 0.859141\, x^{2\,\alpha},$$
$$y_{n+1} = \alpha^3 L_\alpha^{-1} A_n, \quad n \geq 0.$$

Therefore

$$y_1 = -0.381384\, x^{2\,\alpha} + 0.166667\, x^{3\,\alpha} + 0.0416667\, x^{4\,\alpha} + 0.0203047\, x^{5\,\alpha} - 0.00138889\, x^{6\,\alpha}$$
$$+0.000415788\, x^{7\,\alpha} + ...,$$
$$y_2 = 0.0266254\, x^{2\,\alpha} - 0.0127128\, x^{5\,\alpha} + 0.00277778\, x^{6\,\alpha} - 0.000907655\, x^{7\,\alpha} + ...,$$
$$y(x) = y_0 + y_1 + y_2 =$$
$$1 + x^\alpha + 0.504383\, x^{2\,\alpha} + 0.166667\, x^{3\,\alpha} + 0.0416667\, x^{4\,\alpha} + 0.00759191\, x^{5\,\alpha} + 0.00138889\, x^{6\,\alpha}$$
$$-0.000491867\, x^{7\,\alpha} + ....$$

Table 2: Approximate Solution of Example 2 for different values of $\alpha$ and absolute error at $\alpha = 1$

| | Approximate solutions by ADM | | | | Exact | Error |
|---|---|---|---|---|---|---|
| x | $\alpha = 1$ | $\alpha = 0.75$ | $\alpha = 0.50$ | $\alpha = 0.25$ | $\alpha = 1$ | $y_{Exact} - y_{ADM}$ |
| 0.0 | 1.000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.0000 |
| 0.1 | 1.10521 | 1.19476 | 1.37238 | 1.75611 | 1.10517 | 0.00004 |
| 0.3 | 1.35025 | 1.5005 | 1.73058 | 2.09826 | 1.34986 | 0.00039 |
| 0.5 | 1.64979 | 1.81379 | 2.03011 | 2.32102 | 1.64872 | 0.00107 |
| 0.7 | 2.01572 | 2.15187 | 2.3112 | 2.49881 | 2.01375 | 0.00197 |
| 0.9 | 2.46237 | 2.52223 | 2.58519 | 2.65144 | 2.4596 | 0.00277 |




| | | | | |
|---|---|---|---|---|
| —— Exact at($\alpha = 1$) | —— y($\alpha = 0.99$) | —— y($\alpha = 0.98$) | —— y($\alpha = 0.97$) | ——y($\alpha = 1$) |

Figure 2: Approximate solution and Exact solution at ($\alpha = 1$) of $y(x)$ of Example 2 at different values of $\alpha$

**Example 3.** Consider the following problems:

$$D_x^{3\alpha} y = -\alpha^3 e^{x^\alpha} y^2, \tag{25}$$

with boundray condition

$$y(0) = 1, y(1) = \frac{1}{e}, D_x^{2\alpha} y(0) = \alpha^2,$$

where $y(x) = e^{-x^\alpha}$is the exact solution.
Eq.(25) in an operator form (8) becomes

$$L_\alpha = -\alpha^3 e^{x^\alpha} y^2. \tag{26}$$

Applying eq.(11 ) on eq.(26) we get

$$y = 1 - 1.13212\, x^\alpha + 0.5\, x^{2\,\alpha} - \alpha^3 L_\alpha^{-1} e^{x^\alpha} y^2.$$

$$y_0 = 1 - 1.13212\, x^\alpha + 0.5\, x^{2.\,\alpha},$$
$$y_{n+1} = -\alpha^3 L_\alpha^{-1} A_n, \quad n \geq 0.$$

Therefore

$$y_1 = 0.123717\, x^\alpha - 0.166667\, x^{3\,\alpha} + 0.0526767\, x^{4\,\alpha} - 0.00862426\, x^{5\,\alpha} + ...,$$

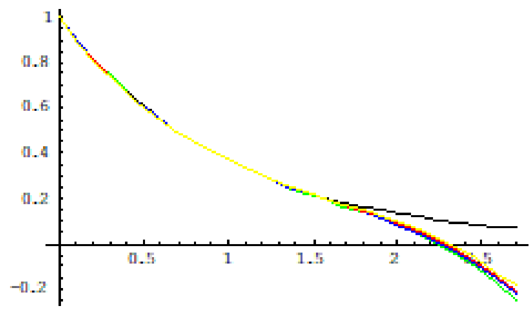
$$y_2 = 0.00765879\, x^\alpha - 0.0103098\, x^{4\,\alpha} + 0.000544853\, x^{5\,\alpha} + ...,$$

$y(x) = y_0 + y_1 + y_2 =$

$1 - 1.00074\, x^\alpha + 0.5\, x^{2\,\alpha} - 0.166667\, x^{3\,\alpha} + 0.0423669\, x^{4\,\alpha} - 0.00807941\, x^{5\,\alpha} + ....$

Table 3: Approximate Solution of Example 3 for different values of $\alpha$ and absolute error at $\alpha = 1$

| | Approximate solutions by ADM | | | | Exact | Error |
|---|---|---|---|---|---|---|
| x | $\alpha = 1$ | $\alpha = 0.75$ | $\alpha = 0.50$ | $\alpha = 0.25$ | $\alpha = 1$ | $y_{Exact} - y_{ADM}$ |
| 0.0 | 1.000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 0.0000 |
| 0.1 | 0.904763 | 0.836955 | 0.728664 | 0.569498 | 0.904837 | 0.000074 |
| 0.3 | 0.7406000 | 0.666451 | 0.577898 | 0.476584 | 0.740818 | 0.000218 |
| 0.5 | 0.60619 | 0.551388 | 0.492605 | 0.430717 | 0.606531 | 0.000341 |
| 0.7 | 0.496127 | 0.464687 | 0.432554 | 0.399894 | 0.496585 | 0.000458 |
| 0.9 | 0.405856 | 0.396151 | 0.386417 | 0.376657 | 0.40657 | 0.000714 |




| | | | | |
|---|---|---|---|---|
| —— Exact at($\alpha = 1$) | —— y($\alpha = 0.99$) | —— y($\alpha = 0.98$) | —— y($\alpha = 0.97$) | ——y($\alpha = 1$) |

Figure 3: Approximate solution and Exact solution at ($\alpha = 1$) of $y(x)$ of Example 3 at different values of $\alpha$

# Conclusion

In this paper, ADM has been successfully applied to finding the Approximate solution of nonlinear fractional differential equations. The MADM is an easy algorithm for solving this kind of equations. The method generates fairly accurate results indicating that it is an effective method.

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