International Journal of Mathematics Trends and Technology
ISSN: 2231 – 5373 /doi:10.14445/22315373/IJMTT-V66I11P502
Volume 66 Issue 11, 13-51, November, 2020

A novel extension to Fourier series for representing combined functions and extension to high precision alternate functions

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Abstract - *This is a new concept to transform a function to a series combining Cosine and Sine functions with Polynomial, Geometric, Sign and Matrix harmonic functions, unlike Fourier transformation which has only Cosine and Sine functions. The inherent nature of the functions used in synthesizing the target functions finds application in data reduction techniques without compromising fidelity and integrity of function. Here, I demonstrate the efficiency of extension in precision error, compression ratio and implementation complexity while applying it to real-world problems such as faster live streaming, prediction of stock market data, and storage of medical imaging data. I have also found methods to transform any discrete function to a continuous function or a continuous function to another function with less error-rate. This is useful in finding interpolation, smoothing or coil of rough functions, where the nature of curves is not known and is also useful in functions in a certain range such as hearing frequency, visible wave-lengths. Further, I have found smoothing transformation which is useful in finding both accurate values and in finding roots and maximum, minimum turning points of discrete points. Also one of the methods is useful in constructing a decorative curve from a given path.*

Keywords — *Function transformation, Fourier series, trigonometric functions, polynomial functions, geometric functions, matrix functions, matrix harmonic functions, prediction, data reduction, faster approximations, extrapolation, interpolation, encryption, function compression, image compression, transformation, compression, smoothness, coil, polygon, quadratic, parabolic, quartic, polynomial, decorative, decorative curve, roots, maximum, minimum, turning points*.

I. INTRODUCTION

A. Concept of Extension to Fourier Series

$$\text{If } \int_a^b T\big(G(n*x)\big)*H(m*x)*dx = 0 \text{ and } \int_a^b T\big(G(n*x)\big)*H(n*x)*dx \neq 0 \qquad (1)$$

for continuous functions, where T is a linear transformation and $\forall\ 0 \le m \le \infty,\ 0 \le n \le \infty,\ m \neq n$

$$\text{or } \sum_{x=1}^{L} T\big(G(n*x)\big)*H(m*x) = 0 \text{ and } \sum_{x=1}^{L} T\big(G(n*x)\big)*H(n*x) \neq 0 \qquad (2)$$

for discrete functions, where T is a linear transformation and $\forall\ 1 \le m \le L,\quad 1 \le n \le L,\quad m \neq n$ then,

One can transform $F(x)$ to another series having $G(n*x)$. For Fourier series, $T\big(G(x)\big) = G(x)$. When G is Cosine, H is Cosine and When G is Sine, H is Sine. Similarly I have found method for finding the appropriate T when G is either Polynomial, Geometric, Sign or Matrix functions separately here and H is Cosine, Sine or any orthogonal function. This can be used for faster approximations, data reductions, extrapolation, interpolation, compression and encryption.

B. Concept of Extension to Higher precision functions

If the discrete function's maximum and minimum and the continuous function's maximum and minimum are at same locations, then we can approximate the discrete function to the same continuous function. Same concept also can be applied not only to discrete function but also to the continuous functions to another continuous function for easy calculations. This can be used for faster approximations, data reductions, extrapolation, interpolation, compression and encryption.

C. Concept of Extension to Faster Smooth functions

$s_r(x) = x - r + |x - r|$ will behave into $s_r(x) = 0$ when $x \le r$ and $s_r(x) = 2*(x-r)$ when $x \ge r$ .Using the summation of powers of this curve, we can smooth the nearby points without looking into whole set of the curve. Thus it could be faster in making discrete or continuous function to smoother curve for easy calculations. This can not only be used for faster approximations, data reductions, extrapolation, interpolation, compression and encryption, but also be used for finding roots of the equation and maximum, minimum turning points of the curve.

II. DERIVATION OF EXTENSION TO FOURIER SERIES

Using integral by part theorem,

$$\int u*v*dx = u*\int v*dx - u'*\iint v*dx*dx + u''*\iiint v*dx*dx*dx + \cdots \tag{3}$$

If $F(x) = P(x)*\cos(x)$,then substituting $u = P(x)$ and $v = cos(x)$ in Equation (3),

$$\int F(x)*dx = P(x)*\sin(x) + P'(x)*\cos(x) - P''(x)*\sin(x) - P'''(x)*\cos(x) + \cdots \tag{4}$$

$$\int F(l*x)*dx = \frac{P(l*x)*\sin(l*x)+P'(l*x)*\cos(l*x)-P''(l*x)*\sin(l*x)+\cdots}{l} \tag{5}$$

$$= J(x)*\sin(l*x) + K(x)*\cos(l*x), \tag{6}$$

where $J(x)$ has same degree as $P(x)$ but $K(x)$ has one degree less than $P(x)$. Same can be applicable for $F(x) = P(x)*\sin(x)$ also. Since we could get separate polynomials $P(x)$ with any degree r, one can find $k_l\ \forall\ 1 \le l \le (r+1)$ such a way that if interval is between x_s and x_f where x_s is the starting point and x_f is the finishing point, then

$$\sum_{l=1}^{r+1} \frac{k_l}{l} * P(l*x_f) - P(l*x_s) = 0 \tag{7}$$

If $x_s = -x_f$, then all even degree co-efficient of $P(l*x_f) - P(l*x_s)$ will be zero and hence it is enough to find $\frac{r}{2}+1$ variables. Hence,

$$\sum_{l=1}^{\frac{r}{2}+1} \frac{k_l}{l} * P(l*x_f) - P(l*x_s) = 0,\ then \tag{8}$$

$$\int_{-\pi}^{\pi} \sum_{l=1}^{\frac{r}{2}+1} k_l * P(l*x)*\cos(n*l*x)*\cos(m*x)*dx = 0,\ \forall\ 0 \le m \le \infty,\ 0 \le n \le \infty,\ m \ne n \text{ and} \tag{9}$$

$$\int_{-\pi}^{\pi} \sum_{l=1}^{\frac{r}{2}+1} k_l * P(l*x)*\sin(n*l*x)*\sin(m*x)*dx = 0\ \forall\ 0 \le m \le \infty,\ 0 \le n \le \infty,\ m \ne n \text{ and} \tag{10}$$

if $P(x)$ is of odd degree r, we need to multiply with $l*x$ to make it as even. Hence

$$\sum_{l=1}^{\frac{r}{2}+\frac{3}{2}} \frac{k_l}{l} * P(l*x_f)*(l*x_f)\ - P(l*x_s)*(l*x_s) = 0 \sum_{l=1}^{\frac{r}{2}+\frac{3}{2}} k_l * P(l*x_f)*(x_f) - P(l*x_s)*(x_s) = 0, \tag{11}$$

$$\int_{-\pi}^{\pi} \sum_{l=1}^{\frac{r}{2}+\frac{3}{2}} k_l * P(l*x)*\cos(n*l*x)*\cos(m*x)*dx = 0\ \forall\ 0 \le m \le \infty,\ 0 \le n \le \infty,\ m \ne n \text{ and} \tag{12}$$

$$\int_{-\pi}^{\pi} \sum_{l=1}^{\frac{r}{2}+\frac{3}{2}} k_l * P(l*x)*\sin(n*l*x)*\sin(m*x)*dx = 0\ \forall\ 0 \le m \le \infty,\ 0 \le n \le \infty,\ m \ne n \text{ and} \tag{13}$$

The property shown in Equation (9),(10),(12) and (13) can be used like Fourier series to find another series. To prove this, Let us start with, if $PCOSX(x) = (a*x+b)*\cos(x)$, $PSINX(x) = (c*x+d)*\sin(x)$, with interval $-\pi$ and π since $P(x)$ is of degree 1, Let us find using the property in Equation (11)

$$\sum_{l=1}^{2} l*\ \ P(l*x_f)*(x_f)\ - P(l*x_s)*(x_s) = 0 \tag{14}$$

$k_1*(a*\pi+b)*\pi - (a*-\pi+b)*-\pi + k_2*(a*2*\pi+b)*\pi - (a*2*-\pi+b)*-\pi = 02*b*\pi*(k_1+k_2) = 0, k_1 = -k_2 if\ \ k_1 = -1,\ then\ \ k_2 = 1.$ Hence,

$$\int_{-\pi}^{\pi} \left(PCOSX(2*n*x)*2*x*\cos(m*x) - PCOSX(n*x)*\ \ x*\cos(m*x)\right)*dx = 0 \tag{15}$$

$$\text{and } \int_{-\pi}^{\pi} \left(PSINX(2*n*x)*2*x*\sin(m*x) - PSINX(n*x)*x*\sin(m*x)\right)*dx = 0 \tag{16}$$

$\forall\ 0 \le m \le \infty,\ 0 \le n \le \infty,\ m \ne n.$

A. Derivation with Polynomial Continuous Function

Hence using this property in the equation (15) and (16), if $f(x)$ is periodic between $-\frac{L}{2}$ and $\frac{L}{2}$, then we can put a series like Fourier as given below:

$$f(x) = (a_0*x+b_0) + \sum_{n=1}^{\infty} \left((a_n*x+b_n)*\cos\left(\frac{2*n*\pi*x}{L}\right) + (c_n*x+d_n)*\sin\left(\frac{2*n*\pi*x}{L}\right)\right) \tag{17}$$

where

$$a_n = \frac{8}{L^3} * \int_{-\frac{L}{2}}^{\frac{L}{2}} \left(f(2*x)*2*x*\cos\left(\frac{4*n*\pi*x}{L}\right) - f(x)*x*\cos\left(\frac{2*n*\pi*x}{L}\right)\right)*dx, \tag{18}$$

$$c_n = \frac{8}{L^3} * \int_{-\frac{L}{2}}^{\frac{L}{2}} \left(f(2*x)*2*x*\sin\left(\frac{4*n*\pi*x}{L}\right) - f(x)*x*\sin\left(\frac{2*n*\pi*x}{L}\right)\right)*dx, \tag{19}$$

$$b_n = \frac{16}{L^3} * \int_{-\frac{L}{2}}^{\frac{L}{2}} \left(G_1(2*x) - G_1(x)\right) * dx \quad (20)$$

where $G_1(x) = \int_d^x dx * \int_c^x \left(dx * f(x) * \cos\left(\frac{2*n*\pi*x}{L}\right)\right)$,

$$d_n = \frac{16}{L^3} * \int_{-\frac{L}{2}}^{\frac{L}{2}} \left(G_2(2*x) - G_2(x)\right) * dx \quad (21)$$

where $G_2(x) = \int_d^x dx * \int_c^x \left(dx * f(x) * \sin\left(\frac{2*n*\pi*x}{L}\right)\right)$,

$$a_0 = \frac{4}{L^3} * \int_{-\frac{L}{2}}^{\frac{L}{2}} \left(f(2*x)*2*x - f(x)*x\right) * dx, \quad (22)$$

$$b_0 = \frac{8}{L^3} * \int_{-\frac{L}{2}}^{\frac{L}{2}} \left(G_3(2*x) - G_3(x)\right) * dx \text{ where } G_3(x) = \int_d^x dx * \int_c^x \left(dx * f(x)\right) \quad (23)$$

In similar way, for quadratic function, it will have same $k_1 = -1$ and $k_2 = 1$.

$$f(x) = (a_0 * x^2 + b_0 * x + c_0) + \sum_{n=1}^{\infty} (a_n * x^2 + b_n * x + c_n) * \cos\left(\frac{2*n*\pi*x}{L}\right) + \sum_{n=1}^{\infty} (d_n * x^2 + e_n * x + f_n) * \sin\left(\frac{2*n*\pi*x}{L}\right) \quad (24)$$

where $a_n = \frac{8}{L^3} * \int_{-\frac{L}{2}}^{\frac{L}{2}} \left(f(2*x) * \cos\left(\frac{4*n*\pi*x}{L}\right) - f(x) * \cos\left(\frac{2*n*\pi*x}{L}\right)\right) * dx,$ (25)

$$d_n = \frac{8}{L^3} * \int_{-\frac{L}{2}}^{\frac{L}{2}} \left(f(2*x) * \sin\left(\frac{4*n*\pi*x}{L}\right) - f(x) * \sin\left(\frac{2*n*\pi*x}{L}\right)\right) * dx, \quad (26)$$

$$b_n = \frac{16}{L^3} * \int_{-\frac{L}{2}}^{\frac{L}{2}} \left(G_1(2*x) - G_1(x)\right) * dx \text{ where } G_1(x) = \int_c^x \left(dx * f(x) * \cos\left(\frac{2*n*\pi*x}{L}\right)\right), \quad (27)$$

$$e_n = \frac{16}{L^3} * \int_{-\frac{L}{2}}^{\frac{L}{2}} \left(G_2(2*x) - G_2(x)\right) * dx \text{ where } G_2(x) = \int_c^x \left(dx * f(x) * \sin\left(\frac{2*n*\pi*x}{L}\right)\right), \quad (28)$$

$$c_n = \left(-a_n * \frac{L^2}{8}\right) + \left(\frac{16}{L^3} * \int_{-\frac{L}{2}}^{\frac{L}{2}} \left(G_3(2*x) - G_3(x)\right) * dx\right) \quad (29)$$

where $G_3(x) = \int_d^x dx * \int_c^x \left(dx * f(x) * \cos\left(\frac{2*n*\pi*x}{L}\right)\right)$,

$$f_n = \left(-d_n * \frac{L^2}{8}\right) + \left(\frac{16}{L^3} * \int_{-\frac{L}{2}}^{\frac{L}{2}} \left(G_4(2*x) - G_4(x)\right) * dx\right) \quad (30)$$

where $G_4(x) = \int_d^x dx * \int_c^x \left(dx * f(x) * \sin\left(\frac{2*n*\pi*x}{L}\right)\right)$,

$$a_0 = \frac{4}{L^3} * \int_{-\frac{L}{2}}^{\frac{L}{2}} \left(f(2*x) - f(x)\right) * dx, \quad (31)$$

$$b_0 = \frac{8}{L^3} * \int_{-\frac{L}{2}}^{\frac{L}{2}} \left(G_5(2*x) - G_5(x)\right) * dx \text{ where } G_5(x) = \int_c^x \left(dx * f(x)\right), \quad (32)$$

$$c_0 = \left(-a_0 * \frac{L^2}{8}\right) + \left(\frac{8}{L^3} * \int_{-\frac{L}{2}}^{\frac{L}{2}} \left(G_6(2*x) - G_6(x)\right) * dx\right) \quad (33)$$

where $G_6(x) = \int_d^x dx * \int_c^x \left(dx * f(x)\right)$

Or we can subtract $\frac{L^2}{8}$ from x^2 co-efficient directly to get c_n, f_n independent of a_n, d_n

$$f(x) = \left(a_0 * \left(x^2 - \frac{L^2}{8}\right) + b_0 * x + c_0\right) + \sum_{n=1}^{\infty} \left(a_n * \left(x^2 - \frac{L^2}{8}\right) + b_n * x + c_n\right) * \cos\left(\frac{2*n*\pi*x}{L}\right) + \sum_{n=1}^{\infty} \left(d_n * \left(x^2 - \frac{L^2}{8}\right) + e_n * x + f_n\right) * \sin\left(\frac{2*n*\pi*x}{L}\right) \quad (34)$$

where $a_n = \frac{8}{L^3} * \int_{-\frac{L}{2}}^{\frac{L}{2}} \left(f(2*x) * \cos\left(\frac{4*n*\pi*x}{L}\right) - f(x) * \cos\left(\frac{2*n*\pi*x}{L}\right)\right) * dx,$ (35)

$$d_n = \frac{8}{L^3} * \int_{-\frac{L}{2}}^{\frac{L}{2}} \left(f(2*x) * \sin\left(\frac{4*n*\pi*x}{L}\right) - f(x) * \sin\left(\frac{2*n*\pi*x}{L}\right)\right) * dx, \quad (36)$$

$$b_n = \frac{16}{L^3} * \int_{-\frac{L}{2}}^{\frac{L}{2}} \left(G_1(2*x) - G_1(x)\right) * dx \text{ where } G_1(x) = \int_c^x \left(dx * f(x) * \cos\left(\frac{2*n*\pi*x}{L}\right)\right), \quad (37)$$

$$e_n = \frac{16}{L^3} * \int_{-\frac{L}{2}}^{\frac{L}{2}} \left(G_2(2*x) - G_2(x)\right) * dx \text{ where } G_2(x) = \int_c^x \left(dx * f(x) * \sin\left(\frac{2*n*\pi*x}{L}\right)\right), \quad (38)$$

$$c_n = \frac{16}{L^3} * \int_{-\frac{L}{2}}^{\frac{L}{2}} \left(G_3(2*x) - G_3(x)\right) * dx \quad (39)$$

where $G_3(x) = \int_d^x dx * \int_c^x \left(dx * f(x) * \cos\left(\frac{2*n*\pi*x}{L}\right)\right)$,

$$f_n = \frac{16}{L^3} * \int_{-\frac{L}{2}}^{\frac{L}{2}} \left(G_4(2*x) - G_4(x)\right) * dx \quad (40)$$

where $G_4(x) = \int_d^x dx * \int_c^x \left(dx * f(x) * \sin\left(\frac{2*n*\pi*x}{L}\right)\right)$,

$$a_0 = \frac{4}{L^3} * \int_{-\frac{L}{2}}^{\frac{L}{2}} \left(f(2*x) - f(x)\right) * dx \quad (41)$$

$$b_0 = \frac{8}{L^3} * \int_{-\frac{L}{2}}^{\frac{L}{2}} \left(G_5(2*x) - G_5(x)\right) * dx \text{ where } G_5(x) = \int_c^x \left(dx * f(x)\right), \quad (42)$$

$$c_0 = \frac{8}{L^3} * \int_{-\frac{L}{2}}^{\frac{L}{2}} \left(G_6(2*x) - G_6(x)\right) * dx \text{ where } G_6(x) = \int_d^x dx * \int_c^x \left(dx * f(x)\right) \quad (43)$$

In general, for degrees of $r = 2*s$, and $r = 2*s-1$, We can find k_m by solving following matrix:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 4 & 9 & 16 & \cdots & (s+1)^2 \\ 1 & 16 & 81 & 256 & \cdots & (s+1)^4 \\ \ldots & \ldots & \ldots & \ldots & \cdots & \ldots \\ 1 & 2^{2*s-2} & 3^{2*s-2} & 4^{2*s-2} & \cdots & (s+1)^{2*s-2} \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ \ldots \\ k_{s+1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \ldots \\ 0 \end{pmatrix} \quad (44)$$

For degree 3 and 4, $s = 2$ we get $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 4 & 9 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ On solving we get $k_1 = 5$, $k_2 = -8$, $k_3 = 3$. Similarly, for degree 5 and 6 we get $k_1 = -2$, $k_2 = 9$, $k_3 = -14$, $k_4 = 7$. To get the k_m, we can go with the concept of having $(e^x + e^{-x} - 2)^s$ in taylor series will start with x^{2*s}, if $f(x) = e^x + e^{-x}$ then the co-efficients of $e^{m*x} + e^{-m*x} =$ co efficents of $f(m*x\ )$which leads to $\sum_{m=1}^{r} \left(k_m * f(m*x)\right)$ will be 0 upto x^{2*s-2}. But this will have constant co-efficients also and to remove constants, we can write two terms as $M*(e^x + e^{-x} - 2)^{s+1} + N*(e^x + e^{-x} - 2)^s$ where M & N will remove the constants.Since $e^x + e^{-x} - 2 = \left(e^{\frac{x}{2}} - e^{-\frac{x}{2}}\right)^2$, $M*\left(e^{\frac{x}{2}} - e^{-\frac{x}{2}}\right)^{2*s+2} + N*\left(e^{\frac{x}{2}} - e^{-\frac{x}{2}}\right)^{2*s}$, using binomial function expansion, we can get the constant coefficient at the midpoint as $M*-1^{s+1}*(2*s+2)C(s+1) + N*-1^s*(2*s)C(s+1) = 0$. This leads to $M = \frac{s+1}{2}$ and $N = (2*s+1)$ using binomial function expansion, $\frac{(s+1)}{2}*\left(e^{\frac{x}{2}} - e^{-\frac{x}{2}}\right)^{2*s+2} + (2*s+1)*\left(e^{\frac{x}{2}} - e^{-\frac{x}{2}}\right)^{2*s}$. We can get $k_m = -1^{s-m+1}*(2*s+2)C(s-m+1)*\frac{m^2}{2*(s+1)} = -1^{s-m+1}*\frac{(2*s+2)!}{(s-m+1)!*(s+m+1)!}*\frac{m^2}{2*(s+1)}$, On simplifying further we get,

$$k_m = -1^{s-m+1}*\frac{m^2*(2*s+1)!}{(s-m+1)!*(s+m+1)!} \quad (45)$$

This is applicable for having degree of $2*s-1$ and $2*s$. We can extend Fourier series for any degree as given below:

For even degree of $2*r$,

$$f(x) = \frac{1}{c_{2*r}}*\left(\sum_{m=0}^{2*r} a_{m_0}*x^m\right) + \quad (46)$$

$$\frac{2}{c_{2*r}}*\left(\sum_{n=1}^{\infty}\left(\left(\sum_{m=0}^{2*r} a_{m_n}*x^m\right)*\cos\left(\frac{2*n*\pi*x}{L}\right)\right)\right) +$$

$$\frac{2}{c_{2*r}}*\left(\sum_{n=1}^{\infty}\left(\left(\sum_{m=0}^{2*r} b_{m_n}*x^m\right)*\sin\left(\frac{2*n*\pi*x}{L}\right)\right)\right) \quad \text{where}$$

$$a_{2*r_n} = \int_{-\frac{L}{2}}^{\frac{L}{2}} dx * \left(\sum_{m=1}^{r+1} k_m * f(m*x) * \cos\left(\frac{2*n*\pi*m*x}{L}\right)\right), \quad (47)$$

$$b_{2*r_n} = \int_{-\frac{L}{2}}^{\frac{L}{2}} dx * \left(\sum_{m=1}^{r+1} k_m * f(m*x) * \sin\left(\frac{2*n*\pi*m*x}{L}\right)\right), \tag{48}$$

$$k_m = -1^{r-m+1} * \frac{m^2*(2*r+1)!}{(r-m+1)!*(r+m+1)!}\ , \tag{49}$$

$$c_{2*r} = \int_{-\frac{L}{2}}^{\frac{L}{2}} dx * (\sum_{m=1}^{r+1} k_m * (m*x)^{2*r}) = \frac{L^{2*r+1}}{2^{2*r}*(2*r+1)} * \sum_{m=1}^{r+1} k_m * m^{2*r+1} \tag{50}$$

For second co-efficient, we need to make double integral

$$a_{2*r-1_n} = 2*r*\int_{-\frac{L}{2}}^{\frac{L}{2}} dx * \left(\sum_{m=1}^{r+1} k_m * G_1(m*x)\right) \tag{51}$$

where $G_1(x) = \int_c^x \left(dx * f(x) * \cos\left(\frac{2*n*\pi*x}{L}\right)\right)$ and

$$b_{2*r-1_n} = 2*r*\int_{-\frac{L}{2}}^{\frac{L}{2}} dx * \left(\sum_{m=1}^{r+1} k_m * G_2(m*x)\right) \tag{52}$$

where $G_2(x) = \int_c^x \left(dx * f(x) * \sin\left(\frac{2*n*\pi*x}{L}\right)\right)$,

For third co-efficient, we need to make triple integral and we need to subtract the first term as it will be increased by 2 degree and hence it will not be zero.

$$a_{2*r-2_n} = -a_{2*r_n} * cc * \frac{(2*r)*(2*r-1)}{(2*r+2)*(2*r+1)} + \left((2*r*)*(2*r-1)*\int_{-\frac{L}{2}}^{\frac{L}{2}} dx * \left(\sum_{m=1}^{r+1} k_m * G_3(m*x)\right)\right) \tag{53}$$

where $G_3(x) = \int_d^x dx * \int_c^x \left(dx * f(x) * \cos\left(\frac{2*n*\pi*x}{L}\right)\right)$ with another constant cc and

$$b_{2*r-2_n} = -b_{2*r_n} * cc * \frac{(2*r)*(2*r-1)}{(2*r+2)*(2*r+1)} + \left((2*r*)*(2*r-1)*\int_{-\frac{L}{2}}^{\frac{L}{2}} dx * \left(\sum_{m=1}^{r+1} k_m * G_4(m*x)\right)\right) \tag{54}$$

where $G_4(x) = \int_d^x dx * \int_c^x \left(dx * f(x) * \sin\left(\frac{2*n*\pi*x}{L}\right)\right)$ with same constant cc and

$$cc = \int_{-\frac{L}{2}}^{\frac{L}{2}} dx * (\sum_{m=1}^{r+1} k_m * x^2 * (m*x)^{2*r}) = \frac{L^{2*r+3}}{2^{2*r+2}*(2*r+3)} * \sum_{m=1}^{r+1} k_m * m^{2*r+3} \tag{55}$$

And so on. Or we can have the polynomial

$$iP_{r_0}(x) = x^{2*r} - \sum_{n=1}^{r} l_n * x^{2*r-2*n} * \left(\frac{L}{2}\right)^n \text{ and } \sum_{m=1}^{r+1} k_m * iP_{r_0}(m*x) \text{ is the polynomial} \tag{56}$$

such a way that triple integral, fifth integral, seventh integral up to $2*r+1$ integral are zero. For example, to find l_1 we need to take triple integral, to find l_2 we need to take fifth integral and so on. To find l_k we can use the following recursion,

$$l_k = \frac{1}{(M(k,k))} * \left((M(k,0)) - \sum_{n=1}^{k-1} l_n * (M(k,n))\right) \text{ where } (M(k,n)) = \frac{(2*(r-n))!}{(2*(r+k-n)!)} * \sum_{m=1}^{r+1} k_m * m^{2*(r+k-n)+1} \tag{57}$$

Since we have arrived k_m by $\frac{(r+1)}{2} * \left(e^{\frac{x}{2}} - e^{-\frac{x}{2}}\right)^{2*r+2} + (2*r+1) * \left(e^{\frac{x}{2}} - e^{-\frac{x}{2}}\right)^{2*r} = \frac{(r+1)}{2} * (e^x - e^{-x} - 2)^{r+1} + (2*r+1)*(e^x - e^{-x} - 2)^r$ on expanding, using tailor series, the co-efficient of $x^{2*r} = 2*r+1$ and $x^{2*r+2} = \frac{(r+2)*(2*r+3)}{12}$ and so on, then equating coefficient of x^{2*r+2} with $iP_{r_0}(x) = x^{2*r} - \sum_{n=1}^{r} l_n * x^{2*r-2*n} * \left(\frac{L}{2}\right)^n$ one can get

$$l_1 = \frac{r*(r+2)*(2*r-1)}{6} \tag{58}$$

and $l_t * \frac{(2*r+2*t)!}{(2*r)!}$ will be another polynomial with r. and equating coefficient of x^{2*r}

$$c_{2*r} = \int_{-\frac{L}{2}}^{\frac{L}{2}} dx * \left(\sum_{m=1}^{r+1} k_m * iP_{r_0}(m*x)^{2*r}\right) = \int_{-\frac{L}{2}}^{\frac{L}{2}} dx * (\sum_{m=1}^{r+1} k_m * (m*x)^{2*r}) \tag{59}$$

$$= \frac{L^{2*r+1}}{2^{2*r}*(2*r+1)} * \sum_{m=1}^{r+1} k_m * m^{2*r+1} = (2*r)! * \left(\frac{L}{2}\right)^{2*r+1},$$

$$l_k = \frac{2}{2*r+1} * \left((M(k,0)) - \sum_{n=1}^{k-1} l_n * (M(k,n))\right) \text{ where } (M(k,n)) = \frac{(2*(r-n))!}{(2*(r+k-n)!)} * \sum_{m=1}^{r+1} k_m * m^{2*(r+k-n)+1} \tag{60}$$

Hence For even degree of $2*r$,

$$f(x) = \frac{1}{c_{2*r}} * \left(\sum_{m=0}^{2*r} a_{m_0} * iP_{r_m}(x)\right) + \tag{61}$$

$$\frac{2}{c_{2*r}} * \left(\sum_{n=1}^{\infty} \left(\left(\sum_{m=0}^{2*r} a_{m_n} * iP_{r_m}(x)\right) * \cos\left(\frac{2*n*\pi*x}{L}\right)\right)\right) +$$

$$\frac{2}{c_{2*r}} * \left(\sum_{n=1}^{\infty} \left(\left(\sum_{m=0}^{2*r} b_{m_n} * iP_{r_m}(x)\right) * \sin\left(\frac{2*n*\pi*x}{L}\right)\right)\right) \text{ where}$$

$$iP_{r_0}(x) = x^{2*r} - \sum_{n=1}^{r} l_n * x^{2*r-2*n} * \left(\frac{L}{2}\right)^n, iP_{r_{m+1}}(x) = iP_{r_m}{}'(x), \quad (62)$$

$$l_k = \frac{1}{(M(k,k))} * \left((M(k,0)) - \sum_{n=1}^{k-1} l_n * (M(k,n))\right) \quad (63)$$

$$= \frac{2}{2*r+1} * \left((M(k,0)) - \sum_{n=1}^{k-1} l_n * (M(k,n))\right)$$

$$\text{where } (M(k,n)) = \frac{(2*(r-n))!}{(2*(r+k-n)!)} * \sum_{m=1}^{r+1} k_m * m^{2*(r+k-n)+1}$$

$$k_m = -1^{r-m+1} * \frac{m^2*(2*r+1)!}{(r-m+1)!*(r+m+1)!}, \quad (64)$$

$$c_{2*r} = \int_{-\frac{L}{2}}^{\frac{L}{2}} dx * \left(\sum_{m=1}^{r+1} k_m * iP_{r_0}(m*x)^{2*r}\right) = \int_{-\frac{L}{2}}^{\frac{L}{2}} dx * \left(\sum_{m=1}^{r+1} k_m * (m*x)^{2*r}\right) \quad (65)$$

$$= \frac{L^{2*r+1}}{2^{2*r}*(2*r+1)} * \sum_{m=1}^{r+1} k_m * m^{2*r+1} = (2*r)! * \left(\frac{L}{2}\right)^{2*r+1},$$

$$a_{m_n} = \int_{-\frac{L}{2}}^{\frac{L}{2}} dx * \left(\sum_{m=1}^{r+1} k_m * cf_m(m*x)\right) \quad (66)$$

$$cf_0(x) = f(x) * \cos\left(\frac{2*n*\pi*x}{L}\right), cf_{m+1}(x) = \int_c^x cf_m(x) * dx \quad (67)$$

$$b_{m_n} = \int_{-\frac{L}{2}}^{\frac{L}{2}} dx * \left(\sum_{m=1}^{r+1} k_m * sf_m(m*x)\right) \quad (68)$$

$$sf_0(x) = f(x) * \sin\left(\frac{2*n*\pi*x}{L}\right), sf_{m+1}(x) = \int_c^x sf_m(x) * dx \quad (69)$$

Hence For odd degree of $2*r-1$,

$$f(x) = \frac{1}{c_{2*r}} * \left(\sum_{m=1}^{2*r} a_{m_0} * iP_{r_m}(x)\right) + \quad (70)$$

$$\frac{2}{c_{2*r}} * \left(\sum_{n=1}^{\infty} \left(\left(\sum_{m=1}^{2*r} a_{m_n} * iP_{r_m}(x)\right) * \cos\left(\frac{2*n*\pi*x}{L}\right)\right)\right) +$$

$$\frac{2}{c_{2*r}} * \left(\sum_{n=1}^{\infty} \left(\left(\sum_{m=1}^{2*r} b_{m_n} * iP_{r_m}(x)\right) * \sin\left(\frac{2*n*\pi*x}{L}\right)\right)\right) \text{ where}$$

$$iP_{r_0}(x) = x^{2*r} - \sum_{n=1}^{r} l_n * x^{2*r-2*n} * \left(\frac{L}{2}\right)^n, iP_{r_{m+1}}(x) = iP_{r_m}{}'(x), \quad (71)$$

$$l_k = \frac{1}{(M(k,k))} * \left((M(k,0)) - \sum_{n=1}^{k-1} l_n * (M(k,n))\right) \quad (72)$$

$$= \frac{2}{2*r+1} * \left((M(k,0)) - \sum_{n=1}^{k-1} l_n * (M(k,n))\right)$$

$$\text{where } (M(k,n)) = \frac{(2*(r-n))!}{(2*(r+k-n)!)} * \sum_{m=1}^{r+1} k_m * m^{2*(r+k-n)+1}$$

$$k_m = -1^{r-m+1} * \frac{m^2*(2*r+1)!}{(r-m+1)!*(r+m+1)!}, \quad (73)$$

$$c_{2*r} = \int_{-\frac{L}{2}}^{\frac{L}{2}} dx * \left(\sum_{m=1}^{r+1} k_m * iP_{r_0}(m*x)^{2*r}\right) = \int_{-\frac{L}{2}}^{\frac{L}{2}} dx * \left(\sum_{m=1}^{r+1} k_m * (m*x)^{2*r}\right) \quad (74)$$

$$= \frac{L^{2*r+1}}{2^{2*r}*(2*r+1)} * \sum_{m=1}^{r+1} k_m * m^{2*r+1} = (2*r)! * \left(\frac{L}{2}\right)^{2*r+1},$$

$$a_{m_n} = \int_{-\frac{L}{2}}^{\frac{L}{2}} dx * \left(\sum_{m=1}^{r+1} k_m * cf_m(m*x)\right) when\ \ m > 1 \quad (75)$$

$$cf_0(x) = f(x) * \cos\left(\frac{2*n*\pi*x}{L}\right), cf_{m+1}(x) = \int_c^x cf_m(x) * dx \quad (76)$$

$$b_{m_n} = \int_{-\frac{L}{2}}^{\frac{L}{2}} dx * \left(\sum_{m=1}^{r+1} k_m * sf_m(m*x)\right) when\ \ m > 1 \quad (77)$$

$$sf_0(x) = f(x) * \sin\left(\frac{2*n*\pi*x}{L}\right), sf_{m+1}(x) = \int_c^x sf_m(x) * dx \quad (78)$$

$$a_{1_n} = \int_{-\frac{L}{2}}^{\frac{L}{2}} dx * \left(\sum_{m=1}^{r+1} k_m * cxf_1(m*x)\right) \text{ multiplying by } x \text{ for first term} \quad (79)$$

$$cxf_1(x) = f(x) * 2*r*x * \cos\left(\frac{2*n*\pi*x}{L}\right) \quad (80)$$

$$b_{1_n} = \int_{-\frac{L}{2}}^{\frac{L}{2}} dx * \left(\sum_{m=1}^{r+1} k_m * sxf_1(m*x)\right) \text{ multiplying by } x \text{ for first term} \tag{81}$$

$$sxf_1(x) = f(x) * 2 * r * x * \sin\left(\frac{2*n*\pi*x}{L}\right) \tag{82}$$

This is not only applicable for cos or sin functions but for any Orthogonal periodic functions, i.e.) $\forall 1 \le k \le r+1$,

$$\int_{-\frac{L}{2}}^{\frac{L}{2}} dx * \left(V_n(k*x) * V_m(k*x)\right) = 0 \text{ when } n \ne m \text{ and, } \int_{-\frac{L}{2}}^{\frac{L}{2}} dx * \left(V_n(k*x) * V_n(k*x)\right) = 1$$

B. Derivation with Polynomial Discrete Function

For discrete functions also, we can use similar way but instead of $\int f$ we need to consider $\sum f$. Let $F(x) = P(x) * \cos\left(\frac{2*\pi*(x-1)}{L} + k*\pi\right) + Q(x) * \sin\left(\frac{2*\pi*(x-1)}{L} + k*\pi\right)$, and if $P(x)$ and $Q(x)$ is of polynomial with degree r and $F(x) \ \exists \ \forall \ 1 \le x \le ((r+1)*L)$ then

$$\text{Let } G(x) = \sum_{r=1}^{x} F(r) \tag{83}$$

Then we need to find

$$\sum_{m=1}^{r+1} k_m * G(m*L) = 0 \tag{84}$$

Since summation will be existing for all polynomial degree, we cannot reduce further as we did in integral of k_m co-efficients. For degree r we get following matrix to be solved

$$\begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & r+1 \\ 1 & 4 & 9 & 16 & \cdots & (r+1)^2 \\ 1 & 8 & 27 & 64 & \cdots & (r+1)^4 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 2^r & 3^r & 4^r & \cdots & (r+1)^r \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ \cdots \\ k_{r+1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \cdots \\ 0 \end{pmatrix} \tag{85}$$

$(e^x - 1)^r$ in taylor series will start with x^r and hence if $f(x) = e^x$ then the co-efficients of e^{m*x} =co efficents of $f(m*x)$ leads to $\sum_{m=1}^{r} \left(k_m * f(m*x)\right)$ will be 0 upto x^{r-1}, terms, since summation will not have a constant term, we can ignore the constant term and hence k_m obeys the binomial distribution and we get,

$$k_m = \ -1^{r-m+1} * \left((r+1)C(m)\right) = \ -1^{r-m+1} * \frac{(r+1)!}{} m! * (r-m+1)! \tag{86}$$

For example, when $r=1$, $k_1 = -2$, $k_2 = 1$ and when $r=2$, $k_1 = 3$, $k_2 = -3$, $k_3 = 1$ and so on. For degree 1, Let $F(x) \ \exists \ \forall \ 1 \le x \le (2*L)$, then

$$F(x) = (a_0 * x + b_0) + \sum_{n=1}^{v} \left(A(n) * (a_n * x + b_n) * \cos\left(\frac{n*\pi*(2*x-2+p)}{(p+1)*L} + k*\pi\right)\right) + \tag{87}$$

$$\sum_{n=1}^{u} \left(2 * (c_n * x + d_n) * \sin\left(\frac{n*\pi*(2*x-2+p)}{(p+1)*L} + k*\pi\right)\right)$$

where if $k \ne \frac{2*w-1}{2}$ and $\forall \ w \subset integer$ then, $p = 0$,

if L is odd, then u and v are $\frac{L-1}{2}$, elseif L is even, then $u = \frac{L}{2}$ and $v = \frac{L}{2} - 1$

if $k = \frac{2*w-1}{2}$ and $\forall \ w \subset integer$ then, $p = 1$, $u = 0$, and $v = L - 1$

$A(n) = \frac{1}{\cos^2(k*\pi)}$ when $n = \frac{L}{2}$ and $k \ne \frac{2*w-1}{2}$,

$A(n) = 2 \ \forall$ when $n \ne \frac{L}{2}$ or $k = \frac{2*w-1}{2}$,

$$a_0 = \frac{\left(G_0(2*L) - 2*G_0(L)\right)}{L^2} , \tag{88}$$

$$G_0(x) = \sum_{r=1}^{x} \left(F(r)\right) , \tag{89}$$

$$a_n = \frac{\left(G_1(2*L) - 2*G_1(L)\right)}{L^2} , \tag{90}$$

$$G_1(x) = \sum_{r=1}^{x} \left(F(r) * \cos\left(\frac{n*\pi*(2*r-2+p)}{(p+1)*L} + k*\pi\right)\right) , \tag{91}$$

$$c_n = \frac{\left(G_2(2*L) - 2*G_2(L)\right)}{L^2} , \tag{92}$$

$$G_2(x) = \sum_{r=1}^{x} \left(F(r) * \sin\left(\frac{n*\pi*(2*r-2+p)}{(p+1)*L} + k*\pi\right)\right) , \tag{93}$$

$$b_0 = -a_0 * (L+1) + \frac{\left(G_3(2*L) - 2*G_3(L)\right)}{L^2} \tag{94}$$

$$G_3(x) = \sum_{m=1}^{x} \left(\sum_{r=1}^{m} F(r)\right) \ or \ , \tag{95}$$

$$G_3(x) = \sum_{r=1}^{x} \left((x-r+1) * F(r)\right) \ , \tag{96}$$

$$b_n = -a_n * (L+1) + \frac{\left(G_4(2*L) - 2*G_4(L)\right)}{L^2} \tag{97}$$

$$G_4(x) = \sum_{m=1}^{x} \left(\sum_{r=1}^{m} F(r) * \cos\left(\frac{n*\pi*(2*r-2+p)}{(p+1)*L} + k*\pi\right)\right) \ or \ , \tag{98}$$

$$G_4(x) = \sum_{r=1}^{x} \left((x-r+1) * F(r) * \cos\left(\frac{n*\pi*(2*r-2+p)}{(p+1)*L} + k*\pi\right)\right) \ , \tag{99}$$

$$d_n = -c_n * (L+1) + \frac{\left(G_5(2*L) - 2*G_5(L)\right)}{L^2} \ , \tag{100}$$

$$G_5(x) = \sum_{m=1}^{x} \left(\sum_{r=1}^{m} F(r) * \sin\left(\frac{n*\pi*(2*r-2+p)}{(p+1)*L} + k*\pi\right)\right) \ or \ , \tag{101}$$

$$G_5(x) = \sum_{r=1}^{x} \left((x-r+1) * F(r) * \sin\left(\frac{n*\pi*(2*r-2+p)}{(p+1)*L} + k*\pi\right)\right) \tag{102}$$

We can also write the same with difference of $-a_n * (L+1)$ and $-c_n * (L+1)$ to the x coefficient as written below

$$F(x) = (a_0 * (x-L-1) + b_0) + \tag{103}$$

$$\sum_{n=1}^{v} A(n) * \left((a_n * (x-L-1) + b_n) * \cos\left(\frac{n*\pi*(2*x-2+p)}{(p+1)*L} + k*\pi\right)\right) +$$

$$\sum_{n=1}^{u} \left(2 * (c_n * (x-L-1) + d_n) * \sin\left(\frac{n*\pi*(2*x-2+p)}{(p+1)*L} + k*\pi\right)\right)$$

where if $k \neq \frac{2*w-1}{2}$ and $\forall\ w \subset integer$ then, $p = 0$,

if L is odd, then u and v are $\frac{L-1}{2}$, elseif L is even, then $u = \frac{L}{2}$ and $v = \frac{L}{2} - 1$

if $k = \frac{2*w-1}{2}$ and $\forall\ w \subset integer$ then, $p = 1$, $u = 0$, and $v = L - 1$

$A(n) = \frac{1}{\cos^2(k*\pi)}$ when $n = \frac{L}{2}$ and $k \neq \frac{2*w-1}{2}$,

$A(n) = 2\ \forall$ when $n \neq \frac{L}{2}$ or $k = \frac{2*w-1}{2}$,

$$a_0 = \frac{\left(G_0(2*L) - 2*G_0(L)\right)}{L^2} \ , \tag{104}$$

$$G_0(x) = \sum_{r=1}^{x} \left(F(r)\right) \ , \tag{105}$$

$$a_n = \frac{\left(G_1(2*L) - 2*G_1(L)\right)}{L^2} \ , \tag{106}$$

$$G_1(x) = \sum_{r=1}^{x} \left(F(r) * \cos\left(\frac{n*\pi*(2*r-2+p)}{(p+1)*L} + k*\pi\right)\right) \ , \tag{107}$$

$$c_n = \frac{\left(G_2(2*L) - 2*G_2(L)\right)}{L^2} \ , \tag{108}$$

$$G_2(x) = \sum_{r=1}^{x} \left(F(r) * \sin\left(\frac{n*\pi*(2*r-2+p)}{(p+1)*L} + k*\pi\right)\right) \ , \tag{109}$$

$$b_0 = \frac{\left(G_3(2*L) - 2*G_3(L)\right)}{L^2} \tag{110}$$

$$G_3(x) = \sum_{m=1}^{x} \left(\sum_{r=1}^{m} F(r)\right) \ or \ , \tag{111}$$

$$G_3(x) = \sum_{r=1}^{x} \left((x-r+1) * F(r)\right) \ , \tag{112}$$

$$b_n = \frac{\left(G_4(2*L) - 2*G_4(L)\right)}{L^2} \tag{113}$$

$$G_4(x) = \sum_{m=1}^{x} \left(\sum_{r=1}^{m} F(r) * \cos\left(\frac{n*\pi*(2*r-2+p)}{(p+1)*L} + k*\pi\right)\right) \ or \ , \tag{114}$$

$$G_4(x) = \sum_{r=1}^{x} \left((x-r+1) * F(r) * \cos\left(\frac{n*\pi*(2*r-2+p)}{(p+1)*L} + k*\pi\right)\right) \ , \tag{115}$$

$$d_n = \frac{\left(G_5(2*L) - 2*G_5(L)\right)}{L^2} \ , \tag{116}$$

$$G_5(x) = \sum_{m=1}^{x} \left(\sum_{r=1}^{m} F(r) * \sin\left(\frac{n*\pi*(2*r-2+p)}{(p+1)*L} + k*\pi\right)\right) \ or \ , \tag{117}$$

$$G_5(x) = \sum_{r=1}^{x} \left((x-r+1) * F(r) * \sin\left(\frac{n*\pi*(2*r-2+p)}{(p+1)*L} + k*\pi\right)\right) \tag{118}$$

Also please note that, one also can simplify $G(2*L)-2*G(L)$, using following identity.

$$if\ \ H(x)=\sum_{r=1}^{x} h(r),\ then\ \ H(2*L)-2*H(L)=\sum_{r=L+1}^{2*L} h(r)-\sum_{r=1}^{L} h(r) \tag{119}$$

Similarly, for quadratic series, Let $F(x)\ \exists\ \forall\ 1\le x\le(3*L)$, then

$$F(x)=\left(a_0*\left(p_1(x)\right)+b_0*\left(p_2(x)\right)+c_0\right)+ \tag{120}$$

$$\sum_{n=1}^{v} A(n)*\left(\left(a_n*\left(p_1(x)\right)+b_n*\left(p_2(x)\right)+c_n\right)*\cos\left(\frac{n*\pi*(2*x-2+p)}{(p+1)*L}+k*\pi\right)\right)+$$

$$\sum_{n=1}^{u}\left(2*\left(d_n*\left(p_1(x)\right)+e_n*\left(p_2(x)\right)+f_n\right)*\sin\left(\frac{n*\pi*(2*x-2+p)}{(p+1)*L}+k*\pi\right)\right)$$

where if $k\ \neq\ \frac{2*w-1}{2}$ and $\forall\ w\ \subset\ integer$ then, $p=0$,

if L is odd, then u and v are $\frac{L-1}{2}$, elseif L is even, then $u=\frac{L}{2}$ and $v=\frac{L}{2}-1$

if $k\ =\ \frac{2*w-1}{2}$ and $\forall\ w\ \subset\ integer$ then, $p=1$, $u=0$, and $v=L-1$

$p_1(x)=x^2+(2-3*L)*x+(2*L^2+3*L+3)$, $p_2(x)=2*x-\left(3*(L+1)\right)$,

$A(n)=\frac{1}{\cos^2(k*\pi)}$ when $n=\frac{L}{2}$ and $k\ \neq\ \frac{2*w-1}{2}$,

$A(n)=2\ \forall$ when $n\ \neq\frac{L}{2}\ or\ k\ =\ \frac{2*w-1}{2}$,

$$a_0=\frac{\left(G_0(3*L)-3*G_0(2*L)+3*G_0(L)\right)}{2*L^3}\ , \tag{121}$$

$$G_0(x)=\sum_{r=1}^{x}\left(F(r)\right)\ , \tag{122}$$

$$a_n=\frac{\left(G_1(3*L)-3*G_1(2*L)+3*G_1(L)\right)}{2*L^3}\ , \tag{123}$$

$$G_1(x)=\sum_{r=1}^{x}\left(F(r)*\cos\left(\frac{n*\pi*(2*r-2+p)}{(p+1)*L}+k*\pi\right)\right)\ , \tag{124}$$

$$d_n=\frac{\left(G_2(3*L)-3*G_2(2*L)+3*G_2(L)\right)}{2*L^3}\ , \tag{125}$$

$$G_2(x)=\sum_{r=1}^{x}\left(F(r)*\sin\left(\frac{n*\pi*(2*r-2+p)}{(p+1)*L}+k*\pi\right)\right)\ , \tag{126}$$

$$b_0=\frac{\left(G_3(3*L)-3*G_3(2*L)+3*G_3(L)\right)}{2*L^3}\ , \tag{127}$$

$$G_3(x)=\sum_{m=1}^{x}\left(\sum_{r=1}^{m}\left(F(r)\right)\right)\ \ or\ \ , \tag{128}$$

$$G_3(x)=\sum_{r=1}^{x}\left((x-r+1)*F(r)\right)\ , \tag{129}$$

$$b_n=\frac{\left(G_4(3*L)-3*G_4(2*L)+3*G_4(L)\right)}{2*L^3}\ , \tag{130}$$

$$G_4(x)=\sum_{m=1}^{x}\left(\sum_{r=1}^{m}\left(F(r)*\cos\left(\frac{n*\pi*(2*r-2+p)}{(p+1)*L}+k*\pi\right)\right)\right)\ \ or\ \ , \tag{131}$$

$$G_4(x)=\sum_{r=1}^{x}\left((x-r+1)*F(r)*\cos\left(\frac{n*\pi*(2*r-2+p)}{(p+1)*L}+k*\pi\right)\right)\ , \tag{132}$$

$$e_n=\frac{\left(G_5(3*L)-3*G_5(2*L)+3*G_5(L)\right)}{2*L^3}\ , \tag{133}$$

$$G_5(x)=\sum_{m=1}^{x}\left(\sum_{r=1}^{m}\left(F(r)*\sin\left(\frac{n*\pi*(2*r-2+p)}{(p+1)*L}+k*\pi\right)\right)\right)\ \ or\ \ , \tag{134}$$

$$G_5(x)=\sum_{r=1}^{x}\left((x-r+1)*F(r)*\sin\left(\frac{n*\pi*(2*r-2+p)}{(p+1)*L}+k*\pi\right)\right), \tag{135}$$

$$c_0=\frac{\left(G_6(3*L)-3*G_6(2*L)+3*G_6(L)\right)}{2*L^3}\ , \tag{136}$$

$$G_6(x)=\sum_{p=1}^{x}\left(\sum_{m=1}^{p}\left(\sum_{r=1}^{m}\left(F(r)\right)\right)\right)\ \ or\ \ , \tag{137}$$

$$G_6(x)=\sum_{r=1}^{x}\left(\frac{(x-r+1)*(x-r+2)}{2}*F(r)\right), \tag{138}$$

$$c_n=\frac{\left(G_7(3*L)-3*G_7(2*L)+3*G_7(L)\right)}{2*L^3}\ , \tag{139}$$

$$G_7(x) = \sum_{p=1}^{x}\left(\sum_{m=1}^{p}\left(\sum_{r=1}^{m}\left(F(r)*\cos\left(\frac{n*\pi*(2*r-2+p)}{(p+1)*L}+k*\pi\right)\right)\right)\right) \ or \ , \quad (140)$$

$$G_7(x) = \sum_{r=1}^{x}\left(\frac{(x-r+1)*(x-r+2)}{2}*F(r)*\cos\left(\frac{n*\pi*(2*r-2+p)}{(p+1)*L}+k*\pi\right)\right), \quad (141)$$

$$f_n = \frac{\left(G_8(3*L)-3*G_8(2*L)+3*G_8(L)\right)}{2*L^3} \ , \quad (142)$$

$$G_8(x) = \sum_{p=1}^{x}\left(\sum_{m=1}^{p}\left(\sum_{r=1}^{m}\left(F(r)*\sin\left(\frac{n*\pi*(2*r-2+p)}{(p+1)*L}+k*\pi\right)\right)\right)\right) \ or \ , \quad (143)$$

$$G_8(x) = \sum_{r=1}^{x}\left(\frac{(x-r+1)*(x-r+2)}{2}*F(r)*\sin\left(\frac{n*\pi*(2*r-2+p)}{(p+1)*L}+k*\pi\right)\right) \quad (144)$$

Also please note that, one also can simplify $G(3*L)-3*G(2*L)+3*G(L)$, using following identity.

$$if \ H(x) = \sum_{r=1}^{x} h(r), \ then \ H(3*L)-3*H(2*L)+3*H(L) = \sum_{r=L+1}^{3*L} h(r) - 2*\sum_{r=2*L+1}^{2*L} h(r) + \sum_{r=1}^{L} h(r) \quad (145)$$

Like the same way that was done in integral to find for any degree r, we can find for discrete functions also if $F(x) \ \exists \ \forall \ 1 \le x \le ((r+1)*L)$ as following:

$$F(x) = \frac{1}{c_r}*\left(\left(\left(\sum_{m=0}^{r} a_{m_0}*sP_{r_m}(x)\right)\right)\right) + \quad (146)$$

$$\frac{1}{c_r}*\left(\sum_{n=1}^{u}\left(\left(\sum_{m=0}^{r} A(n)*a_{m_n}*sP_{r_m}(x)\right)*\cos\left(\frac{n*\pi*(2*x-2+p)}{(p+1)*L}+k*\pi\right)\right)\right) +$$

$$\frac{1}{c_r}*\left(\sum_{n=1}^{v}\left(\left(\sum_{m=0}^{r} 2*b_{m_n}*sP_{r_m}(x)\right)*\sin\left(\frac{n*\pi*(2*x-2+p)}{(p+1)*L}+k*\pi\right)\right)\right)$$

where if $k \neq \frac{2*w-1}{2}$ and $\forall \ w \subset integer$ then, $p=0$,

if L is odd, then u and v are $\frac{L-1}{2}$, elseif L is even, then $u=\frac{L}{2}$ and $v=\frac{L}{2}-1$

if $k = \frac{2*w-1}{2}$ and $\forall \ w \subset integer$ then, $p=1$, $u=0$, and $v=L-1$

$A(n) = \frac{1}{\cos^2(k*\pi)}$ when $n=\frac{L}{2}$ and $k \neq \frac{2*w-1}{2}$,

$A(n) = 2 \ \forall$ when $n \neq \frac{L}{2}$ or $k = \frac{2*w-1}{2}$,

$$sP_{r_0}(x) = \frac{(x+r-1)!}{(x-1)!(r)!} - \sum_{n=1}^{r} l_n*\frac{(x+r-n-1)!}{(x-1)!(r-n)!}, sP_{r_{m+1}}(x) = sP_{r_m}(x+1) - sP_{r_m}(x), \quad (147)$$

and $\sum_{m=1}^{r+1} k_m*sP_{r_0}(m*x)$ is the polynomial such a way that its first cumulative sum, second cumulative of first cumulative sum and sum up to $r-1$ cumulative sum are $zero$.

Also please note that if $S(x,n) = \frac{(x+r-n-1)!}{(x-1)!(r-n)!}$, $then$

$$S(x+1,n) - S(x,n) = \frac{(x+r-n-2)!}{(x-1)!(r-n-1)!} = S(x,n-1),$$

$$and \sum_{m=1}^{x} S(m,n) = \frac{(x+r-n)!}{(x-1)!(r-n+1)!} = S(x,n+1)$$

To find l_1 we need to take first cumulative sum, to find l_2 we need to take second cumulative of first cumulative and so on. To find l_k we can use the following recursion

$$l_k = \frac{1}{(M(k,k))}*\left(\left(M(k,0)\right) - \sum_{n=1}^{k-1} l_n*\left(M(k,n)\right)\right) \text{ where } \left(M(k,n)\right) = \sum_{m=1}^{r} k_m*\frac{(m*L+r+k-n-1)!}{(m*L+k-1)!(r-n)!} \quad (148)$$

Since we have arrived k_m by $(e^x-1)^{r+1}$, on expanding, using tailor series, the co-efficient of $x^{r+1}=1$ and $x^{r+2}=\frac{(r+1)}{2}$ and then for $S(x)=\frac{(x+r+1)!}{(x-1)!(r+2)!}$, cooefficient of $x^{r+1}=\frac{1}{(r+2)!}$ and $x^{r+2}=\frac{1}{2*(r!)}$ and so on. With this one can prove that $l_1 = -\frac{r+1}{2}*(L+1)$ and $l_t*\frac{(r+t)!}{r!}$ will be another polynomial with r and L and by equating coefficient of x^r one can get $c_r = \sum_{m=1}^{r} k_m*\frac{(m*L+r-1)!}{(m*L-1)!(r)!} = L^{r+1}$. Hence we can write

$$F(x) = \frac{1}{L^{r+1}}*\left(\left(\left(\sum_{m=0}^{r} a_{m_0}*sP_{r_m}(x)\right)\right)\right) + \quad (149)$$

$$\frac{1}{L^{r+1}} * \left(\sum_{n=1}^{u} \left(\left(\sum_{m=0}^{r} A(n) * a_{m_n} * sP_{r_m}(x) \right) * \cos\left(\frac{n*\pi*(2*x-2+p)}{(p+1)*L} + k*\pi \right) \right) \right) +$$
$$\frac{1}{L^{r+1}} * \left(\sum_{n=1}^{v} \left(\left(\sum_{m=0}^{r} 2 * b_{m_n} * sP_{r_m}(x) \right) * \sin\left(\frac{n*\pi*(2*x-2+p)}{(p+1)*L} + k*\pi \right) \right) \right)$$

where if $k \neq \frac{2*w-1}{2}$ and $\forall\ w \subset integer$ then, $p = 0$,

if L is odd, then u and v are $\frac{L-1}{2}$, elseif L is even, then $u = \frac{L}{2}$ and $v = \frac{L}{2} - 1$

if $k = \frac{2*w-1}{2}$ and $\forall\ w \subset integer$ then, $p = 1$, $u = 0$, and $v = L - 1$

$A(n) = \frac{1}{\cos^2(k*\pi)}$ when $n = \frac{L}{2}$ and $k \neq \frac{2*w-1}{2}$,

$A(n) = 2\ \forall$ when $n \neq \frac{L}{2}$ or $k = \frac{2*w-1}{2}$,

$$sP_{r_0}(x) = \frac{(x+r-1)!}{(x-1)!(r)!} - \sum_{n=1}^{r} l_n * \frac{(x+r-n-1)!}{(x-1)!(r-n)!}, sP_{r_{m+1}}(x) = sP_{r_m}(x+1) - sP_{r_m}(x) \ , \tag{150}$$

$$l_k = \frac{1}{(M(k,k))} * \left((M(k,0)) - \sum_{n=1}^{k-1} l_n * (M(k,n)) \right) \text{ where } (M(k,n)) = \sum_{m=1}^{r} k_m * \frac{(m*L+r+k-n-1)!}{(m*L+k-1)!(r-n)!} \ , \tag{151}$$

$$k_m = -1^{r-m+1} * ((r+1)C(m)) = -1^{r-m+1} * \frac{(r+1)!}{} m! * (r-m+1)! \ , \tag{152}$$

$$a_{m_n} = \left(\sum_{m=1}^{r+1} k_m * cf_m(m*L) \right) \ , \tag{153}$$

$$cf_0(x) = \sum_{s=1}^{x} F(s) * \cos\left(\frac{n*\pi*(2*s-2+p)}{(p+1)*L} + k*\pi \right), cf_{m+1}(x) = \sum_{s=1}^{x} cf_m(s) \tag{154}$$

$$b_{m_n} = \left(\sum_{m=1}^{r+1} k_m * sf_m(m*L) \right), \tag{155}$$

$$sf_0(x) = \sum_{s=1}^{x} F(s) * \sin\left(\frac{n*\pi*(2*s-2+p)}{(p+1)*L} + k*\pi \right), sf_{m+1}(x) = \sum_{s=1}^{x} sf_m(s) \tag{156}$$

This is not only applicable for cos or sin functions but for any Orthogonal periodic functions, i.e.)$\forall\ 1 \le k \le r+1$, $\sum_{x=1}^{L} (V_n(k*x) * V_m(k*x)) = 0$ when $n \neq m$ and, $\sum_{x=1}^{L} (V_n(k*x) * V_n(k*x)) = 1$ and $V_n(x)\ \exists\ \forall\ 1 \le x \le (r+1)*L$. For making high smoothness function while scaling to this series, we can treat when $r = 0$, instead of $\cos(0*x)$, we have to use $cos(2*\pi*x)$ and $sin(0*x)$ as $sin(\pi*x)$ since $(x-1)*sin(\pi*(x-1)) = 0\ \forall\ 1 \le x \le (2*L+1)$ and derivative of $(x-1)*sin(\pi*(x-1)) = 0$ when $x = 1$ and it is $2*L$, when $x = (2*L+1)$. Hence we can add $d*(x-1)*sin(\pi*(x-1))$ to the curve and $derivative$ at $(2*L+1)$ of this curve $= derivative$ of the curve at $(2*L+1) + 2*d*L = derivative$ of the joining curve at 0. using this we can get the value of d.

C. Derivation with Polynomial Discrete Function for 2 Dimension

Same discrete function can be applied to 2 dimensions also, For degree 1, Let $F(x,y)\ \exists\ \forall\ 1 \le x \le (2*L)$, and $1 \le y \le (2*M)$, then

$$F(x,y) = \left(\left(\left(P(acc_{0_0}, bcc_{0_0}, ccc_{0_0}, dcc_{0_0}, x, y) \right) \right) \right) + \tag{157}$$
$$\left(\sum_{n=1}^{u_c} \left(A(n) * P(acc_{n_0}, bcc_{n_0}, ccc_{n_0}, dcc_{n_0}, x, y) * CCX(n,x) \right) \right) +$$
$$\left(\sum_{m=1}^{v_c} \left(B(m) * P(acc_{0_m}, bcc_{0_m}, ccc_{0_m}, dcc_{0_m}, x, y) * CCY(m,y) \right) \right) +$$
$$\left(\sum_{m=1}^{v_c} \left(\sum_{n=1}^{u_c} \left(A(n) * B(m) * P(acc_{n_m}, bcc_{n_m}, ccc_{n_m}, dcc_{n_m}, x, y) * CCX(n,x) * CCY(m,y) \right) \right) \right) +$$
$$\left(\sum_{n=1}^{u_s} \left(2 * P(asc_{n_0}, bsc_{n_0}, csc_{n_0}, dsc_{n_0}, x, y) * SSX(n,x) \right) \right) +$$
$$\left(\sum_{m=1}^{v_s} \left(2 * P(acs_{0_m}, bcs_{0_m}, ccs_{0_m}, dcs_{0_m}, x, y) * SSY(m,y) \right) \right) +$$
$$\left(\sum_{m=1}^{v_s} \left(\sum_{n=1}^{u_c} \left(A(n) * 2 * P(acs_{n_m}, bcs_{n_m}, ccs_{n_m}, dcs_{n_m}, x, y) * CCX(n,x) * SSY(m,y) \right) \right) \right) +$$
$$\left(\sum_{m=1}^{v_c} \left(\sum_{n=1}^{u_s} \left(2 * B(m) * P(asc_{n_m}, bsc_{n_m}, csc_{n_m}, dsc_{n_m}, x, y) * SSX(n,x) * CCY(m,y) \right) \right) \right) +$$
$$\left(\sum_{m=1}^{v_s} \left(\sum_{n=1}^{u_s} \left(4 * P(ass_{n_m}, bss_{n_m}, css_{n_m}, dss_{n_m}, x, y) * SSX(n,x) * SSY(m,y) \right) \right) \right)$$

where $P(a,b,c,d,x,y) = a*(x-L-1)*(y-M-1) + b*(x-L-1) + c*(y-M-1) + d$,

$$CCX(q,x) = \cos\left(\frac{q*\pi*(2*x-2+p_x)}{(p_x+1)*L} + k*\pi\right) ,$$
$$SSX(q,x) = \sin\left(\frac{q*\pi*(2*x-2+p_x)}{(p_x+1)*L} + k*\pi\right) ,$$
$$CCY(q,y) = \cos\left(\frac{q*\pi*(2*y-2+p_y)}{(p_y+1)*M} + l*\pi\right) ,$$
$$SSY(q,y) = \sin\left(\frac{q*\pi*(2*y-2+p_y)}{(p_y+1)*M} + l*\pi\right) ,$$
if $k \neq \frac{2*w_x-1}{2}$ and $\forall\ w_x \subset integer$ then, $p_x = 0$,
if L is odd, then u_c and u_s are $\frac{L-1}{2}$, elseif L is even, then $u_c = \frac{L}{2}$ and $u_s = \frac{L}{2} - 1$,
if $k = \frac{2*w_x-1}{2}$ and $\forall\ w_x \subset integer$ then, $p_x = 1$, $u_c = 0$, and $u_s = L-1$,
if $l \neq \frac{2*w_y-1}{2}$ and $\forall\ w_y \subset integer$ then, $p_y = 0$,
if M is odd, then v_c and v_s are $\frac{M-1}{2}$, elseif M is even, then $v_c = \frac{M}{2}$ and $v_s = \frac{M}{2} - 1$,
if $l = \frac{2*w_y-1}{2}$ and $\forall\ w_y \subset integer$ then, $p_y = 1$, $v_c = 0$, and $v_s = L-1$,
$$A(q) = \frac{1}{\cos^2(k*\pi)} \text{ when } q = \frac{L}{2} \text{ and } k \neq \frac{2*w_x-1}{2} ,$$
$$A(q) = 2\ \forall \text{ when } q \neq \frac{L}{2} \text{ or } k = \frac{2*w_x-1}{2} ,$$
$$B(q) = \frac{1}{\cos^2(l*\pi)} \text{ when } q = \frac{M}{2} \text{ and } l \neq \frac{2*w_y-1}{2} ,$$
$$B(q) = 2\ \forall \text{ when } q \neq \frac{M}{2} \text{ or } l = \frac{2*w_y-1}{2} ,$$
$$acc_{m_n} = \frac{G_{ccxy}(2*L,2*M)-2*G_{ccxy}(L,2*M)-2*G_{ccxy}(2*L,M)+4*G_{ccxy}(L,M)}{L^2*M^2} , \quad (158)$$
$$G_{ccxy}(p,q) = \sum_{y=1}^{q}\left(\sum_{x=1}^{p} F(x,y)*CCX(n,x)*CCY(m,y)\right) ,$$
$$acs_{m_n} = \frac{G_{csxy}(2*L,2*M)-2*G_{csxy}(L,2*M)-2*G_{csxy}(2*L,M)+4*G_{csxy}(L,M)}{L^2*M^2} , \quad (159)$$
$$G_{csxy}(p,q) = \sum_{y=1}^{q}\left(\sum_{x=1}^{p} F(x,y)*CCX(n,x)*SSY(m,y)\right) ,$$
$$asc_{m_n} = \frac{G_{scxy}(2*L,2*M)-2*G_{scxy}(L,2*M)-2*G_{scxy}(2*L,M)+4*G_{scxy}(L,M)}{L^2*M^2} , \quad (160)$$
$$G_{scxy}(p,q) = \sum_{y=1}^{q}\left(\sum_{x=1}^{p} F(x,y)*SSX(n,x)*CCY(m,y)\right) ,$$
$$ass_{m_n} = \frac{G_{ssxy}(2*L,2*M)-2*G_{ssxy}(L,2*M)-2*G_{ssxy}(2*L,M)+4*G_{ssxy}(L,M)}{L^2*M^2} , \quad (161)$$
$$G_{ssxy}(p,q) = \sum_{y=1}^{q}\left(\sum_{x=1}^{p} F(x,y)*SSX(n,x)*SSY(m,y)\right) ,$$
$$bcc_{m_n} = \frac{G_{ccx}(2*L,2*M)-2*G_{ccx}(L,2*M)-2*G_{ccx}(2*L,M)+4*G_{ccx}(L,M)}{L^2*M^2} , \quad (162)$$
$$G_{ccx}(p,q) = \sum_{y=1}^{q}\left(\sum_{x=1}^{p} (p-x+1)*F(x,y)*CCX(n,x)*CCY(y,m)\right) ,$$
$$bcs_{m_n} = \frac{G_{csx}(2*L,2*M)-2*G_{csx}(L,2*M)-2*G_{csx}(2*L,M)+4*G_{csx}(L,M)}{L^2*M^2} , \quad (163)$$
$$G_{csx}(p,q) = \sum_{y=1}^{q}\left(\sum_{x=1}^{p} (p-x+1)*F(x,y)*CCX(n,x)*SSY(m,y)\right) ,$$
$$asc_{m_n} = \frac{G_{scx}(2*L,2*M)-2*G_{scx}(L,2*M)-2*G_{scx}(2*L,M)+4*G_{scx}(L,M)}{L^2*M^2} , \quad (164)$$
$$G_{scx}(p,q) = \sum_{y=1}^{q}\left(\sum_{x=1}^{p} (p-x+1)*F(x,y)*SSX(n,x)*CCY(m,y)\right) ,$$
$$bss_{m_n} = \frac{G_{ssx}(2*L,2*M)-2*G_{ssx}(L,2*M)-2*G_{ssx}(2*L,M)+4*G_{ssx}(L,M)}{L^2*M^2} , \quad (165)$$
$$G_{ssx}(p,q) = \sum_{y=1}^{q}\left(\sum_{x=1}^{p} (p-x+1)*F(x,y)*SSX(n,x)*SSY(m,y)\right) ,$$
$$ccc_{m_n} = \frac{G_{ccy}(2*L,2*M)-2*G_{ccy}(L,2*M)-2*G_{ccy}(2*L,M)+4*G_{ccy}(L,M)}{L^2*M^2} , \quad (166)$$
$$G_{ccy}(p,q) = \sum_{y=1}^{q}\left(\sum_{x=1}^{p} (q-y+1)*F(x,y)*CCX(n,x)*CCY(m,y)\right) ,$$
$$ccs_{m_n} = \frac{G_{csy}(2*L,2*M)-2*G_{csy}(L,2*M)-2*G_{csy}(2*L,M)+4*G_{csy}(L,M)}{L^2*M^2} , \quad (167)$$
$$G_{csy}(p,q) = \sum_{y=1}^{q}\left(\sum_{x=1}^{p} (q-y+1)*F(x,y)*CCX(n,x)*SSY(m,y)\right) ,$$
$$csc_{m_n} = \frac{G_{scy}(2*L,2*M)-2*G_{scy}(L,2*M)-2*G_{scy}(2*L,M)+4*G_{scy}(L,M)}{L^2*M^2} , \quad (168)$$

$$G_{scy}(p,q) = \sum_{y=1}^{q} \left(\sum_{x=1}^{p} (q-y+1) * F(x,y) * SSX(n,x) * CCY(m,y)\right) ,$$

$$css_{m_n} = \frac{G_{ssy}(2*L,2*M) - 2*G_{ssy}(L,2*M) - 2*G_{ssy}(2*L,M) + 4*G_{ssy}(L,M)}{L^2*M^2} , \quad (169)$$

$$G_{ssy}(p,q) = \sum_{y=1}^{q} \left(\sum_{x=1}^{p} (q-y+1) * F(x,y) * SSX(n,x) * SSY(m,y)\right) ,$$

$$dcc_{m_n} = \frac{G_{cc}(2*L,2*M) - 2*G_{cc}(L,2*M) - 2*G_{cc}(2*L,M) + 4*G_{cc}(L,M)}{L^2*M^2} , \quad (170)$$

$$G_{cc}(p,q) = \sum_{y=1}^{q} \left(\sum_{x=1}^{p} (p-x+1) * (q-y+1) * F(x,y) * CCX(n,x) * CCY(m,y)\right) ,$$

$$dcs_{m_n} = \frac{G_{cs}(2*L,2*M) - 2*G_{cs}(L,2*M) - 2*G_{cs}(2*L,M) + 4*G_{cs}(L,M)}{L^2*M^2} , \quad (171)$$

$$G_{cs}(p,q) = \sum_{y=1}^{q} \left(\sum_{x=1}^{p} (p-x+1) * (q-y+1) * F(x,y) * CCX(n,x) * SSY(m,y)\right) ,$$

$$dsc_{m_n} = \frac{G_{sc}(2*L,2*M) - 2*G_{sc}(L,2*M) - 2*G_{sc}(2*L,M) + 4*G_{sc}(L,M)}{L^2*M^2} , \quad (172)$$

$$G_{sc}(p,q) = \sum_{y=1}^{q} \left(\sum_{x=1}^{p} (p-x+1) * (q-y+1) * F(x,y) * SSX(n,x) * CCY(m,y)\right) ,$$

$$dss_{m_n} = \frac{G_{ss}(2*L,2*M) - 2*G_{ss}(L,2*M) - 2*G_{ss}(2*L,M) + 4*G_{ss}(L,M)}{L^2*M^2} , \quad (173)$$

$$G_{ss}(p,q) = \sum_{y=1}^{q} \left(\sum_{x=1}^{p} (p-x+1) * (q-y+1) * F(x,y) * SSX(n,x) * SSY(m,y)\right)$$

For making high smoothness function while scaling to this series, Like we did for 1 dimension, we need to do for 2 dimensions also; $\left(a_x * (x-1) + b_x * (y-1) + c_x * (x-1) * (y-1) + d_x\right) * sin\left(\pi * (x-1)\right) * CS\left(\frac{2*n*\pi*(y-1)}{L}\right)$ and $\left(a_y * (x-1) + b_y * (y-1) + c_y * (x-1) * (y-1) + d_y\right) * sin\left(\pi * (y-1)\right) * CS\left(\frac{n*\pi*(2*x-2+p)}{(p+1)*L}\right) +$

$$\left(a_{xxy} * (x-1) + b_{xxy} * (y-1) + c_{xxy} * (x-1) * (y-1) + d_{xxy}\right) * sin\left(\pi * (x-1) * (y-1)\right) * CS\left(\frac{n * \pi * (2 * x - 2 + p)}{(p+1) * L}\right),$$

$$\left(a_{yxy} * (x-1) + b_{yxy} * (y-1) + c_{yxy} * (x-1) * (y-1) + d_{yxy}\right) * sin\left(\pi * (x-1) * (y-1)\right) * CS\left(\frac{n * \pi * (2 * x - 2 + p)}{(p+1) * L}\right)$$

where $CS =$ cos or sin, and gradient derivative of x direction, y direction at top edge=0 and gradient derivative of bottom edge should be equal to difference of next curve top edge gradient derivative with gradient derivative of current curve bottom edge, gradient derivative y direction, x direction at left edge=0 and gradient derivative of right edge should be equal to difference of next curve left edge gradient derivative with gradient derivative of current curve right edge and the gradient derivative of x and y direction of extreme top left corner point=0 and the gradient derivative of x and y direction of extreme bottom right corner point should be equal to gradient derivative of x and y direction of next curve extreme top left corner point - gradient derivative of x and y direction of current curve extreme bottom right corner point.

D. Derivation with Sign Discrete Function

Since finding cumulative sum for higher dimensions or even for single dimension is cumbersome, we can simplify by making orthogonal functions. For example for Polynomial of single degree can be written to find orthogonal of

$$S(x) = x - a_0 - \sum_{n=1}^{u} \left(a_n * \cos\left(\frac{n*\pi*(2*x-2+p)}{(p+1)*L} + k*\pi\right)\right) - \sum_{n=1}^{v} \left(b_n * \sin\left(\frac{n*\pi*(2*x-2+p)}{(p+1)*L} + k*\pi\right)\right) \quad (174)$$

if $k \neq \frac{2*w-1}{2}$ and $\forall\ w \subset integer$ then, $p = 0$,

if L is odd, then u and v are $\frac{L-1}{2}$, elseif L is even, then $u = \frac{L}{2}$ and $v = \frac{L}{2} - 1$

if $k = \frac{2*w-1}{2}$ and $\forall\ w \subset integer$ then, $p = 1$, $u = 0$, and $v = L - 1$

We need to find co − efficient a_n and b_n which satisfies

$\sum_{r=1}^{2*L} S(r) * \cos\left(\frac{n*\pi*(2*x-2+p)}{(p+1)*L} + k*\pi\right) = 0$ and

$\sum_{r=1}^{2*L} S(r) * \sin\left(\frac{n*\pi*(2*x-2+p)}{(p+1)*L} + k*\pi\right) = 0$ then

$S(x) * \cos\left(\frac{n*\pi*(2*x-2+p)}{(p+1)*L} + k*\pi\right)$ and

$S(x) * \sin\left(\frac{n*\pi*(2*x-2+p)}{(p+1)*L} + k*\pi\right)$ are orthogonal, using orthogonal property, we can find a_n and b_n as below:

$$a_n = A(n) * \sum_{r=1}^{2*L} S(r) * \cos\left(\frac{n*\pi*(2*x-2+p)}{(p+1)*L} + k*\pi\right), \text{ and} \quad (175)$$

$$b_n = 2 * \sum_{r=1}^{2*L} S(r) * \sin\left(\frac{n*\pi*(2*x-2+p)}{(p+1)*L} + k*\pi\right) = 0, \text{ where} \quad (176)$$

if $k \neq \frac{2*w-1}{2}$ and $\forall\ w \subset integer$ then, $p = 0$,

if L is odd, then u and v are $\frac{L-1}{2}$, elseif L is even, then $u = \frac{L}{2}$ and $v = \frac{L}{2} - 1$

if $k = \frac{2*w-1}{2}$ and $\forall\ w \subset integer$ then, $p = 1$, $u = 0$, and $v = L - 1$

$A(n) = \frac{1}{\cos^2(k*\pi)}$ when $n = \frac{L}{2}$ and $k \neq \frac{2*w-1}{2}$,

$A(n) = 2\ \forall$ when $n \neq \frac{L}{2}$ or $k = \frac{2*w-1}{2}$,

Also interestingly after substituting a_n, we get

$S(r) = 1\ \forall\ 1 \leq r \leq L$ and $S(r) = -1\ \forall\ (L+1) \leq r \leq (2*L)$

Hence, if $F(x)\ \exists\ \forall\ 1 \leq x \leq (2*L)$ and $S(r) = 1\ \forall\ 1 \leq r \leq L$ and $S(r) = -1\ \forall\ (L+1) \leq r \leq (2*L)$, then

$$F(x) = (a_0 * S(x) + b_0) + \sum_{n=1}^{v}\left(A(n) * (a_n * S(x) + b_n) * \cos\left(\frac{n*\pi*(2*x-2+p)}{(p+1)*L} + k*\pi\right)\right) + \quad (177)$$

$$\sum_{n=1}^{u}\left(2 * (c_n * S(x) + d_n) * \sin\left(\frac{n*\pi*(2*x-2+p)}{(p+1)*L} + k*\pi\right)\right)$$

if $k \neq \frac{2*w-1}{2}$ and $\forall\ w \subset integer$ then, $p = 0$,

if L is odd, then u and v are $\frac{L-1}{2}$, elseif L is even, then $u = \frac{L}{2}$ and $v = \frac{L}{2} - 1$

if $k = \frac{2*w-1}{2}$ and $\forall\ w \subset integer$ then, $p = 1$, $u = 0$, and $v = L - 1$

$A(n) = \frac{1}{\cos^2(k*\pi)}$ when $n = \frac{L}{2}$ and $k \neq \frac{2*w-1}{2}$,

$A(n) = 2\ \forall$ when $n \neq \frac{L}{2}$ or $k = \frac{2*w-1}{2}$,

$$a_n = \frac{1}{2*L} * \sum_{r=1}^{2*L}\left(F(r) * S(r) * \cos\left(\frac{n*\pi*(2*x-2+p)}{(p+1)*L} + k*\pi\right)\right), \quad (178)$$

$$c_n = \frac{1}{2*L} * \sum_{r=1}^{2*L}\left(F(r) * S(r) * \sin\left(\frac{n*\pi*(2*x-2+p)}{(p+1)*L} + k*\pi\right)\right), \quad (179)$$

$$b_n = \frac{1}{2*L} * \sum_{r=1}^{2*L}\left(F(r) * \cos\left(\frac{n*\pi*(2*x-2+p)}{(p+1)*L} + k*\pi\right)\right), \quad (180)$$

$$d_n = \frac{1}{2*L} * \sum_{r=1}^{2*L}\left(F(r) * \sin\left(\frac{n*\pi*(2*x-2+p)}{(p+1)*L} + k*\pi\right)\right), \quad (181)$$

$$S(r) = 1\ \forall\ 1 \leq r \leq L \text{ and } S(r) = -1\ \forall\ (L+1) \leq r \leq (2*L) \quad (182)$$

E. Derivation with Sign Continuous Function

Similarly for continuous function, using this property, if $F(x)$ is periodic between $-\frac{L}{2}$ and $\frac{L}{2}$, then $S(x) = Sign(x) = 1\ \forall\ x \geq 0$ and $S(x) = Sign(x) = -1\ \forall\ x \leq 0$. Then We can put a series like Fourier as given below:

$$f(x) = (a_0 * x + b_0) + \sum_{n=1}^{\infty}\left((a_n * S(x) + b_n) * \cos\left(\frac{2*n*\pi*x}{L}\right) + (c_n * S(x) + d_n) * \sin\left(\frac{2*n*\pi*x}{L}\right)\right) \quad (183)$$

where $S(x) = Sign(x) = 1\ \forall\ x \geq 0$ and $S(x) = Sign(x) = -1\ \forall\ x \leq 0$,

$$a_n = \frac{1}{L} * \int_{-\frac{L}{2}}^{\frac{L}{2}}\left(F(2*r) * S(2*r) * \cos\left(\frac{4*n*\pi*r)}{L}\right)\right) \ and \quad (184)$$

$$a_0 = \frac{1}{2*L} * \int_{-\frac{L}{2}}^{\frac{L}{2}}(F(2*r) * S(2*r)) \ and \quad (185)$$

$$c_n = \frac{1}{L} * \int_{-\frac{L}{2}}^{\frac{L}{2}}\left(F(2*r) * S(2*r) * \sin\left(\frac{4*n*\pi*r)}{L}\right)\right) \ and \quad (186)$$

$$b_n = \frac{1}{L} * \int_{-\frac{L}{2}}^{\frac{L}{2}}\left(F(2*r) * \cos\left(\frac{4*n*\pi*r)}{L}\right)\right) \ and \quad (187)$$

$$b_0 = \frac{1}{2*L} * \int_{-\frac{L}{2}}^{\frac{L}{2}} \left(F(2*r)\right) \ and \tag{188}$$

$$d_n = \frac{1}{L} * \int_{-\frac{L}{2}}^{\frac{L}{2}} \left(F(2*r) * \sin\left(\frac{4*n*\pi*r)}{L}\right)\right) \tag{189}$$

G. Derivation with Exponential Geometric Discrete Function

In similar way, we can extend not only for polynomial or sign functions, but also for exponential geometric functions for both discrete and continuous functions as given below. Let us take example of discrete functions.

$$F(x) = \frac{1}{L} * \left(\left(\left(\sum_{m=0}^{r} a_{m_0} * {r_m}^{(x-1)}\right)\right)\right) + \tag{190}$$

$$\frac{1}{L} * \left(\sum_{n=1}^{u} \left(\left(\sum_{m=0}^{r} A(n) * a_{m_n} * {r_m}^{(x-1)}\right) * \cos\left(\frac{n*\pi*(2*x-2+p)}{(p+1)*L} + k*\pi\right)\right)\right) +$$

$$\frac{1}{L} * \left(\sum_{n=1}^{v} \left(\left(\sum_{m=0}^{r} 2 * b_{m_n} * {r_m}^{(x-1)}\right) * \sin\left(\frac{n*\pi*(2*x-2+p)}{(p+1)*L} + k*\pi\right)\right)\right)$$

where if $k \neq \frac{2*w-1}{2}$ and $\forall\ w \subset integer$ then, $p = 0$,

if L is odd, then u and v are $\frac{L-1}{2}$, elseif L is even, then $u = \frac{L}{2}$ and $v = \frac{L}{2} - 1$

if $k = \frac{2*w-1}{2}$ and $\forall\ w \subset integer$ then, $p = 1$, $u = 0$, and $v = L - 1$

$A(n) = \frac{1}{\cos^2(k*\pi)}$ when $n = \frac{L}{2}$ and $k \neq \frac{2*w-1}{2}$,

$A(n) = 2\ \forall$ when $n \neq \frac{L}{2}$ or $k = \frac{2*w-1}{2}$,

where $r_0 = 1$ and having unique r of r_m values of geometric powers for m other than 0.

if $G_{0_0}(q) = \sum_{x=1+(q*L)}^{(q+1)*L} \left(F(x)\right)$, $then$ $G_{0_0}(q)$ is also a geometric series having same ${r_m}^L$ values of geometric powers except for $r_0 = 1$ which is $r*L$

Since Geometric series has a property of $\sum_{q=0}^{r} \left(c_{0_{0_q}} * G_{0_0}(q)\right) = 0$

where $c_{0_{0_q}}$ is the co − efficient of x^q in $\prod_{p=1}^{r} \left(x - (r_p)^L\right)$ Hence one can get coefficient a_{0_0}

$$a_{0_0} = \frac{\sum_{q=0}^{r}\left(c_{0_{0_q}} * G_{0_0}(q)\right)}{\sum_{q=0}^{r}\left(c_{0_{0_q}}\right)} \quad \text{where} \quad c_{0_{0_q}} \text{ is the co} - \text{efficient of } x^q \text{ in } \prod_{p=1}^{r} \left(x - (r_p)^L\right) \tag{191}$$

to get the co − efficient of a_{m_0}

if $G_{m_0}(q) = \sum_{x=1+(q*L)}^{(q+1)*L} \left(F(x)\right) * \left({r_m}^{-(x-1)}\right)$, then $G_{m_0}(q)$ is also a geometric series having same $\left(\frac{r_p}{r_m}\right)^L$ values of geometric powers except for $\frac{r_m}{r_m} = 1$ which is $r*L$

Since Geometric series has a property of $\sum_{q=0}^{r} \left(c_{m_{0_q}} * G_{m_0}(q)\right) = 0$ Hence one can get coefficient a_{m_0}

$$a_{m_0} = \frac{\sum_{q=0}^{r}\left(c_{m_{0_q}} * G_{m_0}(q)\right)}{\sum_{q=0}^{r}\left(c_{m_{0_q}}\right)} \quad \text{where} \quad c_{m_{0_q}} \text{ is the co} - \text{efficientof } x^q \text{ in } \prod_{p=0 \text{ and } p\neq m}^{r} \left(x - \left(\frac{r_p}{r_m}\right)^L\right) \tag{192}$$

to get the co − efficient of a_{m_n}

if $G_{m_n}(q) = \sum_{x=1+(q*L)}^{(q+1)*L} \left(F(x)\right) * ({r_m}^{-(x-1)}) * \cos\left(\frac{n*\pi*(2*x-2+p)}{(p+1)*L} + k*\pi\right)$,

then $G_{m_n}(r)$ is also a geometric series having same

$\left(\frac{r_p}{r_m}\right)^L$ values of geometric powers except for $\frac{r_m}{r_m} = 1$ whichis $r*L$ Hence

$$a_{m_n} = \frac{2*\sum_{q=0}^{r}\left(c_{m_q} * G_{m_n}(q)\right)}{\sum_{q=0}^{r}\left(c_{m_q}\right)} \quad \text{where} \quad c_{m_q} \text{ is the co} - \text{efficient of } x^q \text{ in } \prod_{p=0 \text{ and } p\neq m}^{r} \left(x - \left(\frac{r_p}{r_m}\right)^L\right) \tag{193}$$

Similarly to get the co − efficient of b_{m_n} we need to apply $\sin$ instead of $\cos$, Hence

$$b_{m_n} = \frac{2*\sum_{q=0}^{r}\left(c_{m_q} * H_{m_n}(q)\right)}{\sum_{q=0}^{r}\left(c_{m_q}\right)} \quad \text{where} \quad c_{m_q} \text{ is the co} - \text{efficient of } x^q \text{ in } \prod_{p=0\text{and } p\neq m}^{r} \left(x - \left(\frac{r_p}{r_m}\right)^L\right) \tag{194}$$

$$H_{m_n}(q) = \sum_{x=1+(q*L)}^{(q+1)*L} \left(F(x)\right) * ({r_m}^{-(x-1)}) * \sin\left(\frac{n*\pi*(2*x-2+p)}{(p+1)*L} + k*\pi\right)$$

Hence we can generalize the exponential geometric powers also as given below:

$$F(x) = \frac{1}{L} * \left(\left(\left(\sum_{m=0}^{r} a_{m_0} * {r_m}^{(x-1)}\right)\right)\right) + \tag{195}$$

$$\frac{1}{L} * \left(\sum_{n=1}^{u} \left(\left(\sum_{m=0}^{r} A(n) * a_{m_n} * {r_m}^{(x-1)}\right) * \cos\left(\frac{n*\pi*(2*x-2+p)}{(p+1)*L} + k*\pi\right)\right)\right) +$$

$$\frac{1}{L} * \left(\sum_{n=1}^{v} \left(\left(\sum_{m=0}^{r} 2 * b_{m_n} * {r_m}^{(x-1)}\right) * \sin\left(\frac{n*\pi*(2*x-2+p)}{(p+1)*L} + k*\pi\right)\right)\right)$$

where if $k \neq \frac{2*w-1}{2}$ and $\forall\ w \subset integer$ then, $p = 0$,

if L is odd, then u and v are $\frac{L-1}{2}$, elseif L is even, then $u = \frac{L}{2}$ and $v = \frac{L}{2} - 1$

if $k = \frac{2*w-1}{2}$ and $\forall\ w \subset integer$ then, $p = 1$, $u = 0$, and $v = L - 1$

$A(n) = \frac{1}{\cos^2(k*\pi)}$ when $n = \frac{L}{2}$ and $k \neq \frac{2*w-1}{2}$,

$A(n) = 2\ \forall$ when $n \neq \frac{L}{2}$ or $k = \frac{2*w-1}{2}$,

where $r_0 = 1$ and having unique r of r_m values of geometric powers for m other than 0.

$$a_{m_n} = \frac{\sum_{q=0}^{r}\left(c_{m_q} * G_{m_n}(q)\right)}{\sum_{q=0}^{r}\left(c_{m_q}\right)} \text{ where } c_{m_q} \text{ is the co-efficient of } x^q \text{ in } \prod_{p=0 \text{ and } p\neq m}^{r} \left(x - \left(\frac{r_p}{r_m}\right)^L\right) \tag{196}$$

$$G_{m_n}(q) = \sum_{x=1+(q*L)}^{(q+1)*L} \left(F(x)\right) * ({r_m}^{-(x-1)}) * \cos\left(\frac{n*\pi*(2*x-2+p)}{(p+1)*L} + k*\pi\right) ,$$

$$b_{m_n} = \frac{\sum_{q=0}^{r}\left(c_{m_q} * H_{m_n}(q)\right)}{\sum_{q=0}^{r}\left(c_{m_q}\right)} \text{ where } c_{m_q} \text{ is the co-efficient of } x^q \text{ in } \prod_{p=0 \text{ and } p\neq m}^{r} \left(x - \left(\frac{r_p}{r_m}\right)^L\right) \tag{197}$$

$$H_{m_n}(q) = \sum_{x=1+(q*L)}^{(q+1)*L} \left(F(x)\right) * ({r_m}^{-(x-1)}) * \sin\left(\frac{n*\pi*(2*x-2+p)}{(p+1)*L} + k*\pi\right)$$

To find appropriate geometric terms r_m of the data, one can go with finding the Geometric series of s exponential terms with data of $(s+1)*L$ values for L periodic states. This will have property of co-efficients as given below.

$$t_{n+s*L} = \sum_{r=0}^{s} c_{n_r} * v_{n_r} \text{ where } v_{n_0} = 1 \text{ and } v_{n_p} = t_{n+(s-p)*L} - \sum_{r=0}^{p} c_{p_r} * v_{p_r} \tag{198}$$

where $c_{p_q} = \frac{\sum_{n=1}^{L} t_{n+(s-p)*L} * v_{p_q}}{\sum_{n=1}^{L}\left(v_{p_q}\right)^2}$

Using this, one can obtain the following values

$$c_{n_0} = \frac{\sum_{n=1}^{L} t_{n+s*L}}{L} , \tag{199}$$

$$v_{n_1} = t_{n+(s-1)*L} - \frac{\sum_{n=1}^{L} t_{n+s*L}}{L} , \tag{200}$$

$$c_{n_1} = \frac{\sum_{n=1}^{L} t_{n+(s-1)*L} * v_{n_1}}{\sum_{n=1}^{L}\left(v_{n_1}\right)^2} \tag{201}$$

And so on. Once we get all c_{n_r}, we can again rewrite to the following form

$$t_{n+s*L} = \sum_{r=0}^{s} c_{n_r} * v_{n_r} - c_{n_0} * v_{n_0} = \sum_{r=0}^{s} k_{n_r} * t_{n+r*L} \text{ where } k_{n_s} = 1 , \tag{202}$$

Then, as per geometric series properties, roots of the following equation with variable x will be the geometric terms r_m for the series.

$$\sum_{r=0}^{s} k_{n_r} * x^{r*L} = 0 \text{ having } s \text{ roots} \tag{203}$$

H. Derivation with Exponential Geometric Continuous Function

As we did in the discrete geometric series, we need to take same way for continuous series, but integral $\int$ instead of summation $\sum$.

$$f(x) = \frac{1}{L} * \left(\sum_{m=0}^{2*r} a_{m_0} * {r_m}^x\right) + \tag{204}$$

$$\frac{2}{L} * \left(\sum_{n=1}^{\infty} \left(\left(\sum_{m=0}^{2*r} a_{m_n} * {r_m}^x\right) * \cos\left(\frac{2*n*\pi*x}{L}\right)\right)\right) +$$

$$\frac{2}{L} * \left(\sum_{n=1}^{\infty} \left(\left(\sum_{m=0}^{2*r} b_{m_n} * {r_m}^x\right) * \sin\left(\frac{2*n*\pi*x}{L}\right)\right)\right)$$

where $r_0 = 1$ and having unique r of r_m values of geometric powers for m other than 0 then

$G_{m_n}(q) = \int_{-\frac{L}{2}}^{\frac{L}{2}} \left(F(q*x)\right)*({r_m}^{-q*x})*\cos\left(\frac{2*n*\pi*q*x}{L}\right)*dx$ will have a geometric series of q having both $\left(\frac{r_p}{r_m}\right)^{\frac{L}{2}}$ and $\left(\frac{r_p}{r_m}\right)^{\frac{-L}{2}}$ values of geometric powers except for $\frac{r_m}{r_m}=1$ which is L and since it has both $\left(\frac{r_p}{r_m}\right)^{\frac{L}{2}}$ and $\left(\frac{r_m}{r_p}\right)^{\frac{L}{2}}$, we need to find $\text{co}-\text{efficient}$ by multiplying $\left(x+\frac{1}{x}\right)-\left(\left(\frac{r_p}{r_m}\right)^{\frac{L}{2}}+\left(\frac{r_m}{r_p}\right)^{\frac{L}{2}}\right)$ Hence

$$a_{m_n} = \frac{\sum_{q=1}^{r+1}\left(c_{m_q}*G_{m_n}(q)\right)}{\sum_{q=1}^{r+1}\left(c_{m_q}\right)} \text{ where } c_{m_q} \text{ is the co}-\text{efficient of } \left(x^q+\frac{1}{x^q}\right) \tag{205}$$

in $\left(x+\frac{1}{x}\right)*\left(\prod_{p=0 \text{and } p\neq m}^{r}\left(x+\frac{1}{x}\right)-\left(\left(\frac{r_p}{r_m}\right)^{\frac{L}{2}}+\left(\frac{r_m}{r_p}\right)^{\frac{L}{2}}\right)\right)$ and

$G_{m_n}(q) = \int_{-\frac{L}{2}}^{\frac{L}{2}} \left(F(q*x)\right)*({r_m}^{-q*x})*\cos\left(\frac{2*n*\pi*q*x}{L}\right)*dx$,

$$b_{m_n} = \frac{\sum_{q=1}^{r+1}\left(c_{m_q}*H_{m_n}(q)\right)}{\sum_{q=1}^{r+1}\left(c_{m_q}\right)} \text{ where } c_{m_q} \text{ is the co}-\text{efficient of } \left(x^q+\frac{1}{x^q}\right) \tag{206}$$

in $\left(x+\frac{1}{x}\right)*\left(\prod_{p=0 \text{and } p\neq m}^{r}\left(x+\frac{1}{x}\right)-\left(\left(\frac{r_p}{r_m}\right)^{\frac{L}{2}}+\left(\frac{r_m}{r_p}\right)^{\frac{L}{2}}\right)\right)$ and

$H_{m_n}(q) = \int_{-\frac{L}{2}}^{\frac{L}{2}} \left(F(q*x)\right)*({r_m}^{-q*x})*\sin\left(\frac{2*n*\pi*q*x}{L}\right)*dx$

To find appropriate geometric terms r_m of the data in continuous functions, we need to go only with same discrete approach of finding roots as explained in (198) to (203).

If in case, r_m values are not unique either in discrete or in continuous method, Then it will be combination of both polynomial and geometric powers and one can find combining both the methods those were explained above.We need to find the highest repeater say d, of the r_m and we need to multiply with the polynomial with degree d. First we need to eliminate by multiplying ${r_m}^{-x}$ and then with cumulative sum or multiple integrals with the combination of k_m as explained in the polynomial approach to find the polynomial co-efficient.

I. Derivation with Matrix Harmonic Discrete Function

In similar way, we can extend not only for polynomial, sign or exponential functions, but also for matrix harmonic functions for both discrete and continuous functions as follows. This is applicable to any square matrix, Let us take example of discrete functions with 2 X 2 matrix to prove the concept.

$$\text{Let } C_n(x) = \cos\left(\frac{2*n*\pi*(x-1)}{L}\right) \tag{207}$$

$$S_n(x) = \sin\left(\frac{2*n*\pi*(x-1)}{L}\right) \tag{208}$$

$$A_n(x) = \begin{pmatrix} a_1*C_n(x)+a_2*S_n(x) a_5*C_n(x)+a_6*S_n(x) \\ a_3*C_n(x)+a_4*S_n(x) a_7*C_n(x)+a_8*S_n(x)\end{pmatrix}, \tag{209}$$

$$B_n(x) = \begin{pmatrix} b_1*C_n(x)+b_2*S_n(x) b_5*C_n(x)+b_6*S_n(x) \\ b_3*C_n(x)+b_4*S_n(x) b_7*C_n(x)+b_8*S_n(x)\end{pmatrix}, \tag{210}$$

$$AB_n(x) = \begin{pmatrix} c_1*C_{2*n}(x)+c_2*S_{2*n}(x) c_5*C_{2*n}(x)+c_6*S_{2*n}(x) \\ c_3*C_{2*n}(x)+c_4*S_{2*n}(x) c_7*C_{2*n}(x)+c_8*S_{2*n}(x)\end{pmatrix}, \tag{211}$$

$$BA_n(x) = \begin{pmatrix} d_1*C_{2*n}(x)+d_2*S_{2*n}(x) d_5*C_{2*n}(x)+d_6*S_{2*n}(x) \\ d_3*C_{2*n}(x)+d_4*S_{2*n}(x) d_7*C_{2*n}(x)+d_8*S_{2*n}(x)\end{pmatrix} \tag{212}$$

$$\text{and } AB_n(x) = A_n(x)*B_n(x) \tag{213}$$

$$BA_n(x) = B_n(x)*A_n(x) \text{ then} \tag{214}$$

$$\sum_{x=0}^{L} A_n(x)*A_m(x) = \begin{pmatrix}00\\00\end{pmatrix} \text{ when } n \neq m \tag{215}$$

$$\sum_{x=0}^{L} A_n(x)*A_m(x) = \begin{pmatrix}00\\00\end{pmatrix} \text{ when } n \neq m \tag{216}$$

$$\sum_{x=0}^{L} A_n(x) * B_m(x) = \begin{pmatrix} 00 \\ 00 \end{pmatrix} \text{ when } n \neq m \quad (217)$$

$$\sum_{x=0}^{L} A_n(x) * A_n(x) \neq \begin{pmatrix} 00 \\ 00 \end{pmatrix} \quad (218)$$

$$\sum_{x=0}^{L} B_n(x) * B_n(x) \neq \begin{pmatrix} 00 \\ 00 \end{pmatrix} \text{ and hence one can transform like following} \quad (219)$$

Let $F_N(x) \ \exists \ \forall \ 1 \leq x \leq (L)$, Let $F_D(x) \ \exists \ \forall \ 1 \leq x \leq (L)$, (220)

$$\begin{pmatrix} F_N(x) \\ F_D(x) \end{pmatrix} = \begin{pmatrix} N_{a_0} \\ D_{a_0} \end{pmatrix} + \sum_{n=1}^{L-1} A_n(x) * \begin{pmatrix} N_{a_n} \\ D_{a_n} \end{pmatrix} + \sum_{n=1}^{L-1} B_n(x) * \begin{pmatrix} N_{b_n} \\ D_{b_n} \end{pmatrix} \quad (221)$$

To get this series, using (213) and (214) we need to solve 8 variables of B for a given A matrix
To minimize number of equations and variables, for sxs matrix, we can go with following approach
Let $B = A^{-1} + I_s * V$ which will reduce half of the variables
directly to the equations for each s variables of each row of Vmatrix, where
I_s is the Identity matrix and V is a vector and
r^{th} cell of vector V as $V_r = \sum_{r=1}^{s} V_{V_r} * O_{X_r}(x,n)$,
O_{X_r} is the unique orthogonal function. For $2x2$ matrix, we can write

$$B = A^{-1} + \begin{pmatrix} 10 \\ 01 \end{pmatrix} * \begin{pmatrix} V_{V_1} * C_n(x) + V_{V_2} * S_n(x) \\ V_{V_3} * C_n(x) + V_{V_4} * S_n(x) \end{pmatrix} \text{ then}$$

$$B = A^{-1} + \begin{pmatrix} V_{V_1} * C_n(x) + V_{V_2} * S_n(x) 0 \\ 0 \qquad V_{V_3} * C_n(x) + V_{V_4} * S_n(x) \end{pmatrix}$$

on solving these 4 variables and multiplying common terms, we get

$$b_1 = K * a_7 - M * a_4,\ b_2 = K * a_8 + M * a_3,\ b_3 = -K * L * a_3,\ b_4 = -K * L * a_4 \quad (222)$$

$$b_5 = -K * L * a_5,\ b_6 = -K * L * a_6,\ b_7 = L * a_1 - M * a_6,\ b_8 = L * a_2 + M * a_8 \quad (223)$$

$$K = a_1 * a_4 - a_2 * a_3,\ L = a_5 * a_8 - a_7 * a_6,\ M = (a_1 * a_7 + a_2 * a_8) - (a_3 * a_5 + a_4 * a_6) \quad (224)$$

and hence we get the following series

$$\begin{pmatrix} F_N(x) \\ F_D(x) \end{pmatrix} = \begin{pmatrix} N_{a_0} \\ D_{a_0} \end{pmatrix} + \sum_{n=1}^{L-1} A_n(x) * \begin{pmatrix} N_{a_n} \\ D_{a_n} \end{pmatrix} + \sum_{n=1}^{L-1} B_n(x) * \begin{pmatrix} N_{b_n} \\ D_{b_n} \end{pmatrix} \text{ where} \quad (225)$$

$$\begin{pmatrix} N_{a_0} \\ D_{a_0} \end{pmatrix} = \begin{pmatrix} \sum_{x=1}^{L} F(x) \\ \sum_{x=1}^{L} G(x) \end{pmatrix} \quad (226)$$

$$\begin{pmatrix} k_{a_n} \\ l_{a_n} \end{pmatrix} = \begin{pmatrix} \sum_{x=1}^{L} (a_1 * F_N(x) + a_3 * F_D(x)) * C_n(x) + (a_2 * F_N(x) + a_4 * F_D(x)) * S_n(x) \\ \sum_{x=1}^{L} (a_5 * F_N(x) + a_7 * F_D(x)) * C_n(x) + (a_6 * F_N(x) + a_8 * F_D(x)) * S_n(x) \end{pmatrix} \quad (227)$$

$$\begin{pmatrix} N_{a_n} \\ D_{a_n} \end{pmatrix} = \begin{pmatrix} (a_3 * a_5 + a_4 * a_6 + {a_7}^2 + {a_8}^2) * k_{a_n} - (a_1 * a_3 + a_2 * a_4 + a_3 * a_7 + a_4 * a_8) * l_{a_n} \\ (a_1 * a_5 + a_2 * a_6 + a_7 * a_5 + a_8 * a_6) * k_{a_n} - ({a_1}^2 + {a_2}^2 + a_3 * a_5 + a_4 * a_6) * l_{a_n} \end{pmatrix}$$

(228)

$$\begin{pmatrix} k_{b_n} \\ l_{b_n} \end{pmatrix} = \begin{pmatrix} \sum_{x=1}^{L} (b_1 * F_N(x) + b_3 * F_D(x)) * C_n(x) + (b_2 * F_N(x) + b_4 * F_D(x)) * S_n(x) \\ \sum_{x=1}^{L} (b_5 * F_N(x) + b_7 * F_D(x)) * C_n(x) + (b_6 * F_N(x) + b_8 * F_D(x)) * S_n(x) \end{pmatrix} \quad (229)$$

$$\begin{pmatrix} N_{b_n} \\ D_{b_n} \end{pmatrix} = \begin{pmatrix} (b_3 * b_5 + b_4 * b_6 + {b_7}^2 + {b_8}^2) * k_{b_n} - (b_1 * a_3 + b_2 * b_4 + b_3 * b_7 + b_4 * b_8) * l_{b_n} \\ (b_1 * b_5 + b_2 * b_6 + b_7 * b_5 + b_8 * b_6) * k_{b_n} - ({b_1}^2 + {b_2}^2 + b_3 * b_5 + b_4 * b_6) * l_{b_n} \end{pmatrix} \quad (230)$$

$$C_n(x) = \cos\left(\frac{2*n*\pi*(x-1)}{L}\right) \quad (231)$$

$$S_n(x) = \sin\left(\frac{2*n*\pi*(x-1)}{L}\right) \quad (232)$$

$$A_n(x) = \begin{pmatrix} a_1 * C_n(x) + a_2 * S_n(x) a_5 * C_n(x) + a_6 * S_n(x) \\ a_3 * C_n(x) + a_4 * S_n(x) a_7 * C_n(x) + a_8 * S_n(x) \end{pmatrix}, \quad (233)$$

$$B_n(x) = \begin{pmatrix} b_1 * C_n(x) + b_2 * S_n(x) b_5 * C_n(x) + b_6 * S_n(x) \\ b_3 * C_n(x) + b_4 * S_n(x) b_7 * C_n(x) + b_8 * S_n(x) \end{pmatrix}, \quad (234)$$

$$b_1 = K * a_7 - M * a_4,\ b_2 = K * a_8 + M * a_3,\ b_3 = -K * L * a_3,\ b_4 = -K * L * a_4 \quad (235)$$

$$b_5 = -K * L * a_5,\ b_6 = -K * L * a_6,\ b_7 = L * a_1 - M * a_6,\ b_8 = L * a_2 + M * a_8 \quad (236)$$

$$K = a_1 * a_4 - a_2 * a_3,\ L = a_5 * a_8 - a_7 * a_6,\ M = (a_1 * a_7 + a_2 * a_8) - (a_3 * a_5 + a_4 * a_6) \quad (237)$$

J. Derivation with Matrix Harmonic Continuous Function

In similar to the discrete function, the same property applies for continuous functions. Let us take the example of continuous functions with 2 X 2 matrix. Let $F_N(x), F_D(x)$ are periodic between $-\frac{L}{2}$ and $\frac{L}{2}$, then

$$\begin{pmatrix} F_N(x) \\ F_D(x) \end{pmatrix} = \begin{pmatrix} N_{a_0} \\ D_{a_0} \end{pmatrix} + \sum_{n=1}^{\infty} A_n(x) * \begin{pmatrix} N_{a_n} \\ D_{a_n} \end{pmatrix} + \sum_{n=1}^{\infty} B_n(x) * \begin{pmatrix} N_{b_n} \\ D_{b_n} \end{pmatrix} \text{ where} \tag{238}$$

$$\begin{pmatrix} N_{a_0} \\ D_{a_0} \end{pmatrix} = \begin{pmatrix} \int_{-\frac{L}{2}}^{\frac{L}{2}} (F(x)) * dx \\ \int_{-\frac{L}{2}}^{\frac{L}{2}} (G(x)) * dx \end{pmatrix} \tag{239}$$

$$\begin{pmatrix} k_{a_n} \\ l_{a_n} \end{pmatrix} = \begin{pmatrix} \int_{-\frac{L}{2}}^{\frac{L}{2}} ((a_1 * F_N(x) + a_3 * F_D(x)) * C_n(x) + (a_2 * F_N(x) + a_4 * F_D(x)) * S_n(x)) * dx \\ \int_{-\frac{L}{2}}^{\frac{L}{2}} ((a_5 * F_N(x) + a_7 * F_D(x)) * C_n(x) + (a_6 * F_N(x) + a_8 * F_D(x)) * S_n(x)) * dx \end{pmatrix} \tag{240}$$

$$\begin{pmatrix} N_{a_n} \\ D_{a_n} \end{pmatrix} = \begin{pmatrix} (a_3 * a_5 + a_4 * a_6 + a_7{}^2 + a_8{}^2) * k_{a_n} - (a_1 * a_3 + a_2 * a_4 + a_3 * a_7 + a_4 * a_8) * l_{a_n} \\ (a_1 * a_5 + a_2 * a_6 + a_7 * a_5 + a_8 * a_6) * k_{a_n} - (a_1{}^2 + a_2{}^2 + a_3 * a_5 + a_4 * a_6) * l_{a_n} \end{pmatrix} \tag{241}$$

$$\begin{pmatrix} k_{b_n} \\ l_{b_n} \end{pmatrix} = \begin{pmatrix} \int_{-\frac{L}{2}}^{\frac{L}{2}} ((b_1 * F_N(x) + b_3 * F_D(x)) * C_n(x) + (b_2 * F_N(x) + b_4 * F_D(x)) * S_n(x)) * dx \\ \int_{-\frac{L}{2}}^{\frac{L}{2}} ((b_5 * F_N(x) + b_7 * F_D(x)) * C_n(x) + (b_6 * F_N(x) + b_8 * F_D(x)) * S_n(x)) * dx \end{pmatrix} \tag{242}$$

$$\begin{pmatrix} N_{b_n} \\ D_{b_n} \end{pmatrix} = \begin{pmatrix} (b_3 * b_5 + b_4 * b_6 + b_7{}^2 + b_8{}^2) * k_{b_n} - (b_1 * a_3 + b_2 * b_4 + b_3 * b_7 + b_4 * b_8) * l_{b_n} \\ (b_1 * b_5 + b_2 * b_6 + b_7 * b_5 + b_8 * b_6) * k_{b_n} - (b_1{}^2 + b_2{}^2 + b_3 * b_5 + b_4 * b_6) * l_{b_n} \end{pmatrix} \tag{243}$$

$$C_n(x) = \cos\left(\frac{2*n*\pi*(x-1)}{L}\right) \tag{244}$$

$$S_n(x) = \sin\left(\frac{2*n*\pi*(x-1)}{L}\right) \tag{245}$$

$$A_n(x) = \begin{pmatrix} a_1 * C_n(x) + a_2 * S_n(x) a_5 * C_n(x) + a_6 * S_n(x) \\ a_3 * C_n(x) + a_4 * S_n(x) a_7 * C_n(x) + a_8 * S_n(x) \end{pmatrix}, \tag{246}$$

$$B_n(x) = \begin{pmatrix} b_1 * C_n(x) + b_2 * S_n(x) b_5 * C_n(x) + b_6 * S_n(x) \\ b_3 * C_n(x) + b_4 * S_n(x) b_7 * C_n(x) + b_8 * S_n(x) \end{pmatrix}, \tag{247}$$

$$b_1 = K * a_7 - M * a_4,\ b_2 = K * a_8 + M * a_3,\ b_3 = -K * L * a_3,\ b_4 = -K * L * a_4 \tag{248}$$

$$b_5 = -K * L * a_5,\ b_6 = -K * L * a_6,\ b_7 = L * a_1 - M * a_6,\ b_8 = L * a_2 + M * a_8 \tag{249}$$

$$K = a_1 * a_4 - a_2 * a_3,\ L = a_5 * a_8 - a_7 * a_6,\ M = (a_1 * a_7 + a_2 * a_8) - (a_3 * a_5 + a_4 * a_6) \tag{250}$$

and one can get for both continuous and discrete harmonic functions by, if

$$\begin{pmatrix} F_N(x) \\ F_D(x) \end{pmatrix} = \begin{pmatrix} G_N(x) \\ G_D(x) \end{pmatrix}, \text{ then, harmonic property is } \frac{F_N(x)}{F_D(x)} = \frac{G_N(x)}{G_D(x)} \tag{251}$$

K. Derivation with Combined Functions for both Discrete or Continuous of Matrix, Exponential, Sign, Polynomial Functions

We can also combine, all the functions with matrix harmonic functions for both discrete and continuous functions as follows.

$$\begin{pmatrix} F_N(x) \\ F_D(x) \end{pmatrix} = \begin{pmatrix} N_{a_0} \\ D_{a_0} \end{pmatrix} + \sum_{n=1}^{\infty} A_n(x) * \begin{pmatrix} N_{a_n} \\ D_{a_n} \end{pmatrix} + \sum_{n=1}^{\infty} B_n(x) * \begin{pmatrix} N_{b_n} \\ D_{b_n} \end{pmatrix} \tag{252}$$

Here $A_n(x), B_n(x)$ are same matrix as when we used to find harmonic functions, but instead of treating, $\begin{pmatrix} N_{a_r} \\ D_{a_r} \end{pmatrix}$ as constants, we need to consider them as the combined functions. Then we need to find co-efficient of combined functions with the same approach of first we need to eliminate by multiplying ${r_m}^{-x}$ and then with cumulative sum or multiple integrals with the combination of k_m as explained in the polynomial approach to find the polynomial co-efficient. Since $Sign$ functions are orthogonal, we can just multiply $Sign$ function to get co-efficient of $Sign$ functions. Thus we could get all the co-efficient of combined functions. Since $A_n(x), B_n(x)$ has 8 unknown variables where in 2 will go for scaling and other 6 unknown variables will decide the multi-variant of functions such as stock market, gold rate and environmental conditions.

III. DERIVATION OF HIGH PRECISION CURVE

A. Derivation of Coil Curve for Discrete Functions

Let us assume a function like

$$f(x) = \frac{\left(a(x)*(p(x))^2+b(x)*(q(x))^2\right)}{\left(c(x)*(p(x))^2+d(x)*(q(x))^2\right)} = \frac{(a*p^2+b*q^2)}{(c*p^2+d*q^2)} \text{ Then} \tag{253}$$

$$f'(x) = \frac{\left((2*a*p*p'+a'*p^2+2*b*q*q'+b'*q^2)*(c*p^2+d*q^2)\right)-\left((a*p^2+b*q^2)*(2*c*p*p'+c'*p^2+2*d*q*q'+d'*q^2)\right)}{(c*p^2+d*q^2)^2} \tag{254}$$

$$= \frac{\left(\left(p*q*(2*(b*c-a*d)*(p*q'-p'*q)+(b'*c+a'*d-b*c'-a*d\prime)*p*q)\right)\right)+\left((a'*c-a*c')*p^4+(b'*d-b*d')*q^4\right)}{(c*p^2+d*q^2)^2}$$

if we solve $a'*c-a*c'=0$ and $b'*d-b*d'=0$, we get $a=V*c$ and $b=W*d$ where V and W are constants, then this function will have maximum or minimum at $p=0$ or $q=0 => p(x)=0$ or $q(x)=0$. Also one can prove that if $c(x)*(p(x))^2+d(x)*(q(x))^2 \neq 0$, then $f(x)$ is continuous, which means, $c(x)*(p(x))^2+d(x)*(q(x))^2$ can be 0 only when $\frac{(p(x))^2}{(q(x))^2} = -\frac{d(x)}{c(x)}$. This leads to if $c(x)*d(x)>0$, then $\frac{(p(x))^2}{(q(x))^2}$ will never exists as it is negative. Hence $c(x)*(p(x))^2+d(x)*(q(x))^2 \neq 0$ when $c(x)*d(x)>0$. On substituting $a(x)=V*c(x)$ and $b(x)=W*d(x)$, let us rewrite $f(x)$ to another form.

$$f(x) = f_0(x) = \frac{\left(a(x)*R*(p(x))^2+b(x)*S*(q(x))^2\right)}{\left(a(x)*T*(p(x))^2+b(x)*U*(q(x))^2\right)} \text{ then} \tag{255}$$

if $(a(x)*b(x)*T*U)>0$, then $f(x)$ is continuous and also has maximum value and minimum value when $p(x)=0$ or $q(x)=0$ respectively. Let $f_1(x)$ has values of $\frac{gmaxN}{gmaxD}$ as maximum and $\frac{gminN}{gminD}$ as minimum between $xLow$ and $xHigh$.

On considering $a(x)=\frac{gmaxN}{gmaxD}-f_1(x)$, $b(x)=f_1(x)-\frac{gminN}{gminD}$, and T,U are positive constants, then $(a(x)*b(x)*T*U)>0$ between $xLow$ and $xHigh$.

Let $f(x)$ has values of $\frac{fmaxN}{fmaxD}$ as maximum at $Xmax$ and $\frac{fminN}{fminD}$ as minimum at $Xmin$ between $xLow$ and $xHigh$.

Then $p(x)=x-Xmin, q(x)=x-Xmax$ is the minimal polynomial satisfying $p(x)=0$ and $q(x)=0$. Hence,

$$f_0(x) = \frac{\left(\left(\frac{gmaxN}{gmaxD}-f_1(x)\right)*R*(x-Xmin)^2\right)+\left(\left(f_1(x)-\frac{gminN}{gminD}\right)*S*(x-Xmax)^2\right)}{\left(\left(\frac{gmaxN}{gmaxD}-f_1(x)\right)*T*(x-Xmin)^2\right)+\left(\left(f_1(x)-\frac{gminN}{gminD}\right)*U*(x-Xmax)^2\right)} \tag{256}$$

when $x=Xmin, f(Xmin)=\frac{S}{U}$, hence $S=A*fminN$ and $U=A*fminD$ where $A\neq 0$ and when $x=Xmax, f(Xmax)=\frac{R}{T}$, hence $R=B*fmaxN$ and $T=B*fmaxD$ where $B\neq 0$ and hence

$$f_0(x) = \frac{\left(\left(\frac{gmaxN}{gmaxD}-f_1(x)\right)*B*fmaxN*(x-Xmin)^2\right)+\left(\left(f_1(x)-\frac{gminN}{gminD}\right)*A*fminN*(x-Xmax)^2\right)}{\left(\left(\frac{gmaxN}{gmaxD}-f_1(x)\right)*B*fmaxD*(x-Xmin)^2\right)+\left(\left(f_1(x)-\frac{gminN}{gminD}\right)*A*fminD*(x-Xmax)^2\right)} \tag{257}$$

if $\frac{K(x)}{L(x)}=\frac{M}{N}$, then minimum error solving will be (258)

$\sum_{x=xLow}^{xHigh}\left(M-K(x)\right)^2+\left(N-L(x)\right)^2+\left(Z(x)*\left(M*L(x)-N*K(x)\right)\right)^2=0$, where

$Z(x)$ will decide the deviation of curve and we can assume 0 for less computation

Let us assume, approximately $f_1(x)=$ average of $\frac{gminN}{gminD},\frac{gmaxN}{gmaxD}$ then approximately

$f_0(x)=\frac{fN_0(x)}{fD_0(x)}=\frac{\left(B*fmaxN*(x-Xmin)^2\right)+\left(A*fminN*(x-Xmax)^2\right)}{\left(B*fmaxD*(x-Xmin)^2\right)+\left(A*fminD*(x-Xmax)^2\right)}$ By using (258) leads to (259)

$K=fN_0(x), L=fD_0(x), M=(B*fmaxN*(x-Xmin)^2)+(A*fminN*(x-Xmax)^2)$,

$N=(B*fmaxD*(x-Xmin)^2)+(A*fminD*(x-Xmax)^2)$ on solving using (258) we assume

$B=A=1$ orformoreaccurarcy, weget $B=N1*D2-D1*ND$, $A=D1*N2-N1*ND$ where (260)

$N1 = \sum_{x=xLow}^{xHigh} (fN_0(x) * fMaxN + fD_0(x) * fMaxD) * (x - Xmin)^2$,
$D1 = \sum_{x=xLow}^{xHigh} (fN_0(x) * fMinN + fD_0(x) * fMinD) * (x - Xmax)^2$,
$ND = \sum_{x=xLow}^{xHigh} (fMaxN * fMinN + fMaxD * fMinD) * \left((x - Xmin) * (x - xMax)\right)^2$,
$N2 = \sum_{x=xLow}^{xHigh} (fMaxN^2 + fMaxD^2) * (x - Xmin)^4$,
$D2 = \sum_{x=xLow}^{xHigh} (fMinN^2 + fMinD^2) * (x - Xmax)^4$, then
If we solve $f_1(x)$ in terms of $f_0(x)$ from the equation reference(258), (257), we get

$$f_1(x) = \frac{\left(C*\frac{gmaxN}{gmaxD}*B*(x-Xmin)^2\right)+\left(D*\frac{gminN}{gminD}*A*(x-Xmax)^2\right)}{(C*B*(x-Xmin)^2)+(D*A*(x-Xmax)^2)} \tag{261}$$

where $C = (-f_0(x) * fmaxD) + (fmaxN), D = (f_0(x) * fminD) - (fminN)$
to simplify let us also consider, $gminN, gminD, gmaxN, gmaxD$
assame as $fminN, fminD, fmaxN, fmaxD$. Then, we can rewrite,

$$f_1(x) = \frac{\left(\left(\frac{fmaxN}{fmaxD}-f_0(x)\right)*B*fmaxN*(x-Xmin)^2\right)+\left(\left(f_0(x)-\frac{fminN}{fminD}\right)*A*fminN*(x-Xmax)^2\right)}{\left(\left(\frac{fmaxN}{fmaxD}-f_0(x)\right)*B*fmaxD*(x-Xmin)^2\right)+\left(\left(f_0(x)-\frac{fminN}{fminD}\right)*A*fminD*(x-Xmax)^2\right)} \tag{262}$$

Hence $f_1(x)$ will have maximum and minimum other than $Xmax$ and $Xmin$ because when $x = Xmax$ and $x = Xmin, f_1(x) = \frac{0}{0}$

Hence Let us take derivative of $f_1(x)$ to see when it approaches $x = Xmax$ and $x = Xmin$

$$f_1'(x) = \frac{\left(A*B*K*\left(\left(\frac{fmaxN}{fmaxD}-f_0(x)\right)*\left(f_0(x)-\frac{fminN}{fminD}\right)*L-f'(x)*M*K\right)\right)}{\left(\left(\frac{fmaxN}{fmaxD}-f_0(x)\right)*B*fmaxD*(x-Xmin)^2+\left(f_0(x)-\frac{fminN}{fminD}\right)*A*fminD*(x-Xmax)^2\right)^2} \tag{263}$$

where $K = (x - Xmin) * (x - Xmax)$,
$L = 2 * (fmaxN * fminD - fmaxD * fminN) * (Xmax - Xmin)$,
$M = \frac{(fmaxN*fminD-fmaxD*fminN)^2}{(fmaxD*fminD)}$,
On dividing $K^2 = (x - Xmin)^2 * (x - Xmax)^2$ on Numerator and denominator, we get,

$$= \frac{A*B*\left(\frac{\left(\frac{fmaxN}{fmaxD}-f_0(x)\right)}{(x-Xmax)}*\frac{\left(f_0(x)-\frac{fminN}{fminD}\right)}{(x-Xmin)}*L-f'(x)*M\right)}{\left(\frac{\left(\frac{fmaxN}{fmaxD}-f_0(x)\right)}{(x-Xmax)}*B*fmaxD*(x-Xmin)+\frac{\left(f_0(x)-\frac{fminN}{fminD}\right)}{(x-Xmin)}*A*fminD*(x-Xmax)\right)^2} \tag{264}$$

Hence $f_1'(x)$ will have maximum and minimum other than $(x = Xmax)$ and $(x = Xmin)$ because when $x = Xmax$ and $x = Xmin, f_1'(x) = \frac{0}{0}$ and approaches $\frac{f'max'N}{(f'max'D)}$ and $\frac{f'min'N}{(f'min'D)}$ respectively, and we can take $f_1(x)$ to take next maximum and minimum value. Hence compute $f_1(x)$ for all values except when $x = Xmax$ and $x = Xmin$. Get the maximum and minimum of the rest of the values. Let it has values of $\frac{f_1MaxN}{f_1MaxD}$ as maximum and $\frac{f_1MinN}{f_1MinD}$ as minimum at when $x = x_1max$ and $x = x_1min$, and let

$$g_1H = \frac{f_1MaxN}{f_1MaxD} - f_2(x), g_1L = f_2(x) - \frac{f_1MaxN}{f_1MaxD}, x_1H = (x - x_1max)^2, x_1L = (x - x_1min)^2, \text{ andif} \tag{265}$$

we get B_1, A_1 using (258), (258)(260) then

$$f_1(x) = \frac{g_1H*f_1MaxN*x_1L*B_1+g_1L*f_1MinN*x_1H*A_1}{g_1H*f_1MaxD*x_1L*B_1+g_1L*f_1MinD*x_1H*A_1} \tag{266}$$

$$f_1H = \frac{f_1MaxN}{f_1MaxD} - f_1(x), f_1L = f_1(x) - \frac{f_1MinN}{f_1MinD}, \text{ then} \tag{267}$$

$$f_2(x) = \frac{f_1H*f_1MaxN*x_1L*B_1+f_1L*f_1MinN*x_1H*A_1}{f_1H*f_1MaxD*x_1L*B_1+f_1L*f_1MinD*x_1H*A_1} \tag{268}$$

Here we won't be able to compute values for taken maximum and minimum values at $Xmin, Xmax, x_1max, x_1min$ and so on. Hence compute values of $f_r(x)$ for all x except for already found maximum and minimum locations, and Let

$$x_{(r-1)}H = \left(x - x_{(r-1)}max\right)^2, x_{(r-1)}L = \left(x - x_{(r-1)}min\right)^2, f_{(r-1)}H = \frac{f_{(r-1)}MaxN}{f_{(r-1)}MaxD} - f_{(r-1)}(x), f_{(r-1)}L = f_{(r-1)}(x) - \frac{f_{(r-1)}MinN}{f_{(r-1)}MinD},$$

We need to get $B_{(r-1)}, A_{(r-1)}$ using $(258), (260)$ then

$$f_r(x) = \frac{f_{(r-1)}H*f_{(r-1)}MaxN*x_{(r-1)}L*B_{(r-1)}+f_{(r-1)}L*f_{(r-1)}MinN*x_{(r-1)}H*A_{(r-1)}}{f_{(r-1)}H*f_{(r-1)}MaxD*x_{(r-1)}L*B_{(r-1)}+f_{(r-1)}L*f_{(r-1)}MinD*x_{(r-1)}H*A_{(r-1)}} \text{ and} \quad (269)$$

$$f_0(x) = f(x) \quad (270)$$

Recursively repeat until either one value or two values are left out, If it is left by one value then that is the constant function. If there are two values left, then it is average of these two values. If in case, there exists same maximum or minimum for more than one location,

then minimal polynomial satisfying $p(x) = 0$ and $q(x) = 0$ will be $\prod_{t=1}^{m} (x - xMax_t)$ and $\prod_{t=1}^{n} (x - xMin_t)$ respectively, and then we need to apply this to the equation and you will get following recursive relations. Let,

$$x_{(r-1)}H = \prod_{t=1}^{m_{(r-1)}} \left(x - x_{(r-1)}max_t\right)^2, \quad (271)$$

$$x_{(r-1)}L = \prod_{t=1}^{n_{(r-1)}} \left(x - x_{(r-1)}min_t\right)^2, \quad (272)$$

$$f_{(r-1)}H = \frac{f_{(r-1)}MaxN}{f_{(r-1)}MaxD} - f_{(r-1)}(x), \quad (273)$$

$$f_{(r-1)}L = f_{(r-1)}(x) - \frac{f_{(r-1)}MinN}{f_{(r-1)}MinD}, \quad (274)$$

We need to get $B_{(r-1)}, A_{(r-1)}$ using $(258), (260)$ then (275)

$$f_r(x) = \frac{f_{(r-1)}H*f_{(r-1)}MaxN*x_{(r-1)}L*B_{(r-1)}+f_{(r-1)}L*f_{(r-1)}MinN*x_{(r-1)}H*A_{(r-1)}}{f_{(r-1)}H*f_{(r-1)}MaxD*x_{(r-1)}L*B_{(r-1)}+f_{(r-1)}L*f_{(r-1)}MinD*x_{(r-1)}H*A_{(r-1)}} \text{ and} \quad (276)$$

$$f_0(x) = f(x) \quad (277)$$

Once we found all $f_r(x)$ functions using this recursive relation, then,
we need to obtain reverse recursion to get as following

$$g_{(r)}H = \frac{f_{(r)}MaxN}{f_{(r)}MaxD} - f_{(r+1)}(x), \quad (278)$$

$$g_{(r)}L = f_{(r+1)}(x) - \frac{f_{(r)}MinN}{f_{(r)}MinD}, \quad (279)$$

$$f_r(x) = \frac{g_{(r)}H*f_{(r)}MaxN*x_{(r)}L*B_{(r)}+g_{(r)}L*f_{(r)}MinN*x_{(r)}H*A_{(r)}}{g_{(r)}H*f_{(r)}MaxD*x_{(r)}L*B_{(r)}+g_{(r)}L*f_{(r)}MinD*x_{(r)}H*A_{(r)}} \text{ and} \quad (280)$$

$$f_0(x) = f(x) = \frac{g_0H*f_0MaxN*x_0L*B_0+g_0L*f_0MinN*x_0H*A_0}{g_0H*f_0MaxD*x_0L*B_0+g_0L*f_0MinD*x_0H*A_0} \quad (281)$$

The same function can be extended for multi-dimension also, Let there are d dimensions, and then minimal polynomial satisfying $p(x_1, x_2, x_3, \ldots, x_d)^2 = 0$ and $q(x_1, x_2, x_3, \ldots, x_d)^2 = 0$ will be $\prod_{t=1}^{m} (\sum_{s=1}^{d} (x_s - x_sMax_t)^2)$ and $\prod_{t=1}^{n} (\sum_{s=1}^{d} (x_s - x_sMin_t)^2)$ respectively. Hence Let,

$$x_{(r-1)}H = \prod_{t=1}^{m_{(r-1)}} \left(\sum_{s=1}^{d} \left(x_s - x_{(r-1)_s}max_t\right)^2\right), \quad (282)$$

$$x_{(r-1)}L = \prod_{t=1}^{n_{(r-1)}} \left(\sum_{s=1}^{d} \left(x_s - x_{(r-1)_s}min_t\right)^2\right), \quad (283)$$

$$f_{(r-1)}H = \frac{f_{(r-1)}MaxN}{f_{(r-1)}MaxD} - f_{(r-1)}(x_1, x_2, x_3, \ldots, x_d), \quad (284)$$

We need to get $B_{(r-1)}, A_{(r-1)}$ using $(258), (260)$ then (285)

$$f_{(r-1)}L = f_{(r-1)}(x_1, x_2, x_3, \ldots, x_d) - \frac{f_{(r-1)}MinN}{f_{(r-1)}MinD}, \quad (286)$$

$$f_r = f_r(x_1, x_2, x_3, \ldots, x_d) \quad (287)$$

$$f_r = \frac{f_{(r-1)}H*f_{(r-1)}MaxN*x_{(r-1)}L*B_{(r-1)}+f_{(r-1)}L*f_{(r-1)}MinN*x_{(r-1)}H*A_{(r-1)}}{f_{(r-1)}H*f_{(r-1)}MaxD*x_{(r-1)}L*B_{(r-1)}+f_{(r-1)}L*f_{(r-1)}MinD*x_{(r-1)}H*A_{(r-1)}} \text{ and} \quad (288)$$

$$f_0 = f_0(x_1, x_2, x_3, \ldots, x_d) = f(x_1, x_2, x_3, \ldots, x_d) \quad (289)$$

Once we found all $f_r(x_1, x_2, x_3, \ldots, x_d)$ functions using this recursive relation,
Recursively repeat until either one value or two values are left out, If it is left by one value then that is the constant function. If there are two values left, then it is average of these two values and we need to obtain reverse recursion to get as following

$$g_{(r)}H = \frac{f_{(r)}MaxN}{f_{(r)}MaxD} - f_{(r+1)}(x_1, x_2, x_3, \ldots, x_d), \quad (290)$$

$$g_{(r)}L = f_{(r+1)}(x_1, x_2, x_3, \ldots, x_d) - \frac{f_{(r)}MinN}{f_{(r)}MinD}, \tag{291}$$

$$f_r = f_r(x_1, x_2, x_3, \ldots, x_d) \tag{292}$$

$$f_r = \frac{g_{(r)}H * f_{(r)}MaxN * x_{(r)}L * B_{(r)} + g_{(r)}L * f_{(r)}MinN * x_{(r)}H * A_{(r)}}{g_{(r)}H * f_{(r)}MaxD * x_{(r)}L * B_{(r)} + g_{(r)}L * f_{(r)}MinD * x_{(r)}H * A_{(r)}} \text{ and} \tag{293}$$

$$f_0 = f_0(x_1, x_2, x_3, \ldots, x_d) = f(x_1, x_2, x_3, \ldots, x_d) \tag{294}$$

$$f_0 = \frac{g_0H * f_0MaxN * x_0L * B_0 + g_0L * f_0MinN * x_0H * A_0}{g_0H * f_0MaxD * x_0L * B_0 + g_0L * f_0MinD * x_0H * A_0} \tag{295}$$

Let us explain the concept with single dimension, Let the function be, $f(x) = |\sqrt{25 - x^2}|$

Let us take discrete values from this continuous function as $f_0(x) = f(x) = f(-5) = 0, f(-4) = 3, f(-3) = 4, f(0) = 5, f(3) = 4, f(4) = 3, f(5) = 0$

Here maximum is at 0 with value of 5 and minimum at both $-5,5$ with value of 0

$B_0 = 6876157,\ A_0 = 439192749$ using the equation reference (276) We need to calculate values other than $-5,0,5$ for the following function.

$f_1(x) = \frac{(5-f_0(x))*5*6876157*((x+5)*(x-5))^2}{(5-f_0(x))*1*6876157*((x+5)*(x-5))^2+(f_0(x)-0)*1*439192749*(x-0)^2}$,Then $f_1(-4) = f_1(4) = \frac{2784843585}{11097594693}, f_1(-3) = f_1(3) = \frac{2200370240}{4392808789}$ Since only two values are left, we need to take average of the maximum and minimum value. So $f_2(x) = \frac{\frac{2784843585}{11097594693}+\frac{2200370240}{4392808789}}{2}$, if in case it is left by one value, then $f_2(x) = leftvalue$, Hence using the equation reference (280), we need to recur back as following methods.

$$f_1(x) = \frac{\left(\frac{2200370240}{4392808789} - \frac{\frac{2784843585}{11097594693}+\frac{2200370240}{4392808789}}{2}\right) * 2200370240 * ((x+4)*(x-4))^2 + \left(\frac{\frac{2784843585}{11097594693}+\frac{2200370240}{4392808789}}{2} - \frac{2784843585}{11097594693}\right) * 2784843585 * ((x+3)*(x-3))^2}{\left(\frac{2200370240}{4392808789} - \frac{\frac{2784843585}{11097594693}+\frac{2200370240}{4392808789}}{2}\right) * 4392808789 * ((x+4)*(x-4))^2 + \left(\frac{\frac{2784843585}{11097594693}+\frac{2200370240}{4392808789}}{2} - \frac{2784843585}{11097594693}\right) * 11097594693 * ((x+3)*(x-3))^2}$$

$$= \frac{2200370240 * ((x+4)*(x-4))^2 + 2784843585 * ((x+3)*(x-3))^2}{4392808789 * ((x+4)*(x-4))^2 + 11097594693 * ((x+3)*(x-3))^2}$$

$$f_0(x) = \frac{(5 - f_1(x)) * 5 * 6876157 * ((x+5)*(x-5))^2}{(5 - f_1(x)) * 1 * 6876157 * ((x+5)*(x-5))^2 + (f_1(x) - 0) * 1 * 439192749 * (x-0)^2} =$$

$$= f(x)$$

$$= \frac{(3952734741 * ((x+4)*(x-4))^2 + 10540625976 * ((x+3)*(x-3))^2) * 5 * ((x+5)*(x-5))^2}{(3952734741 * ((x+4)*(x-4))^2 + 10540625976 * ((x+3)*(x-3))^2) * ((x+5)*(x-5))^2 + (64 * ((x+4)*(x-4))^2 + 81 * ((x+3)*(x-3))^2) * 439192749 * x^2}$$

We also need not to calculate B_0 , A_0 and assumed to be 1, then, same transformation will lead to

$f_1(x) = \frac{((5-f_0(x))*5*((x+5)*(x-5))^2}{((5-f_0(x))*1*((x+5)*(x-5))^2+(f_0(x)-0)*1*(x-0)^2}$, Then $f_1(-4) = f_1(4) = \frac{27}{7}, f_1(-3) = f_1(3) = \frac{320}{73}$

Since only two values are left, we need to take average of the maximum and minimum value. So $f_2(x) = \frac{\frac{320}{73}+\frac{27}{7}}{2}$, if in case it is left by one value, then $f_2(x) = leftvalue$, Hence using the equation reference (280), we need to recur back as following methods.

$$f_1(x) = \frac{\left(\frac{320}{73} - \frac{\frac{320}{73}+\frac{27}{7}}{2}\right) * 320 * ((x+4)*(x-4))^2 + \left(\frac{\frac{320}{73}+\frac{27}{7}}{2} - \frac{27}{7}\right) * 27 * ((x+3)*(x-3))^2}{\left(\frac{320}{73} - \frac{\frac{320}{73}+\frac{27}{7}}{2}\right) * 73 * ((x+4)*(x-4))^2 + \left(\frac{\frac{1280}{73}+\frac{27}{7}}{2} - \frac{27}{7}\right) * 7 * ((x+3)*(x-3))^2}$$

$$= \frac{320 * ((x+4)*(x-4))^2 + 27 * ((x+3)*(x-3))^2}{73 * ((x+4)*(x-4))^2 + 7 * ((x+3)*(x-3))^2}$$

$$f_0(x) = \frac{(5 - f_1(x)) * 5 * ((x+5)*(x-5))^2}{(5 - f_1(x)) * 1 * ((x+5)*(x-5))^2 + (f_1(x) - 0) * 1 * (x-0)^2}$$

$$= f(x)$$

$$= \frac{(45 * ((x+4)*(x-4))^2 + 8 * ((x+3)*(x-3))^2) * 5 * ((x+5)*(x-5))^2}{(45 * ((x+4)*(x-4))^2 + 8 * ((x+3)*(x-3))^2) * 1 * ((x+5)*(x-5))^2 + (320 * ((x+4)*(x-4))^2 + 27 * ((x+3)*(x-3))^2) * 1 * x^2}$$

Hence We could transform discrete values or even the continuous function to another continuous function. Since this function have maximum or minimum at every point, this will be of coily nature.

B. Derivation of Continuous Harmonic Curve for Discrete Function

Similar way to get smoothness of the function we need to go with following identities,

$$\text{if } f(x) = f_0(x) = \frac{(a(x)*R*(|p(x)|)+b(x)*S*(|q(x)|))}{(a(x)*T*(|p(x)|)+b(x)*U*(|q(x)|))} \tag{296}$$

then this function will not have any maximum or minimum when $p = 0$ or when $q = 0$. Similarly

$$\text{if } f(x) = f_0(x) = \frac{\left(a(x)*R*(p(x))^2+b(x)*S*(|q(x)|)\right)}{\left(a(x)*T*(p(x))^2+b(x)*U*(|q(x)|)\right)} \tag{297}$$

then this function will have maximum or minimum when $p = 0$ but not when $q = 0$. To find whether the given point is having maximum or minimum, we can go with the principle of direction between points, If neighboring point has the same direction, then this point will not have maximum or minimum, If neighboring point changes the direction or horizontal, then there is atleast one chance of maximum or minimum value presence. If in case, minimum or maximum value happens to be at the beginning or end of the location, we can assume that it changes it direction. The same concept can be extended for multi-dimension also, Let there are d dimensions, and then minimal polynomial satisfying if the direction at neighboring points changes at maximum and minimum locations, $p(x_1, x_2, x_3, \ldots, x_d)^2 = 0$ and $q(x_1, x_2, x_3, \ldots, x_d)^2 = 0$ will be $\prod_{t=1}^{m} (\sum_{s=1}^{d} (x_s - x_sMax_t)^2)$ and $\prod_{t=1}^{n} (\sum_{s=1}^{d} (x_s - x_sMin_t)^2)$ respectively. If in case one of the value do not change, then $|q(x_1, x_2, x_3, \ldots, x_d)| = 0$ will be $\prod_{t=1}^{n} (\sum_{s=1}^{d} |(x_s - x_sMin_t)|)$. If both of the values do not change their directions, then $|p(x_1, x_2, x_3, \ldots, x_d)| = 0$ and $|q(x_1, x_2, x_3, \ldots, x_d)| = 0$ will be $\prod_{t=1}^{m} (\sum_{s=1}^{d} |(x_s - x_sMax_t)|)$ and $\prod_{t=1}^{n} (\sum_{s=1}^{d} |(x_s - x_sMin_t)|)$ respectively. Hence Let,

$$x_{(r-1)}H = \prod_{t=1}^{m_{(r-1)}} \left(\sum_{s=1}^{d} \left(|x_s - x_{(r-1)_s}max_t|\right)^{p_t}\right) \text{ where } p_t = 2 \text{ if neighboring points direction change, else } p_t = 1, \tag{298}$$

$$x_{(r-1)}L = \prod_{t=1}^{n_{(r-1)}} \left(\sum_{s=1}^{d} \left(|x_s - x_{(r-1)_s}min_t|\right)^{p_t}\right) \text{ where } p_t = 2 \text{ if neighboring points direction change, else } p_t = 1, \tag{299}$$

$$f_{(r-1)}H = \frac{f_{(r-1)}MaxN}{f_{(r-1)}MaxD} - f_{(r-1)}(x_1, x_2, x_3, \ldots, x_d), \tag{300}$$

$$\text{We need to get } B_{(r-1)}, A_{(r-1)} \text{ using } (258), (258)(260) \text{ then} \tag{301}$$
$$= \text{Any arbitrary value, which will decide the degreeof coil or smoothness,}$$

$$f_{(r-1)}L = f_{(r-1)}(x_1, x_2, x_3, \ldots, x_d) - \frac{f_{(r-1)}MinN}{f_{(r-1)}MinD}, \tag{302}$$

$$f_r = f_r(x_1, x_2, x_3, \ldots, x_d) \tag{303}$$

$$f_r = \frac{f_{(r-1)}H*f_{(r-1)}MaxN*x_{(r-1)}L*B_{(r-1)}+f_{(r-1)}L*f_{(r-1)}MinN*x_{(r-1)}H*A_{(r-1)}}{f_{(r-1)}H*f_{(r-1)}MaxD*x_{(r-1)}L*B_{(r-1)}+f_{(r-1)}L*f_{(r-1)}MinD*x_{(r-1)}H*A_{(r-1)}} \text{ and} \tag{304}$$

$$f_0 = f_0(x_1, x_2, x_3, \ldots, x_d) = f(x_1, x_2, x_3, \ldots, x_d) \tag{305}$$

Once we found all $f_r(x_1, x_2, x_3, \ldots, x_d)$ functions using this recursive relation, recursively repeat until either one value or two values are left out, If it is left by one value then that is the constant function. If there are two values left, then it is average of these two values and we need to obtain reverse recursion to get as following

$$g_{(r)}H = \frac{f_{(r)}MaxN}{f_{(r)}MaxD} - f_{(r+1)}(x_1, x_2, x_3, \ldots, x_d), \tag{306}$$

$$g_{(r)}L = f_{(r+1)}(x_1, x_2, x_3, \ldots, x_d) - \frac{f_{(r)}MinN}{f_{(r)}MinD}, \tag{307}$$

$$f_r = f_r(x_1, x_2, x_3, \ldots, x_d) \tag{308}$$

$$f_r = \frac{g_{(r)}H*f_{(r)}MaxN*x_{(r)}L*B_{(r)}+g_{(r)}L*f_{(r)}MinN*x_{(r)}H*A_{(r)}}{g_{(r)}H*f_{(r)}MaxD*x_{(r)}L*B_{(r)}+g_{(r)}L*f_{(r)}MinD*x_{(r)}H*A_{(r)}} \text{ and} \tag{309}$$

$$f_0 = f_0(x_1, x_2, x_3, \ldots, x_d) = f(x_1, x_2, x_3, \ldots, x_d) \tag{310}$$

$$f_0 = \frac{g_0H*f_0MaxN*x_0L*B_0+g_0L*f_0MinN*x_0H*A_0}{g_0H*f_0MaxD*x_0L*B_0+g_0L*f_0MinD*x_0H*A_0} \tag{311}$$

Let us explain the smoothness concept with single dimension with the same function, $f(x) = |\sqrt{25 - x^2}|$ and take same discrete values from this continuous function as $f_0(x) = f(x) = f(-5) = 0, f(-4) = 3, f(-3) = 4, f(0) = 5, f(3) = 4, f(4) = 3, f(5) = 0$. In this function only there is direction change at 0 and there are no other points have direction changes. $B_0 = 367909$, $A_0 = 1015509$ using the equation reference (276) We need to calculate values other than $-5,0,5$ for the following function.

$f_1(x) = \frac{((5-f_0(x))*367909*5*|((x+5)*(x-5))|}{((5-f_0(x))*367909*1*|((x+5)*(x-5))|+(f_0(x)-0)*1015509*1*(x-0)^2}$,Then $f_1(-4) = f_1(4) = \frac{16555905}{27683397}, f_1(-3) = f_1(3) = \frac{7358180}{10611217}$.

Since only two values are left, we need to take average of the maximum and minimum value. So $f_2(x) =$

$\frac{\frac{7358180}{10611217}+\frac{16555905}{27683397}}{2}$, if in case it is left by one value, then $f_2(x) = leftvalue$, Hence using the equation reference (309), we need to recur back as following methods.

$f_1(x)$

$$= \frac{\left(\frac{7358180}{10611217} - \frac{\frac{7358180}{10611217}+\frac{16555905}{27683397}}{2}\right) * 7358180 * |((x+4)*(x-4))| + \left(\frac{\frac{7358180}{10611217}+\frac{16555905}{27683397}}{2} - \frac{16555905}{27683397}\right) * 16555905 * |((x+3)*(x-3))|}{\left(\frac{7358180}{10611217} - \frac{\frac{7358180}{10611217}+\frac{16555905}{27683397}}{2}\right) * 10611217 * |((x+4)*(x-4))| + \left(\frac{\frac{7358180}{10611217}+\frac{16555905}{27683397}}{2} - \frac{16555905}{27683397}\right) * 27683397 * |((x+3)*(x-3))|}$$

$$= \frac{7358180 * |((x+4)*(x-4))| + 16555905 * |((x+3)*(x-3))|}{10611217 * |((x+4)*(x-4))| + 27683397 * |((x+3)*(x-3))|}$$

$$f_0(x) = \frac{(5 - f_1(x)) * 367909 * 5 * |((x+5)*(x-5))|}{(5 - f_1(x)) * 367909 * 1 * |((x+5)*(x-5))| + (f_1(x) - 0) * 1015509 * 1 * (x-0)^2}$$

$= f(x)$

$$= \frac{(9139581 * |((x+4)*(x-4))| + 24372216 * |((x+3)*(x-3))|) * 5 * |((x+5)*(x-5))|}{(9139581 * |((x+4)*(x-4))| + 24372216 * |((x+3)*(x-3))|) * 1 * |((x+5)*(x-5))| + (4 * |((x+4)*(x-4))| + 9 * |((x+3)*(x-3))|) * 1015509 * 1 * x^2}$$

Also let us not calculate $B_0,\ A_0$ and assumed to be 1, then, same transformation will lead to

$f_1(x) = \frac{((5-f_0(x))*5*|((x+5)*(x-5))|}{((5-f_0(x))*1*|((x+5)*(x-5))|+(f_0(x)-0)*1*(x-0)^2}$,Then $f_1(-4) = f_1(4) = \frac{15}{11}, f_1(-3) = f_1(3) = \frac{20}{13}$.

Since only two values are left, we need to take average of the maximum and minimum value. So $f_2(x) = \frac{\frac{20}{13}+\frac{15}{11}}{2}$, if in case it is left by one value, then $f_2(x) = leftvalue$, Hence using the equation reference (309), we need to recur back as following methods.

$$f_1(x) = \frac{\left(\frac{20}{13} - \frac{\frac{20}{13}+\frac{15}{11}}{2}\right) * 20 * |((x+4)*(x-4))| + \left(\frac{\frac{20}{13}+\frac{15}{11}}{2} - \frac{15}{11}\right) * 15 * |((x+3)*(x-3))|}{\left(\frac{20}{13} - \frac{\frac{20}{13}+\frac{15}{11}}{2}\right) * 13 * |((x+4)*(x-4))| + \left(\frac{\frac{20}{13}+\frac{15}{11}}{2} - \frac{15}{11}\right) * 11 * |((x+3)*(x-3))|}$$

$$= \frac{20 * |((x+4)*(x-4))| + 15 * |((x+3)*(x-3))|}{13 * |((x+4)*(x-4))| + 11 * |((x+3)*(x-3))|}$$

$$f_0(x) = \frac{(5 - f_1(x)) * 5 * |((x+5)*(x-5))|}{(5 - f_1(x)) * 1 * |((x+5)*(x-5))| + (f_1(x) - 0) * 1 * (x-0)^2}$$

$$= f(x) = \frac{(9*|((x+4)*(x-4))|+8*|((x+3)*(x-3))|)*5*|((x+5)*(x-5))|}{(9*|((x+4)*(x-4))|+8*|((x+3)*(x-3))|)*1*|((x+5)*(x-5))|+(4*|((x+4)*(x-4))|+3*|((x+3)*(x-3))|)*1*x^2}$$

C. Derivation of High Smooth and High precision Harmonic Curve for Discrete Functions

Since finding A, B is time consuming, let us go back and change the foundation equation as per reference (255),

Let $fN_1(x)$, $fD_1(x)$ has positive values or zero between $xLow$ and $xHigh$. On considering $a(x) = gN_1(x)$, $b(x) = gD_1(x)$ and T, U are positive constants, then $(a(x) * b(x) * T * U) > 0$ between $xLow$ and $xHigh$.

Let $f(x)$ has values of $\frac{fmaxN}{fmaxD}$ as maximum at $Xmax$ and $\frac{fminN}{fminD}$ as minimum at $Xmin$ between $xLow$ and $xHigh$.

Then $p(x) = x - Xmin, q(x) = x - Xmax$ is the minimal polynomial satisfying $p(x) = 0$ and $q(x) = 0$. Hence,

$$f_0(x) = \frac{\left((gN_1(x))*R*(x-Xmin)^2\right)+\left((gD_1(x))*S*(x-Xmax)^2\right)}{\left((gN_1(x))*T*(x-Xmin)^2\right)+\left((gD_1(x))*U*(x-Xmax)^2\right)} \quad (312)$$

when $x = Xmin, f(Xmin) = \frac{S}{U}$, hence $S = A * fminN$ and $U = A * fminD$ where $A \neq 0$ and when $x = Xmax, f(Xmax) = \frac{R}{T}$, hence $R = B * fmaxN$ and $T = B * fmaxD$ where $B \neq 0$ and hence

$$f_0(x) = \frac{\left((gN_1(x))*B*fmaxN*(x-Xmin)^2\right)+\left((gD_1(x))*A*fminN*(x-Xmax)^2\right)}{\left((gN_1(x))*B*fmaxD*(x-Xmin)^2\right)+\left((gD_1(x))*A*fminD*(x-Xmax)^2\right)} \quad (313)$$

Now we can group $\left(gN_1(x)\right) * B = fN_1(x), \left(gD_1(x)\right) * A = fD_1(x)$ and hence there is no need to compute A, B . Let us rewrite $f_0(x)$ as fraction of functions $\frac{gN_0(x)}{gD_0(x)} = \frac{sign(gD_0(x))*gN_0(x)}{|gD_0(x)|} = \frac{fN_0(x)}{fD_0(x)}$,

$$f_0(x) = \frac{fN_0(x)}{fD_0(x)} = \frac{\left((fN_1(x))*fmaxN*(x-Xmin)^2\right)+\left((fD_1(x))*fminN*(x-Xmax)^2\right)}{\left((fN_1(x))*fmaxD*(x-Xmin)^2\right)+\left((fD_1(x))*fminD*(x-Xmax)^2\right)} \text{ which leads to} \quad (314)$$

$$\frac{fN_1(x)}{fD_1(x)} = \frac{\left(fminD*fN_0(x)-fminN*fD_0(x)\right)*(x-Xmax)^2}{\left(fmaxN*fD_0(x)-fmaxD*fN_0(x)\right)*(x-Xmin)^2} \quad (315)$$

Here $\frac{fN_1(x)}{fD_1(x)}$ will have values other than $Xmax$ and $Xmin$ because when $x = Xmax$, and $x = Xmin$, $\frac{fN_1(x)}{fD_1(x)} = \frac{0}{0}$. Hence compute values other than $Xmax$ and $Xmin$ and find the next maximum and minimum values and so on. Like we did in above subsections, we can extend for multi-dimensions. Let there are d dimensions, and then minimal polynomial satisfying if the direction at neighboring points changes at maximum and minimum locations, $p(x_1, x_2, x_3, \ldots, x_d)^2 = 0$ and $q(x_1, x_2, x_3, \ldots, x_d)^2 = 0$ will be $\prod_{t=1}^{m} \left(\sum_{s=1}^{d} (x_s - x_sMax_t)^2\right)$ and $\prod_{t=1}^{n} \left(\sum_{s=1}^{d} (x_s - x_sMin_t)^2\right)$ respectively. If in case one of the value do not change, then $|q(x_1, x_2, x_3, \ldots, x_d)| = 0$ will be $\prod_{t=1}^{n} \left(\sum_{s=1}^{d} |(x_s - x_sMin_t)|\right)$. If both of the values do not change their directions, then $|p(x_1, x_2, x_3, \ldots, x_d)| = 0$ and $|q(x_1, x_2, x_3, \ldots, x_d)| = 0$ will be $\prod_{t=1}^{m} \left(\sum_{s=1}^{d} |(x_s - x_sMax_t)|\right)$ and $\prod_{t=1}^{n} \left(\sum_{s=1}^{d} |(x_s - x_sMin_t)|\right)$ respectively. Hence Let,

$$x_{(r-1)}H = \prod_{t=1}^{m_{(r-1)}} \left(\sum_{s=1}^{d} \left(|x_s - x_{(r-1)_s}max_t|\right)^{p_t}\right) \text{ where } p_t = 2 \text{ if neighboring points direction change, else } p_t = 1, \quad (316)$$

$$x_{(r-1)}L = \prod_{t=1}^{n_{(r-1)}} \left(\sum_{s=1}^{d} \left(|x_s - x_{(r-1)_s}min_t|\right)^{p_t}\right) \text{ where } p_t = 2 \text{ if neighboring points direction change, else } p_t = 1, \quad (317)$$

$$fN_r = fN_r(x_1, x_2, x_3, \ldots, x_d),\ fD_r = fD_r(x_1, x_2, x_3, \ldots, x_d) \quad (318)$$

$$\frac{fN_r}{fD_r} = \frac{\left(f_{(r-1)}MinD*fN_{(r-1)} - f_{(r-1)}MinN*fD_{(r-1)}\right)*x_{(r-1)}H}{\left(f_{(r-1)}MaxN*fD_{(r-1)} - f_{(r-1)}MaxD*fN_{(r-1)}\right)*x_{(r-1)}L} \text{ and} \quad (319)$$

$$\frac{fN_0}{fD_0} = \frac{fN_0(x_1, x_2, x_3, \ldots, x_d)}{fD_0(x_1, x_2, x_3, \ldots, x_d)} = \frac{fN(x_1, x_2, x_3, \ldots, x_d)}{fD(x_1, x_2, x_3, \ldots, x_d)} \quad (320)$$

Once we found all $f_r(x_1, x_2, x_3, \ldots, x_d)$ functions using this recursive relation, recursively repeat until either one value or two values are left out, If it is left by one value then that is the constant function. If there are two values left and assume $\frac{\text{fMaxN}}{\text{fMaxD}}, \frac{\text{fMinN}}{\text{fMinD}}$ then it is $\frac{\text{fMinD}}{\text{fMaxD}}$ and we need to obtain reverse recursion to get as following

$$\frac{fN_r}{fD_r} = \frac{fN_{(r+1)}*f_{(r)}MaxN*x_{(r)}L + fD_{(r+1)}*f_{(r)}MinN*x_{(r)}H}{fN_{(r+1)}*f_{(r)}MaxD*x_{(r)}L*B_{(r)} + fD_{(r+1)}*f_{(r)}MinD*x_{(r)}H} \text{ and} \quad (321)$$

$$\frac{fN_0}{fD_0} = \frac{fN_0(x_1, x_2, x_3, \ldots, x_d)}{fD_0(x_1, x_2, x_3, \ldots, x_d)} = \frac{fN(x_1, x_2, x_3, \ldots, x_d)}{fD(x_1, x_2, x_3, \ldots, x_d)} \quad (322)$$

$$\frac{fN_0}{fD_0} = \frac{fN_{(1)}*f_{(0)}MaxN*x_{(0)}L + fD_{(1)}*f_{(0)}MinN*x_{(0)}H}{fN_{(1)}*f_{(0)}MaxD*x_{(0)}L*B_{(0)} + fD_{(1)}*f_{(0)}MinD*x_{(0)}H} \quad (323)$$

Let us explain the optimal smoothness concept with single dimension with the same function, $f(x) = |\sqrt{25 - x^2}|$ and take same discrete values from this continuous function as $\frac{fN_0(x)}{fD_0(x)} = \frac{f(x)}{1}, fN_0(-5) = 0, fN_0(-4) = 3, fN_0(-3) = 4, fN_0(0) = 5, fN_0(3) = 4, fN_0(4) = 3, fN_0(5) = 0$. In this function only there is direction change at 0 and there are no other points have direction changes.Hence using the equation reference (315) We need to calculate values other than $-5,0,5$ for the following function.

$\frac{fN_1(x)}{fD_1(x)} = \frac{((f_0(x))*(x-0)^2}{((5-f_0(x))*|((x+5)*(x-5))|}$,Then $\frac{fN_1(-4)}{fD_1(-4)} = \frac{fN_1(4)}{fD_1(4)} = \frac{8}{3}, \frac{fN_1(-3)}{fD_1(-3)} = \frac{fN_1(3)}{fD_1(3)} = \frac{9}{4}$.

Since only two values are left, we need to take $\frac{fMinD}{fMaxD}$,So $\frac{fN_2(x)}{fD_2(x)} = \frac{4}{3}$, if in case it is left by one value, then $\frac{fN_2(x)}{fD_2(x)} = leftvalue$, Hence using the equation reference (321), we need to recur back as following methods.

$$\frac{fN_1(x)}{fD_1(x)} = \frac{\left(fN_2(x)\right)*8*(|(x-3)*(x+3)|) + \left(fD_2(x)\right)*9*(|(x-4)*(x+4)|)}{\left(fN_2(x)\right)*3*(|(x-3)*(x+3)|) + \left(fD_2(x)\right)*4*(|(x-4)*(x+4)|)} = \frac{32*(|(x-3)*(x+3)|)+27*(|(x-4)*(x+4)|)}{12*(|(x-3)*(x+3)|)+12*(|(x-4)*(x+4)|)} \text{ and}$$

$$\frac{fN_0(x)}{fD_1(x)} = \frac{\left(fN_1(x)\right)*5*(|(x-5)*(x+5)|)}{\left(fN_1(x)\right)*1*(|(x-5)*(x+5)|) + \left(fD_1(x)\right)*1*((x-0)^2)} = f(x)$$

$$= \frac{\left(32*(|(x-3)*(x+3)|) + 27*(|(x-4)*(x+4)|)\right)*5*(|(x-5)*(x+5)|)}{\left(32*(|(x-3)*(x+3)|) + 27*(|(x-4)*(x+4)|)\right)*1*(|(x-5)*(x+5)|) + \left(12*(|(x-3)*(x+3)|) + 12*(|(x-4)*(x+4)|)\right)*1*((x-0)^2)}$$

If in case, we have approximated with normal polynomial method, then, it would have been termed as

$$f(x)=\frac{-3*(x+5)*(x+3)*(x)*(x-3)*(x-4)*(x-5)}{2016}+\frac{4*(x+5)*(x+4)*(x)*(x-3)*(x-4)*(x-5)}{2016}+\frac{-5*(x+5)*(x+4)*(x+3)*(x-3)*(x-4)*(x-5)}{3600}+\frac{4*(x+5)*(x+4)*(x+3)*(x)*(x-4)*(x-5)}{2016}+\frac{-3*(x+5)*(x+4)*(x+3)*(x)*(x-3)*(x-5)}{2016}$$

Let us compare each functions approximations from the following table

x	**Actual Value** $\lvert\sqrt{25-x^2}\rvert$	**Normal Polynomial Function**	**Error in Polynomial Function**	**Coil Function**	**Error in Coil Function**	**Continuous Function**	**Error in Continuous Function**	**Harmonic Function**	**Error in Harmonic Function**
0.5	4.9749	4.9629	-0.0120	4.9859	-0.0110	4.9797	-0.0048	4.9790	-0.0041
1.0	4.8990	4.8571	-0.0418	4.9405	-0.0416	4.9173	-0.0183	4.9145	-0.0155
1.5	4.7697	4.6973	-0.0724	4.8532	-0.0835	4.8074	-0.0377	4.8011	-0.0314
2.0	4.5826	4.5000	-0.0826	4.7015	-0.1189	4.6395	-0.0570	4.6284	-0.0458
2.5	4.3301	4.2732	-0.0570	4.4425	-0.1124	4.3929	-0.0627	4.3769	-0.0468
3.5	3.570	3.6191	+0.0484	3.3274	+0.2434	3.6220	-0.0513	3.5888	-0.0181
4.5	2.1794	1.9138	-0.2657	0.9054	+1.2740	1.8972	+0.2822	1.8725	+0.3070

You can note that harmonic function achieved from my method is more accurate than normal polynomial function, Hence we could transform discrete values or even the continuous function to another continuous smooth function.

D. Derivation of High Smooth Orthogonal Matrix Harmonic Curve for Continuous Functions

To transform continuous functions without taking discrete values, Let there are d dimensions, and then minimal polynomial satisfying points at maximum and minimum locations, $(|p(x_1,x_2,x_3,\ldots,x_d)|)^u=0$ and $(|q(x_1,x_2,x_3,\ldots,x_d)|)^v=0$ will be $\prod_{t=1}^{m}(\sum_{s=1}^{d}(x_s-x_sMax_t)^{p_t})$ and $\prod_{t=1}^{n}(\sum_{s=1}^{d}(x_s-x_sMin_t)^{p_t})$ respectively. $p_t=1$ When $f'\neq 0$, and $p_t=2$ When $f''\neq 0$ and $f'=0$, and $p_t=3$ when $f'''\neq 0, f''=0$ and $f'=0$ and so on, We need to take maximum or minimum by solving all the points of $f'(x)=0$ and also to consider the beginning point and ending point for maximum or minimum value. If in case minimum and maximum happened to be beginning point or ending point and it is not part of the points of $f'(x)=0$ then we can assume that $p_t=1$ Since $f'\neq 0$. Let us go with following identity,

$$f_0(x)=\frac{\left(\left(\frac{fmaxN}{fmaxD}-f_1(x)\right)*B*fmaxN*(x-Xmin)^2\right)+\left(\left(f_1(x)-\frac{fminN}{fminD}\right)*A*fminN*(x-Xmax)^2\right)}{\left(\left(\frac{fmaxN}{fmaxD}-f_1(x)\right)*B*fmaxD*(x-Xmin)^2\right)+\left(\left(f_1(x)-\frac{fminN}{fminD}\right)*A*fminD*(x-Xmax)^2\right)} \quad (324)$$

which can be written as matrix as below

$$\begin{pmatrix} f_0(x) \\ 1 \end{pmatrix}=\begin{pmatrix} -P(x) & R(x) \\ Q(x) & P(x) \end{pmatrix}*\begin{pmatrix} f_1(x) \\ 1 \end{pmatrix} \text{ where} \quad (325)$$

$P(x)=B*fmaxN*(x-Xmin)^2-A*fminN*(x-Xmax)^2$

$Q(x)=\left(\frac{fmaxN^2}{fmaxD}*B*(x-Xmin)^2\right)-\left(\frac{fminN^2}{fminD}*A*(x-Xmax)^2\right)$

$R(x)=-(fmaxD*B*(x-Xmin)^2)+(fminD*A*(x-Xmax)^2)$

Let $A=L*fminD$ and $B=K*fmaxD, f_0(x)=\frac{f_{N_0}(x)}{f_{D_0}(x)}, f_1(x)=\frac{f_{N_1}(x)}{f_{D_1}(x)}$ then

$$\begin{pmatrix} f_{N_0}(x) \\ f_{D_0}(x) \end{pmatrix}=\begin{pmatrix} -P_0(x) & R_0(x) \\ Q_0(x) & P_0(x) \end{pmatrix}*\begin{pmatrix} f_{N_1}(x) \\ f_{D_1}(x) \end{pmatrix} \text{ where} \quad (326)$$

$P_0(x)=K*fmaxN*fmaxD*(x-Xmin)^2-L*fminN*fminD*(x-Xmax)^2$

$Q_0(x)=K*(fmaxN^2*(x-Xmin)^2)-L*(fminN^2*(x-Xmax)^2)$

$R_0(x)=-K*(fmaxD^2*(x-Xmin)^2)+L*(fminD^2*(x-Xmax)^2)$ which leads to

$$\begin{pmatrix} f_{N_1}(x) \\ f_{D_1}(x) \end{pmatrix}=\begin{pmatrix} -P_0(x) & R_0(x) \\ Q_0(x) & P_0(x) \end{pmatrix}*\begin{pmatrix} f_{N_0}(x) \\ f_{D_0}(x) \end{pmatrix} \quad (327)$$

now get maximum and minimum of this function and substitute back, then you will get

$$\begin{pmatrix} f_{N_0}(x) \\ f_{D_0}(x) \end{pmatrix} = \begin{pmatrix} -P_0(x) R_0(x) \\ Q_0(x) \;\; P_0(x) \end{pmatrix} * \begin{pmatrix} -P_1(x) R_1(x) \\ Q_1(x) \;\; P_1(x) \end{pmatrix} * \begin{pmatrix} f_{N_2}(x) \\ f_{D_2}(x) \end{pmatrix} \text{ and so on, Hence} \quad (328)$$

$$\begin{pmatrix} f_{N_0}(x) \\ f_{D_0}(x) \end{pmatrix} = \left(\prod_{r=0}^{u-1} \begin{pmatrix} -P_r(x) R_r(x) \\ Q_r(x) \;\; P_r(x) \end{pmatrix} \right) * \begin{pmatrix} f_{N_u}(x) \\ f_{D_u}(x) \end{pmatrix} \text{ where} \quad (329)$$

$$P_v(x) = K_v * f_v maxN * f_v maxD * (x - X_v min)^2 - L_v * f_v minN * f_v minD * (x - X_v max)^2$$
$$Q_v(x) = K_v * (f_v maxN^2 * (x - X_v min)^2) - L_v * (f_v minN^2 * (x - X_v max)^2)$$
$$R_v(x) = -K_v * (f_v maxD^2 * (x - X_v min)^2) + L_v * (f_v minD^2 * (x - X_v max)^2)$$

$\frac{K_v}{L_v}$ is the arbitrary value which will decide the degree of smoothness, and (330)

$$\begin{pmatrix} f_{N_v}(x) \\ f_{D_v}(x) \end{pmatrix} = \left(\prod_{r=0}^{v-1} \begin{pmatrix} -P_{v-r-1}(x) R_{v-r-1}(x) \\ Q_{v-r-1}(x) \;\; P_{v-r-1}(x) \end{pmatrix} \right) * \begin{pmatrix} f_{N_0}(x) \\ f_{D_0}(x) \end{pmatrix} \quad (331)$$

Here also, we can apply the same technique when there is no maximum or minimum found as we did for discrete transformation. Hence for multi − dimension,

$$\begin{pmatrix} f_{N_0}(x_1, x_2, x_3, \dots, x_d) \\ f_{D_0}(x_1, x_2, x_3, \dots, x_d) \end{pmatrix} = \left(\prod_{r=0}^{u-1} \begin{pmatrix} -P_r(x_1, x_2, x_3, \dots, x_d) R_r(x_1, x_2, x_3, \dots, x_d) \\ Q_r(x_1, x_2, x_3, \dots, x_d) \;\; P_r(x_1, x_2, x_3, \dots, x_d) \end{pmatrix} \right) * \quad (332)$$
$$\begin{pmatrix} f_{N_u}(x_1, x_2, x_3, \dots, x_d) \\ f_{D_u}(x_1, x_2, x_3, \dots, x_d) \end{pmatrix} \text{ where}$$

$$P_v(x_1, x_2, x_3, \dots, x_d) = K_v * f_v maxN * f_v maxD * \left(x_{(r-1)} L\right) - L_v * f_v minN * f_v minD * \left(x_{(r-1)} H\right)$$
$$Q_v(x_1, x_2, x_3, \dots, x_d) = K_v * \left(f_v maxN^2 * \left(x_{(r-1)} L\right)\right) - L_v * \left(f_v minN^2 * \left(x_{(r-1)} H\right)\right)$$
$$R_v(x_1, x_2, x_3, \dots, x_d) = -K_v * \left(f_s maxD^2 * \left(x_{(r-1)} L\right)\right) + L_s * \left(f_v minD^2 * \left(x_{(r-1)} H\right)\right)$$

$\frac{K_s}{L_s}$ is the arbitrary value which will decide the degree of smoothness, and (333)

$$x_{(r-1)} H = \prod_{t=1}^{m_{(r-1)}} \left(\sum_{s=1}^{d} \left(|x_s - x_{(r-1)_s} max_t| \right)^{p_t} \right) \text{ where if } f_r = \frac{f_{N_r}(x_1, x_2, x_3, \dots, x_d)}{f_{D_r}(x_1, x_2, x_3, \dots, x_d)} \quad (334)$$

$p_t = 1$ When $f_r{}' \neq 0$, $p_t = 2$ When $f_r{}'' \neq 0$ $f_r{}' = 0$, $p_t = 3$ When $f_r{}''' \neq 0, f_r{}'' = 0$ $f_r{}' = 0$ and so on,

$$x_{(r-1)} L = \prod_{t=1}^{n_{(r-1)}} \left(\sum_{s=1}^{d} \left(|x_s - x_{(r-1)_s} min_t| \right)^{p_t} \right) \text{ where if } f_r = \frac{f_{N_r}(x_1, x_2, x_3, \dots, x_d)}{f_{D_r}(x_1, x_2, x_3, \dots, x_d)} \quad (335)$$

$p_t = 1$ When $f_r{}' \neq 0$, $p_t = 2$ When $f_r{}'' \neq 0$ $f_r{}' = 0$, $p_t = 3$ When $f_r{}''' \neq 0, f_r{}'' = 0$ $f_r{}' = 0$ and so on,

$$\begin{pmatrix} f_{N_s}(x_1, x_2, x_3, \dots, x_d) \\ f_{D_s}(x_1, x_2, x_3, \dots, x_d) \end{pmatrix} = \left(\prod_{r=0}^{s-1} \begin{pmatrix} -P_{s-r-1}(x_1, x_2, x_3, \dots, x_d) R_{s-r-1}(x_1, x_2, x_3, \dots, x_d) \\ Q_{s-r-1}(x_1, x_2, x_3, \dots, x_d) \;\; P_{s-r-1}(x_1, x_2, x_3, \dots, x_d) \end{pmatrix} \right) * \quad (336)$$
$$\begin{pmatrix} f_{N_0}(x_1, x_2, x_3, \dots, x_d) \\ f_{D_0}(x_1, x_2, x_3, \dots, x_d) \end{pmatrix}$$

Thus we could transform both discrete and continuous multi-dimensional functions to another smooth functions.

E. Derivation of High Smooth and High Precision non Orthogonal Matrix Harmonic Curve for Continuous Functions

Since finding A, B is time consuming, to transform continuous functions we can go with the identities (314), (315),

$$f_0(x) = \frac{fN_0(x)}{fD_0(x)} = \frac{\left((fN_1(x)) * fmaxN * (x - Xmin)^2\right) + \left((fD_1(x)) * fminN * (x - Xmax)^2\right)}{\left((fN_1(x)) * fmaxD * (x - Xmin)^2\right) + \left((fD_1(x)) * fminD * (x - Xmax)^2\right)} \text{ which leads to}$$

$$\frac{fN_1(x)}{fD_1(x)} = \frac{\left(fminD * fN_0(x) - fminN * fD_0(x)\right) * (x - Xmax)^2}{\left(fmaxN * fD_0(x) - fmaxD * fN_0(x)\right) * (x - Xmin)^2} \text{ which can be written as matrix as below.}$$

$$\begin{pmatrix} fN_0(x) \\ fD_0(x) \end{pmatrix} = \begin{pmatrix} fmaxN * (x - Xmin)^2 fminN * (x - Xmax)^2 \\ fmaxD * (x - Xmin)^2 fminD * (x - Xmax)^2 \end{pmatrix} * \begin{pmatrix} fN_1(x) \\ fD_1(x) \end{pmatrix} \text{ which leads to} \quad (337)$$

$$\begin{pmatrix} fN_1(x) \\ fD_1(x) \end{pmatrix} = \begin{pmatrix} fminD * (x - Xmax)^2 & -fminN * (x - Xmax)^2 \\ -fmaxD * (x - Xmin)^2 fmaxN * (x - Xmin)^2 \end{pmatrix} * \begin{pmatrix} fN_0(x) \\ fD_0(x) \end{pmatrix} \text{ and as a recursion, get into} \quad (338)$$

$$\begin{pmatrix} fN_r(x) \\ fD_r(x) \end{pmatrix} = \begin{pmatrix} fmin_{(r-1)} D * \left(x - Xmax_{(r-1)}\right)^2 & -fmin_{(r-1)} N * \left(x - Xmax_{(r-1)}\right)^2 \\ -fmax_{(r-1)} D * \left(x - Xmin_{(r-1)}\right)^2 fmax_{(r-1)} N * \left(x - Xmin_{(r-1)}\right)^2 \end{pmatrix} * \begin{pmatrix} fN_{(r-1)}(x) \\ fD_{(r-1)}(x) \end{pmatrix} \quad (339)$$

and after substituting back the recursion, we get

$$\begin{pmatrix} f_{N_0}(x) \\ f_{D_0}(x) \end{pmatrix} = \left(\prod_{r=0}^{u-1} \begin{pmatrix} fmax_{(r-1)}N * \left(x - Xmin_{(r-1)}\right)^2 fmin_{(r-1)}N * \left(x - Xmax_{(r-1)}\right)^2 \\ fmax_{(r-1)}D * \left(x - Xmin_{(r-1)}\right)^2 fmin_{(r-1)}D * \left(x - Xmax_{(r-1)}\right)^2 \end{pmatrix} \right) * \begin{pmatrix} f_{N_u}(x) \\ f_{D_u}(x) \end{pmatrix} \quad (340)$$

Here also, we can apply the same technique when there is no maximum or minimum found as we did for discrete transformation. Hence for multi $-$ dimension,

$$\begin{pmatrix} f_{N_0}(x_1, x_2, x_3, \dots, x_d) \\ f_{D_0}(x_1, x_2, x_3, \dots, x_d) \end{pmatrix} = \left(\prod_{r=0}^{u-1} \begin{pmatrix} fmax_{(r-1)}N * \left(x_{(r-1)}L\right) fmin_{(r-1)}N * \left(x_{(r-1)}H\right) \\ fmax_{(r-1)}D * \left(x_{(r-1)}L\right) fmin_{(r-1)}D * \left(x_{(r-1)}H\right) \end{pmatrix} \right) * \quad (341)$$

$$\begin{pmatrix} f_{N_u}(x_1, x_2, x_3, \dots, x_d) \\ f_{D_u}(x_1, x_2, x_3, \dots, x_d) \end{pmatrix} \text{ where}$$

$$\begin{pmatrix} f_{N_s}(x_1, x_2, x_3, \dots, x_d) \\ f_{D_s}(x_1, x_2, x_3, \dots, x_d) \end{pmatrix} = \left(\prod_{r=0}^{s-1} \begin{pmatrix} fmin_{(s-r-1)}D * \left(x_{(s-r-1)}H\right) & -fmin_{(s-r-1)}N * \left(x_{(s-r-1)}H\right) \\ -fmax_{(s-r-1)}D * \left(x_{(s-r-1)}L\right) fmax_{(s-r-1)}N * \left(x_{(s-r-1)}L\right) \end{pmatrix} \right) * \quad (342)$$

$$\begin{pmatrix} f_{N_0}(x_1, x_2, x_3, \dots, x_d) \\ f_{D_0}(x_1, x_2, x_3, \dots, x_d) \end{pmatrix} \text{ and}$$

$$x_{(r-1)}H = \prod_{t=1}^{m_{(r-1)}} \left(\sum_{s=1}^{d} \left(|x_s - x_{(r-1)_s} max_t|\right)^{p_t} \right) \text{ where if } f_r = \frac{f_{N_r}(x_1, x_2, x_3, \dots, x_d)}{f_{D_r}(x_1, x_2, x_3, \dots, x_d)} \quad (343)$$

$p_t = 1$ When $f_r{}' \neq 0$, $p_t = 2$ When $f_r{}'' \neq 0$ $f_r{}' = 0$, $p_t = 3$ When $f_r{}''' \neq 0, f_r{}'' = 0$ $f_r{}' = 0$ and so on,

$$x_{(r-1)}L = \prod_{t=1}^{n_{(r-1)}} \left(\sum_{s=1}^{d} \left(|x_s - x_{(r-1)_s} min_t|\right)^{p_t} \right) \text{ where if } f_r = \frac{f_{N_r}(x_1, x_2, x_3, \dots, x_d)}{f_{D_r}(x_1, x_2, x_3, \dots, x_d)} \quad (344)$$

$p_t = 1$ When $f_r{}' \neq 0$, $p_t = 2$ When $f_r{}'' \neq 0$ $f_r{}' = 0$, $p_t = 3$ When $f_r{}''' \neq 0, f_r{}'' = 0$ $f_r{}' = 0$ and s oon

We need to take maximum or minimum by solving all the points of $f'(x) = 0$ and also to consider the beginning point and ending point for maximum or minimum value. If in case minimum and maximum happened to be beginning point or ending point and it is not part of the points of $f'(x) = 0$ then we can assume that $p_t = 1$ Since $f' \neq 0$. Thus we could transform both discrete and continuous multi-dimensional functions to another smooth functions.

F. Derivation of High Smooth and High Precision Polynomial Curve for Continuous Functions

Since keep on finding maximum and minimum for continuous function will be time consuming, we can minimize computation using following technique.

if $L(x, m, d) = \left(\sum_{s=0}^{d-1} a_{m_s} * (x - x_m)^s\right) * \prod_{p=0 \text{ and } p \neq m}^{n} \left(x - x_p\right)^d$ then $L(x_r, m, d) = 0 \ \forall \ r, \ 0 \leq r \leq n$ and $r \neq m$, but $L(x_m, m, d) \neq 0$, $L'(x_r, m, d) = 0 \ \forall \ r, \ 0 \leq r \leq n$ and $r \neq m$, but $L'(x_m, m, d) \neq 0$, (345)
$L''(x_r, m, d) = 0 \ \forall \ r, \quad 0 \leq r \leq n$ and $r \neq m$, but $L''(x_m, m, d) \neq 0$ and so on up to $(d-1)^{th}$ derivative.

We can get these co $-$ efficients by equating the derivatives for the following generalized function

$$f(x) = \sum_{m=0}^{n} \left(\left(\sum_{s=0}^{d-1} a_{m_s} * (x - x_m)^s\right) * \prod_{p=0 \text{ and } p \neq m}^{n} \left(x - x_p\right)^d \right) \quad (346)$$

Let us explain the concept by taking degree $d = 2$

if $L(x, m) = \left(a_m + b_m * (x - x_m)\right) * \prod_{p=0 \text{ and } p \neq m}^{n} \left(x - x_p\right)^2$ then $L(x_r, m) = 0 \ \forall \ r, \ 0 \leq r \leq n$ and $r \neq m$, (347)

$L(x_m, m) = a_m * \prod_{p=0 \text{ and } p \neq m}^{n} \left(x - x_p\right)^2$ Andalso $L'(x_r, m) = 0 \ \forall \ r, \ 0 \leq r \leq n$ and $r \neq m$,

$$L'(x_m, m) = \left(\prod_{p=0 \text{ and } p \neq m}^{n} \left(x - x_p\right)^2\right) * \left(b_m + 2 * a_m * \left(\sum_{p=0 \text{ and } p \neq m}^{n} \frac{1}{x - x_p}\right)\right)$$

Hence we can get if $f(x_r)$ and $f'(x_r) \exists \forall 0 \leq r \leq n$ has values as following.

$$f(x) = \sum_{m=0}^{n} \left(\left(a_m + b_m * (x - x_m)\right) * \prod_{p=0 \text{ and } p \neq m}^{n} \left(x - x_p\right)^2 \right) \text{ where} \quad (348)$$

$$a_m = f(x_m) * \prod_{p=0 \text{ and } p \neq m}^{n} \frac{1}{(x_m - x_p)^2}, \ b_m = f'(x_m) * \prod_{p=0 \text{ and } p \neq m}^{n} \frac{1}{(x_m - x_p)^2} - 2 * a_m * \sum_{p=0 \text{ and } p \neq m}^{n} \frac{1}{x_m - x_p} \quad (349)$$

Hence in general we can obtain co efficients by equating the derivatives for the following generalized function

$$f(x) = \sum_{m=0}^{n} \left(\left(\sum_{s=0}^{d-1} a_{m_s} * (x - x_m)^s\right) * \prod_{p=0 \text{ and } p \neq m}^{n} \left(x - x_p\right)^d \right) \quad (350)$$

G. Derivation of High Smooth and High Precision Polynomial Harmonic Curve for Continuous Functions

Like the same way as polynomial curve for the continuous function, instead of computing

$$f(x) = \sum_{m=0}^{n} \left(\left(\sum_{s=0}^{d-1} a_{m_s} * (x - x_m)^s \right) * \prod_{p=0 \text{ and } p \neq m}^{n} \left(x - x_p \right)^d \right), \text{ we need to go with} \quad (351)$$

$$g_1(x) + f(x) * g_2(x) + \left(g_3(x) + f(x) * g_4(x)\right) * \left(\sum_{m=0}^{n} \left(\sum_{s=0}^{c} a_{m_s} * (x - x_m)^s \right) * \prod_{p=0 \text{ and } p \neq m}^{n} \left(x - x_p \right)^d \right) =$$

$$\left(g_5(x) + f(x) * g_6(x)\right) * \left(\sum_{m=0}^{n} \left(\sum_{s=0}^{d-c-2} b_{m_s} * (x - x_m)^s \right) * \prod_{p=0 \text{ and } p \neq m}^{n} \left(x - x_p \right)^d \right) \quad (352)$$

Which will have same number of variables as we did in polynomial curve and once we found all the variables a_{m_s}, b_{m_s}

Let $LT(x) = \sum_{m=0}^{n} \left(\sum_{s=0}^{c} a_{m_s} * (x - x_m)^s \right) * \prod_{p=0 \text{ and } p \neq m}^{n} \left(x - x_p \right)^d$ and

Let $RT(x) = \sum_{m=0}^{n} \left(\sum_{s=0}^{d-c-2} b_{m_s} * (x - x_m)^s \right) * \prod_{p=0 \text{ and } p \neq m}^{n} \left(x - x_p \right)^d$ then

$$f(x) = \frac{g_5(x) * RT(x) - g_3(x) * LT(x) - g_1(x)}{-g_6(x) * RT(x) + g_4(x) * LT(x) + g_2(x)} \quad (353)$$

Let us explain the concept by taking degree $d = 2, c = 0, g_1(x) = g_3(x) = g_6(x) = 0, g_2(x) = g_4(x) = g_5(x) = 1$

if $f(x) + f(x) * \left(\sum_{m=0}^{n} \left(a_m * \prod_{p=0 \text{ and } p \neq m}^{n} \left(x - x_p \right)^2 \right) \right) = \sum_{m=0}^{n} \left(b_m * \prod_{p=0 \text{ and } p \neq m}^{n} \left(x - x_p \right)^2 \right)$, then when $x = x_m$ (354)

$$f(x_m) + a_m * f(x_m) * \left(\prod_{p=0 \text{ and } p \neq m}^{n} \left(x_m - x_p \right)^2 \right) = b_m * \left(\prod_{p=0 \text{ and } p \neq m}^{n} \left(x_m - x_p \right)^2 \right) \text{ and on taking derivative} \quad (355)$$

$$f'(x_m) + a_m * \left(\prod_{p=0 \text{ and } p \neq m}^{n} \left(x_m - x_p \right)^2 \right) * \left(f'(x_m) + 2 * f(x_m) * \left(\sum_{p=0 \text{ and } p \neq m}^{n} \frac{1}{x_m - x_p} \right) \right) \quad (356)$$

$$= b_m * \left(\prod_{p=0 \text{ and } p \neq m}^{n} \left(x_m - x_p \right)^2 \right) * 2 * \left(\sum_{p=0 \text{ and } p \neq m}^{n} \frac{1}{x - x_p} \right) \quad (357)$$

once we solve a_m, b_m via these two linear equations, one can get

$$f(x) = \frac{\sum_{m=0}^{n} \left(b_m * \prod_{p=0 \text{ and } p \neq m}^{n} (x - x_p)^2 \right)}{\left(1 + \sum_{m=0}^{n} \left(a_m * \prod_{p=0 \text{ and } p \neq m}^{n} (x - x_p)^2 \right) \right)} \text{ where} \quad (358)$$

$$a_m = \frac{f'(x_m) - 2 * f(x_m) * \left(\sum_{p=0 \text{ and } p \neq m}^{n} \frac{1}{x_m - x_p} \right)}{f'(x_m) * \left(\prod_{p=0 \text{ and } p \neq m}^{n} (x_m - x_p)^2 \right)} \text{ and } b_m = \frac{2 * \left(f(x_m)\right)^2 * \left(\sum_{p=0 \text{ and } p \neq m}^{n} \frac{1}{x_m - x_p} \right)}{f'(x_m) * \left(\prod_{p=0 \text{ and } p \neq m}^{n} (x_m - x_p)^2 \right)} \quad (359)$$

This will have solution if $f'(x_m) \neq 0$, but having solution even if $f'(x_m) = \pm\infty$. We need to adjust g_k where $1 \leq k \leq 6$ functions so that we can get proper solution for a_{m_s}, b_{m_s}. For the above example, Instead of $g_5(x) = 1$, Let us consider $g_5(x) = \sum_{m=0}^{n} \left(b_m * \prod_{p=0 \text{ and } p \neq m}^{n} \left(x - x_p \right) \right)$.

if $f(x) + f(x) * \left(\sum_{m=0}^{n} \left(a_m * \prod_{p=0 \text{ and } p \neq m}^{n} \left(x - x_p \right)^2 \right) \right) = \sum_{m=0}^{n} \left(b_m * \prod_{p=0 \text{ and } p \neq m}^{n} \left(x - x_p \right)^3 \right)$, then when $x = x_m$ (360)

$$f(x_m) + a_m * f(x_m) * \left(\prod_{p=0 \text{ and } p \neq m}^{n} \left(x_m - x_p \right)^2 \right) = b_m * \left(\prod_{p=0 \text{ and } p \neq m}^{n} \left(x_m - x_p \right)^3 \right) \text{ and on taking derivative} \quad (361)$$

$$f'(x_m) + a_m * \left(\prod_{p=0 \text{ and } p \neq m}^{n} \left(x_m - x_p \right)^2 \right) * \left(f'(x_m) + 2 * f(x_m) * \left(\sum_{p=0 \text{ and } p \neq m}^{n} \frac{1}{x_m - x_p} \right) \right) \quad (362)$$

$$= b_m * \left(\prod_{p=0 \text{ and } p \neq m}^{n} \left(x_m - x_p \right)^3 \right) * 3 * \left(\sum_{p=0 \text{ and } p \neq m}^{n} \frac{1}{x - x_p} \right) \quad (363)$$

once we solve a_m, b_m via these two linear equations, one can get

$$f(x) = \frac{\sum_{m=0}^{n} \left(b_m * \prod_{p=0 \text{ and } p \neq m}^{n} (x - x_p)^3 \right)}{\left(1 + \sum_{m=0}^{n} \left(a_m * \prod_{p=0 \text{ and } p \neq m}^{n} (x - x_p)^2 \right) \right)} \text{ where} \quad (364)$$

$$a_m = \frac{f'(x_m) - 3 * f(x_m) * \left(\sum_{p=0 \text{ and } p \neq m}^{n} \frac{1}{x_m - x_p} \right)}{\left(f'(x_m) - f(x_m) * \left(\sum_{p=0 \text{ and } p \neq m}^{n} \frac{1}{x_m - x_p} \right) \right) * \left(\prod_{p=0 \text{ and } p \neq m}^{n} (x_m - x_p)^2 \right)} \text{ and} \quad (365)$$

$$b_m = \frac{2 * \left(f(x_m)\right)^2 * \left(\sum_{p=0 \text{ and } p \neq m}^{n} \frac{1}{x_m - x_p} \right)}{\left(f'(x_m) - f(x_m) * \left(\sum_{p=0 \text{ and } p \neq m}^{n} \frac{1}{x_m - x_p} \right) \right) * \left(\prod_{p=0 \text{ and } p \neq m}^{n} (x_m - x_p)^3 \right)} \quad (366)$$

Thus we could change the denominator by adjusting $g_5(x)$. Instead of changing whole degree, Even we can just change only to the denominator where it is becoming zero to have both variables to right function.To make this generalized let us have the shorten version as below.

$$f(x) + f(x) * \left(\sum_{m=0}^{n} \left(\sum_{s=0}^{c_m} a_{m_s} * (x - x_m)^s\right) * \prod_{p=0 \text{ and } p\neq m}^{n} \left(x - x_p\right)^d\right) = \left(\sum_{m=0}^{n} \left(\sum_{s=0}^{d-c_m-2} b_{m_s} * (x - x_m)^s\right) * \prod_{p=0 \text{ and } p\neq m}^{n} \left(x - x_p\right)^d\right) \quad (367)$$

Which will have same number of variables and c_m depends upon the value of $f'(x_m)$.

Once we find all the variables, a_{m_s}, b_{m_s}, we can get

$$f(x) = \frac{\sum_{m=0}^{n} \left(\sum_{s=0}^{d-c_m-2} b_{m_s} * (x - x_m)^s\right) * \prod_{p=0 \text{ and } p\neq m}^{n} (x - x_p)^d}{1 + \left(\sum_{m=0}^{n} \left(\sum_{s=0}^{c_m} a_{m_s} * (x - x_m)^s\right) * \prod_{p=0 \text{ and } p\neq m}^{n} (x - x_p)^d\right)} \quad (368)$$

Let us explain the concept by taking degree $d = 2, c_m = 0$ or -1 if

$$f(x) + f(x) * \left(\sum_{m=0}^{n} \left(a_m * \prod_{p=0 \text{ and } p\neq m}^{n} \left(x - x_p\right)^2\right)\right) = \sum_{m=0}^{n} \left(\left(b_m + c_m * (x - x_m)\right) * \prod_{p=0 \text{ and } p\neq m}^{n} \left(x - x_p\right)^2\right), \quad (369)$$

then when $x = x_m$,

$$f(x_m) + a_m * f(x_m) * \left(\prod_{p=0 \text{ and } p\neq m}^{n} \left(x_m - x_p\right)^2\right) = b_m * \left(\prod_{p=0 \text{ and } p\neq m}^{n} \left(x_m - x_p\right)^2\right) \quad (370)$$

and on taking derivative

$$f'(x_m) + a_m * \left(\prod_{p=0 \text{ and } p\neq m}^{n} \left(x_m - x_p\right)^2\right) * \left(f'(x_m) + 2 * f(x_m) * \left(\sum_{p=0 \text{ and } p\neq m}^{n} \frac{1}{x_m - x_p}\right)\right) \quad (371)$$

$$= b_m * \left(\prod_{p=0 \text{ and } p\neq m}^{n} \left(x_m - x_p\right)^2\right) * 2 * \left(\sum_{p=0 \text{ and } p\neq m}^{n} \frac{1}{x - x_p}\right) + c_m * \left(\prod_{p=0 \text{ and } p\neq m}^{n} \left(x_m - x_p\right)^2\right) \quad (372)$$

once we solve a_m, b_m via these two linear equations, one can get

$$f(x) = \frac{\sum_{m=0}^{n} \left(\left(b_m + c_m * (x - x_m)\right) * \prod_{p=0 \text{ and } p\neq m}^{n} (x - x_p)^2\right)}{\left(1 + \sum_{m=0}^{n} \left(a_m * \prod_{p=0 \text{ and } p\neq m}^{n} (x - x_p)^2\right)\right)} \text{ where} \quad (373)$$

if $f'(x_m)$ nearing 0, then $a_m = 0$, (374)

$$b_m = f(x_m) * \prod_{p=0 \text{ and } p\neq m}^{n} \frac{1}{(x_m - x_p)^2},$$

$$c_m = f'(x_m) * \prod_{p=0 \text{ and } p\neq m}^{n} \frac{1}{(x_m - x_p)^2} - 2 * b_m * \sum_{p=0 \text{ and } p\neq m}^{n} \frac{1}{x_m - x_p} \quad (375)$$

otherwise if $f'(x_m)$ nearing $\pm\infty$, then $c_m = 0$, (376)

$$a_m = \frac{f'(x_m) - 2 * f(x_m) * \left(\sum_{p=0 \text{ and } p\neq m}^{n} \frac{1}{x_m - x_p}\right)}{f'(x_m) * \left(\prod_{p=0 \text{ and } p\neq m}^{n} (x_m - x_p)^2\right)} \text{ and } b_m = \frac{2 * \left(f(x_m)\right)^2 * \left(\sum_{p=0 \text{ and } p\neq m}^{n} \frac{1}{x_m - x_p}\right)}{f'(x_m) * \left(\prod_{p=0 \text{ and } p\neq m}^{n} (x_m - x_p)^2\right)} \quad (377)$$

Hence without finding maximum and minimum but at the same time, we could get alternate high precision and smooth curve at random points. Also we are able to get a solution even if $f'(x_m) = \pm\infty$.

IV. DERIVATION OF FASTER SMOOTH OR DECORATIVE TRANSFORMATION TECHNIQUE

A. Derivation of Simple Polygonal Curve

Here even though the accuracy is great, computation time to calculate the values are higher. Hence Let us go with the identity, $s_r(x) = x - r + |x - r|$ which will behave like $s_r(x) = 0$ when $x \leq r$ and $s_r(x) = 2 * (x - r)$ when $x \geq r$.

Then we can write a discrete function if $f(x) \exists \forall 0 \leq x \leq (n-1)$ has values as following.

$$f(x) = f(0) + \left(f(1) - f(0)\right) * x + \sum_{r=1}^{n-2} a_r * (x - r + |x - r|) \text{ then} \quad (378)$$

$$f(m-1) = f(0) + \left(f(1) - f(0)\right) * (m-1) + \sum_{r=1}^{m-2} a_r * (2 * (m - 1 - r)) \quad (379)$$

$$f(m) = f(0) + \left(f(1) - f(0)\right) * (m) + \sum_{r=1}^{m-1} a_r * (2 * (m - r)) \quad (380)$$

$$f(m+1) = f(0) + \left(f(1) - f(0)\right) * (m+1) + \sum_{r=1}^{m} a_r * (2 * (m + 1 - r)) \quad (381)$$

on calculating $f(m-1) - 2 * f(m) + f(m+1)$ will lead to $2 * a_m$

$$a_m = \frac{f(m-1) - 2 * f(m) + f(m+1)}{2} \text{ hence} \quad (382)$$

$$f(x) = f(0) + \left(f(1) - f(0)\right) * x + \sum_{r=1}^{n-2} \frac{f(r-1) - 2 * f(r) + f(r+1)}{2} * (x - r + |x - r|) \quad (383)$$

This function will be linear in between integer values and hence will look like a polygonal curve.

B. Derivation of Smooth Quadratic (Parabolic) Curve

To make the polygonal curve to be smooth, let us take the curve as the following $s_r(x) = (x - r + |x - r|)^2$ which will behave like $s_r(x) = 0$ when $x \le r$ and $s_r(x) = 4 * (x - r)^2$ when $x \ge r$. $s_r'(x) = 0$ when $x \le r$ and $s_r'(x) = 8 * (x - r)$ when $x \ge r.$ Then we can write a discrete function if $f(x) \exists \forall 0 \le x \le (n-1)$ has values as following.

$$f(x) = f(0) * (1 + k * (x * (x - 1))) + (f(1) - f(0)) * (x^2 + l * (x * (x - 1))) + \sum_{r=1}^{n-2} a_r * \frac{(x - r + |x - r|)^2}{4} \quad (384)$$

We get following pattern

$$a_1 = (3 - 2 * k + 2 * l) * f(0) - 2 * (2 + l) * f(1) + f(2) \quad (385)$$

$$a_2 = -2 * (2 - k + l) * f(0) + (7 + 2 * l) * f(1) - 4 * f(2) + f(3) \quad (386)$$

$$a_3 = 2 * (2 - k + l) * f(0) - 2 * (4 + l) * f(1) + 7 * f(2) - 4 * f(3) + f(4) \quad (387)$$

from when $r = 4, a_4$ it follows the following pattern

$$a_r = f(r) - 4 * f(r - 1) + 7 * f(r - 2) + 8 * \sum_{s=3}^{r-2} (-1)^s * f(r - s) + (-1)^r * 2 * ((4 + l) * f(1) - (2 - k + l) * f(0)) \quad (388)$$

and if we need to generalize with one formula $\forall r$, then we can assume,

$$a_r = -a_{r-1} - f(r - 2) + 3 * f(r - 1) - 3 * f(r) + f(r + 1) \text{ and} \quad (389)$$

$$a_{r-1} = -a_r - f(r - 2) + 3 * f(r - 1) - 3 * f(r) + f(r + 1) \text{ and which leads to} \quad (390)$$

$$f(-1) = (1 + 2 * k - 2 * l) * f(0) + 2 * l * f(1) \quad (391)$$

$$f(-2) = 6 * (k - l) * f(0) + 2 * l * f(1) \quad (392)$$

and to generailize for multi − dimension, we can equate the following functions

$$x^2 + l * (x * (x - 1)) = \frac{(x - r + |x - r|)^2}{4} + \frac{\sin(\pi * r)}{r} * l * (x * (x - 1)) \text{ when } r = 0 \quad (393)$$

$$1 + k * (x * (x - 1)) = \frac{(x - r + |x - r|)^2}{4} + \frac{\sin(\pi * (r + 1))}{(r + 1)} * ((k - 1) * (x * (x - 1)) - 3 * x) \quad (394)$$

when $r = -1$ and hence we can write in general as following

$$f(x) = \sum_{r=0}^{n-1} a_r * \left(\frac{(x - r + 1 + |x - r + 1|)^2}{4} + \frac{\sin(\pi * (r - 1))}{(r - 1)} * l * (x * (x - 1))\right) \quad (395)$$

$$- \sum_{r=0}^{n-1} a_r * \left(\frac{\sin(\pi * r)}{r} * ((k - 1) * (x * (x - 1)) - 3 * x)\right) \text{ where}$$

$$a_r = -a_{r-1} - f(r - 2) + 3 * f(r - 1) - 3 * f(r) + f(r + 1) \text{ and } a_0 = 1, \quad (396)$$

$$f(-1) = (1 + 2 * k - 2 * l) * f(0) + 2 * l * f(1) \quad (397)$$

$$f(-2) = 6 * (k - l) * f(0) + 2 * l * f(1) \quad (398)$$

For discrete function, if we know $f(-2), f(-1)$ and $f(1) \ne 0$ using above method $f(-1) = (1 + 2 * k - 2 * l) * f(0) + 2 * l * f(1)$, $f(-2) = 6 * (k - l) * f(0) + 2 * l * f(1)$ we can compute $l = \frac{3 * (f(-1) - f(0)) - f(-2)}{4 * (f(1))}$, $k = \frac{((f(1) - 3 * f(0)) * (f(-1) - f(0)) - ((f(1) - (f(0)) * f(-2))}{4 * (f(1))}$ and For continuous function, we get optimal k, l by taking integral at the interval and find by least error matching values. In similar way, For Multidimensional functions, if $f(x_1, x_2, x_3, \dots, x_d) \exists \forall 0 \le x_s \le (n_s - 1) \forall 1 \le s \le d$ has values for d dimensions, same can be written as following. Let

$$T(x, r) = \left(\frac{(x - r + 1 + |x - r + 1|)^2}{4} + \frac{\sin(\pi * (r - 1))}{(r - 1)} * l * (x * (x - 1))\right) \quad (399)$$

$$+ \left(\frac{\sin(\pi * r)}{r} * ((k - 1) * (x * (x - 1)) - 3 * x)\right) \text{ then}$$

$$f(x_1, x_2, x_3, \dots, x_d) = \sum_{r_d=0}^{n_d-1} \left(\dots \sum_{r_3=0}^{n_3-1} \left(\sum_{r_2=0}^{n_2-1} \left(\sum_{r_1=0}^{n_1-1} \left(a_{x_d} r_d * \prod_{s=1}^{d} T(x_d, r_d)\right)\right)\right)\right) \text{ where} \quad (400)$$

$$a_{x_d} r_d = -a_{x_d}(r_d - 1)_d - f_d(r_d - 2) + 3 * f_d(r_d - 1) - 3 * f_d(r_d) + f_d(r_d + 1) \text{ and } a_{x_d} 0_d = 1, \quad (401)$$

$$f_d = f(x_1, x_2, x_3, \dots, x_d) \quad (402)$$

$$f_d(-1) = (1 + 2 * k_d - 2 * l_d) * f_d(0) + 2 * l_d * f_d(1) \quad (403)$$

$$f_d(-2) = 6 * (k_d - l_d) * f_d(0) + 2 * l_d * f_d(1) \quad (404)$$

This function will be smooth since it is parabolic curve in between integer intervals and at integer interval, the slope is same since $s_r'(x) = 0$. For continuous functions we can compute k_d, l_d using least error method after taking integral at the intervals.

C. Derivation of Faster Smooth Quadratic (Parabolic) Curve

Here also there are more computation needed to get co-efficient. Instead of $s_r(x) = (x - r)^2$, if we get $s_r(x) = (x - r) * (x - r + 1) => (Curve1)$ or $= (x - r) => (Curve2)$ or $= 1 => (Curve3)$, then computation terms will be reduced. Let us assume, $u = x - r,\ 0 < a < b \le 1$, then

$$k*u^2+l*(u-a)^2+m*(u-b)^2=n*(u)*(u+1) \;=>\; (Curve1),$$
$$k*u^2+l*(u-a)^2+m*(u-b)^2=n*(u) \;=>\; (Curve2),$$
$k*u^2+l*(u-a)^2+m*(u-b)^2=n \;=>\; (Curve3)$, then, on solving each co-efficient, we get
$$(b-a)*(a+b+2*a*b)*u^2-(b*(u-a))^2+(a*(u-b))^2=2*(b-a)*a*b*u*(u+1) => (Curve1),$$
$$(b-a)*(a+b)*u^2-(b*(u-a))^2+(a*(u-b))^2=2*(b-a)*a*b*u => (Curve2),$$
$$(b-a)*u^2-b*((u-a))^2+((u-b))^2=(b-a)*a*b => (Curve3), \quad (405)$$
where $u=x-r,\ 0<a<b\le 1$

Then to make the polygonal curve to be smooth, let us take example of curve with $(Curve1)$ as the following

If $T(x)=x+|x|,\ 0<a<b\le 1$ then

$s_r(x)=\frac{(b-a)*(a+b+2*a*b)*(T(x-r))^2-(b*T(x-r-a))^2+(a*T(x-r-b))^2}{16*(b-a)*a*b}$ will behave like following,

$s_r(x)=0$ when $x\le r$, $s_r(x)=\frac{(b-a)*(a+b+2*a*b)*(x-r)^2}{4*(b-a)*a*b}$ when $x\ge r$ and $x\le(r+a)$,

$s_r(x)=\frac{(b-a)*(a+b+2*a*b)*(x-r)^2-(b*(x-r-a))^2}{4*(b-a)*a*b}$ when $x\ge(r+a)$ and $x\le(r+b)$, $s_r(x)=\frac{(x-r)*(x-r+1)}{2}$ when $x\ge(r+b)$.

$s_r{}'(x)=0$ when $x\le r$, $s_r{}'(x)=\frac{(b-a)*(a+b+2*a*b)*(x-r)}{2*(b-a)*a*b}$ when $x\ge r$ and $x\le(r+a)$,

$s_r{}'(x)=\frac{(b-a)*(a+b+2*a*b)*(x-r)-(b^2*(x-r-a))}{2*(b-a)*a*b}$ when $x\ge(r+a)$ and $x\le(r+b)$, $s_r{}'(x)=\frac{2*(x-r)+1}{2}$ when $x\ge(r+b)$.

For monotonous incremental curve, all slope should be positive, then at interval $x\ge(r+a)$ and $x\le(r+b)$, $(b-a)*(a+b+2*a*b)-b^2\ge 0$, $a\le\frac{2*b^2}{2*b+1}$ $a\le\frac{2}{3}$, $b>\frac{1}{2}$, if $a=\frac{2*b^2}{2*b+1}$, then the slope is a straight line and if $b=1, a=\frac{2}{3}$, then curve will have minimum turning points.

Then we can write a discrete function if $f(x)\exists\forall 0\le x\le(n-1)$ has values as following.

$$f(x)=f(0)+(f(1)-f(0))*x+k*(x*(x-1))+\sum_{r=1}^{n-2} a_r*s_r(x) \quad (406)$$
Where $s_r(x)=\frac{((b-a)*(a+b+2*a*b))*(T(x-r))^2-(b*T(x-r-a))^2+(a*T(x-r-b))^2}{16*(b-a)*a*b}$

$T(x)=x+|x|,\ 0<a<b\le 1,\ a\le\frac{2*b^2}{2*b+1}$ Then we get following pattern (407)

$$a_1=f(0)-2*f(1)+f(2)-2*k \quad (408)$$
from when $r=2, a_2$ it follows the following pattern
$$a_r=-f(r-2)+3*f(r-1)-3*f(r)+f(r+1) \quad (409)$$
and if we need to generalize with one formula $\forall r$, then we can assume,
$$f(-1)=2*f(0)-1*f(1)+2*k \quad (410)$$
$$f(-2)=2*f(0)-3*f(1)+6*k \quad (411)$$
and to generailize for multidimension, we can equate the following functions
$$x+\frac{k*(x*(x-1))}{(f(1)-f(0))}=s_r(x)+\frac{\sin(\pi*r)}{2*r}*((\frac{2*k*(x*(x-1))}{(f(1)-f(0))}-1)) \text{ when } r=0 \quad (412)$$
$$1=s_r(x)-\frac{\sin(\pi*(r+1))}{2*(r+1)}*(x^2+3*x) \text{ when } r=-1 \quad (413)$$
and hence we can write in general as following
$$f(x)=\sum_{r=0}^{n-1} a_r*\left(s_r(x)+\frac{\sin(\pi*(r-1))}{2*(r-1)}*(2*\frac{k*(x*(x-1))}{(f(1)-f(0))}-1)\right) \quad (414)$$
$$-\sum_{r=0}^{n-1} a_r*\left(\frac{\sin(\pi*r)}{2*r}*(x^2+3*x)\right)$$
Where $s_r(x)=\frac{((b-a)*(a+b+2*a*b))*(T(x-r))^2-(b*T(x-r-a))^2+(a*T(x-r-b))^2}{16*(b-a)*a*b}$

$T(x)=x+|x|,\ 0<a<b\le 1,\ a\le\frac{2*b^2}{2*b+1}$ and we get (415)

$$a_r=-f(r-2)+3*f(r-1)-3*f(r)+f(r+1) \text{ and } a_0=1, \quad (416)$$
$$f(-1)=2*f(0)-1*f(1)+2*k \quad (417)$$
$$f(-2)=2*f(0)-3*f(1)+6*k \quad (418)$$

If we notice, we won't be able to get optimal k as we did in previous methods from $f(-1), f(-2)$ .But we can get by least error method by assuming $a_r=0$ and taking the optimal parabolic curve. By this, we will get

$k=\frac{\sum_{r=0}^{n-1}(f(x)-f(0)-(f(1)-f(0))*x)*(x*(x-1))}{\sum_{r=0}^{n-1}(x*(x-1))^2}$. For continuous function, we get optimal k, a, b by taking integral at the interval and find by least error matching values. Please note that if we have considered other curves, then $k=0$ and

$$a_r=f(r-1)-2*f(r)+f(r+1) \text{ if we have taken with } (Curve2),$$

$a_r = -f(r) + f(r+1)$if we have taken with $(Curve3)$.

In similar way, For Multidimensional functions, if

$f(x_1, x_2, x_3, \ldots, x_d) \exists \forall 0 \le x_s \le (n_s - 1) \forall 1 \le s \le d$ has values for d dimensions, same can be written as following.

$$f(x_1, x_2, x_3, \ldots, x_d) = \sum_{r_d=0}^{n_d-1} \left(\ldots \sum_{r_3=0}^{n_3-1} \left(\sum_{r_2=0}^{n_2-1} \left(\sum_{r_1=0}^{n_1-1} \left(a_{x_d} r_d * \prod_{s=1}^{d} s(x_d, r_d) \right) \right) \right) \right) \tag{419}$$

Where $s(x,r) = s_r(x) + t_r(x)$ and

$$s_r(x_d) = \frac{((b_d - a_d)*(a_d + b_d + 2*a_d*b_d))*(T(x_d - r_d))^2 - (b_d * T(x_d - r_d - a_d))^2 + (a_d * T(x_d - r_d - b_d))^2}{16*(b_d - a_d)*a_d*b_d}$$

$$t_r(x_d) = \frac{\sin(\pi*(r_d - 1))}{2*(r_d - 1)} * \left(\frac{2*k_d*(x_d*(x_d - 1))}{(f_d(1) - f_d(0))} - 1\right) - \frac{\sin(\pi*r_d)}{2*r_d} * (x_d^2 + 3*x_d)$$

$$T(x) = x + |x|,\ 0 < a < b \le 1,\ a \le \frac{2*b^2}{2*b+1} \text{ and we get if} \tag{420}$$

$$f_d(x) = f(x_1, x_2, x_3, \ldots, x_d), \text{ then} \tag{421}$$

$$a_{x_d} r_d = -f_d(r_d - 2) + 3*f_d(r_d - 1) - 3*f_d(r_d) + f_d(r_d + 1) \text{ and } a_{x_d} 0_d = 1, \tag{422}$$

$$f_d(-1) = 2*f_d(0) - 1*f_d(1) + 2*k_d, \tag{423}$$

$$f_d(-2) = 2*f_d(0) - 3*f_d(1) + 6*k_d \tag{424}$$

This function will be smooth since it is parabolic curve in between integer intervals and at integer interval, the slope is same since $s_r'(x) = 0$. For continuous functions we can compute k_d, a, b using least error method after taking integral at the intervals.

If in case, we do not have values at equal intervals, then also we can use the same approach, but scaled with the distance as follows. If $T(x) = x + |x|,\ 0 < a_r < b_r \le x_{(r+1)} - x_r$ then

$$s_r(x) = \frac{(b_r - a_r)*((x_r - x_{(r-1)})*(a_r + b_r) + 2*a_r*b_r)*(T(x - x_r))^2 - (x_r - x_{(r-1)})*(b_r * T(x - x_r - a_r))^2 + (x_r - x_{(r-1)})*(a_r * T(x - x_r - b_r))^2}{8*(x_{(r+1)} - x_r)*(x_{(r+1)} - x_{(r-1)})*(b_r - a_r)*a_r*b_r}$$

will behave like following,

$s_r(x) = 0$ when $x \le x_r$, $s_r(x) = \frac{(b_r - a_r)*((x_r - x_{(r-1)})*a_r + b_r + 2*a_r*b_r)*(x - x_r)^2}{2*(x_{(r+1)} - x_r)*(x_{(r+1)} - x_{(r-1)})*(b_r - a_r)*a_r*b_r}$ when $x \ge x_r$ and $x \le (x_r + a_r)$, $s_r(x) = \frac{(b_r - a_r)*((x_r - x_{(r-1)})*a_r + b_r + 2*a_r*b_r)*(x - x_r)^2 - (b_r*(x - x_r - a_r))^2}{2*(x_{(r+1)} - x_r)*(x_{(r+1)} - x_{(r-1)})*(b_r - a_r)*a_r*b_r}$ when $x \ge (x_r + a_r)$ and $x \le (x_r + b_r)$, $s_r = \frac{(x - x_r)*(x - x_{(r-1)})}{(x_{(r+1)} - x_r)*(x_{(r+1)} - x_{(r-1)})}$ when $x \ge (x_r + b_r)$.

$s_r'(x) = 0$ when $x \le x_r$, $s_r'(x) = \frac{(b_r - a_r)*((x_r - x_{(r-1)})*(a_r + b_r) + 2*a_r*b_r)*(x - x_r)}{(x_{(r+1)} - x_r)*(x_{(r+1)} - x_{(r-1)})*(b_r - a_r)*a_r*b_r}$ when $x \ge x_r$ and $x \le (x_r + a_r)$, $s_r'(x) = \frac{(b_r - a_r)*((x_r - x_{(r-1)})*(a_r + b_r) + 2*a_r*b_r)*(x - x_r) - (b_r^2*(x - x_r - a_r))}{(x_{(r+1)} - x_r)*(x_{(r+1)} - x_{(r-1)})*(b_r - a_r)*a_r*b_r}$ when $x \ge (x_r + a_r)$ and $x \le (x_r + b_r)$, $s_r'(x) = \frac{2*x - (x_r + x_{(r-1)})}{(x_{(r+1)} - x_r)*(x_{(r+1)} - x_{(r-1)})}$ when $x \ge (x_r + b_r)$.

For monotonous incremental curve, all slope should be positive, then at interval $x \ge (x_r + a_r)$ and $x \le (x_r + b_r)$, $(b_r - a_r)*((x_r - x_{(r-1)})*(a_r + b_r) + 2*a_r*b_r) - (x_r - x_{(r-1)})*b_r^2 \ge 0$, $a_r \le \frac{2*b_r^2}{2*b_r + (x_r - x_{(r-1)})}$ $a_r \le \frac{2}{2 + (x_r - x_{(r-1)})}$, $b_r > \frac{(x_r - x_{(r-1)})}{2}$, if $a = \frac{2*b_r^2}{2*b_r + (x_r - x_{(r-1)})}$, then the slope is a straight line and if $b = x_{(r+1)} - x_r, a = \frac{2*(x_{(r+1)} - x_r)^2}{2*x_{(r+1)} - x_r - x_{(r-1)}}$, then curve will have minimum turning points.

Then we can write a discrete function if $f(x) \exists for\ x_0 < x_2 < x_3 <\ldots< x_r \forall 1 \le r \le (n-1)$ has values as following.

$$f(x) = f(x_0) + (f(x_1) - f(x_0)) * \left(\left(\frac{x - x_0}{x_1 - x_0}\right)\right) + k*((x - x_0)*(x - x_1)) + \sum_{r=1}^{n-2} a_r * s_r(x) \tag{425}$$

Where $s_r(x) = \frac{(b_r - a_r)*((x_r - x_{(r-1)})*(a_r + b_r) + 2*a_r*b_r)*(T(x - x_r))^2}{8*(x_{(r+1)} - x_r)*(x_{(r+1)} - x_{(r-1)})*(b_r - a_r)*a_r*b_r} + \frac{-(x_r - x_{(r-1)})*(b_r*T(x - x_r - a_r))^2 + (x_r - x_{(r-1)})*(a_r*T(x - x_r - b_r))^2}{8*(x_{(r+1)} - x_r)*(x_{(r+1)} - x_{(r-1)})*(b_r - a_r)*a_r*b_r}$

$$T(x) = x + |x|,\ 0 < a_r < b_r \le x_{(r+1)} - x_r,\ a_r \le \frac{2*b_r^2}{2*b_r + (x_r - x_{(r-1)})} \text{ and we get} \tag{426}$$

$$a_1 = f(x_2) - (f(x_0) + (f(x_1) - f(x_0)) * \left(\left(\frac{x_2 - x_0}{x_1 - x_0}\right)\right) + k*((x_2 - x_0)*(x_2 - x_1))) \tag{427}$$

$$a_r = \frac{F(r-1,r,r+1)*f(x_{r-2})+F(r-2,r,r+1)*f(x_{r-1})}{F(r-2,r-1,r)} \quad (428)$$
$$\frac{+F(r-2,r-1,r+1)*f(x_r)+F(r-2,r-1,r)*f(x_{r+1})}{F(r-2,r-1,r)} \text{ when } r>1, \text{ where}$$
$$F(u,v,w) = ((f(x_u))^2 * f(x_v) + (f(x_v))^2 * f(x_w) + (f(x_w))^2 * f(x_u)) \quad (429)$$
$$-((f(x_v))^2 * f(x_u) + (f(x_w))^2 * f(x_v) + (f(x_u))^2 * f(x_w)),$$
$$k = \frac{\sum_{r=0}^{n-1}(f(x_r)-f(x_0)-(f(x_1)-f(x_0))*(x_r-x_0))*((x_r-x_0)*(x_r-x_1))}{\sum_{r=0}^{n-1}((x_r-x_0)*(x_r-x_1))^2} \quad (430)$$

This kind of smoothing with Quadratic(Parabolic) curve can be used to find approximate roots of the equation, if the root lies between the given range of points. Since derivative is linear, this can be also used to find approximate turning points of maximum and minimum between range of points

D. Derivation of Faster Smooth Polynomial Curve

Same concept can be extended to higher degree of polynomial also, For Cubical curve, if we get $s_r(x) = (x-r)*(x-r+1)*(x-r+2) => (Curve1)$ or $= (x-r)*(x-r+1) => (Curve2)$ or $(x-r) => (Curve3)$ or $= 1 => (Curve4)$, then computation terms will be reduced. Let us assume, $u = x-r$, then

$$k*u^2 + l*u^3 + m*(u-1)^2 + n*(u-1)^3 = p*(u)*(u+1)*(u+2) \;=>\; (Curve1),$$
$$k*u^2 + l*u^3 + m*(u-1)^2 + n*(u-1)^3 = p*(u)*(u+1) \;=>\; (Curve2),$$
$$k*u^2 + l*u^3 + m*(u-1)^2 + n*(u-1)^3 = p*(u) \;=>\; (Curve3),$$
$$k*u^2 + l*u^3 + m*(u-1)^2 + n*(u-1)^3 = p \;=>\; (Curve4),$$ then, on solving each co-efficient, we get
$$7*u^2 - u^3 + 2*(u-1)^2 + 2*(u-1)^3 = (u)*(u+1)*(u+2) => (Curve1),$$
$$3*u^2 - u^3 + (u-1)^2 + (u-1)^3 = (u)*(u+1) => (Curve2),$$
$$2*u^2 - u^3 + (u-1)^2 + (u-1)^3 = (u) => (Curve3),$$
$$3*u^2 - 2*u^3 + 3*(u-1)^2 + 2*(u-1)^3 = 1 => (Curve4),$$

Let us expalin the concept by taking parameters from $(Curve1)$.

If $T(x) = x + |x|$, then $s_r(x) = \frac{-(T(x-r))^3+7*(T(x-r))^2+2*(T(x-r-1))^2+2*(T(x-r-1))^3}{48}$ will behave like following,

$s_r(x) = 0$ when $x \le r$, $s_r(x) = \frac{(x-r)^2*(7-(x-r))}{6}$ when $x \ge r$ and $x \le (r+1)$, $s_r(x) = \frac{(x-r)*(x-r+1)*(x-r+2)}{6}$ when $x \ge (r+1)$.

$s_r'(x) = 0$ when $x \le r$, $s_r'(x) = \frac{(x-r)*(14-3*(x-r))}{6}$ when $x \ge r$ and $x \le (r+1)$, $s_r'(x) = \frac{3*(x-r)^2+6*(x-r)+2}{6}$ when $x \ge (r+1)$.

Then we can write a discrete function if $f(x) \exists \forall 0 \le x \le (n-1)$ has values as following.

$$f(x) = f(0) + (f(1)-f(0))*x + (f(2)-2*f(1)+f(0))*(\tfrac{x*(x-1)}{2}) \quad (431)$$
$$+k*(x*(x-1)*(x-2)) + \textstyle\sum_{r=2}^{n-2} a_r * s_r(x)$$

Where $s_r(x) = \frac{-(T(x-r))^3+7*(T(x-r))^2+2*(T(x-r-1))^2+2*(T(x-r-1))^3}{48}$,

$T(x) = x + |x|$, and we get

$$a_2 = -f(0) + 3*f(1) - 3*f(2) + f(3) - 6*k, \quad (432)$$
$$a_r = f(r-3) - 4*f(r-2) + 6*f(r-1) - 4*f(r) + f(r+1) \text{ when } r>2 \text{ and}, \quad (433)$$
$$k = \frac{\sum_{r=0}^{n-1}(f(x)-f(0)-(f(1)-f(0))*x-(f(2)-2*f(1)+f(0))*(\frac{x*(x-1)}{2}))*(x*(x-1)*(x-2))}{\sum_{r=0}^{n-1}(x*(x-1)*(x-2))^2} \quad (434)$$

Please note that if we have considered other curves, then, $k = 0$ and,

$a_r = -f(r-2) + 3*f(r-1) - 3*f(r) + f(r+1)$ if we have taken with $(Curve2)$,

$a_r = f(r-1) - 2*f(r) + f(r+1)$ if we have taken with $(Curve3)$,

$a_r = -f(r) + f(r+1)$ if we have taken with $(Curve4)$

If in case, we do not have values at equal intervals, then also we can use the same approach, but scaled with the distance as follows. If $T(x) = x + |x|$, then

$$s_r(x) = \frac{((x_{(r+1)}-x_r)^2-(x_r-x_{(r-1)})*(x_r-x_{(r-2)}))*(T(x-x_r))^3}{8*(x_{(r+1)}-x_r)^2*(x_{(r+1)}-x_{(r-2)})*(x_{(r+1)}-x_{(r-1)})*(x_{(r+1)}-x_r)} + \frac{(((x_{(r-1)}+x_{(r-2)}-2*x_r)*(x_{(r+1)}-x_r)^2+2*(x_r-x_{(r-1)})*(x_r-x_{(r-2)})*(T(x-x_r))^2}{8*(x_{(r+1)}-x_r)^2*(x_{(r+1)}-x_{(r-2)})*(x_{(r+1)}-x_{(r-1)})*(x_{(r+1)}-x_r)} + \frac{(x_r-x_{(r-1)})*(x_r-x_{(r-2)})*((x_{(r+1)}-x_r)*(T(x-x_{(r+1)}))^2+(x_r-x_{(r-1)})*(x_r-x_{(r-2)})*(T(x-x_{(r+1)}))^3}{8*(x_{(r+1)}-x_r)^2*(x_{(r+1)}-x_{(r-2)})*(x_{(r+1)}-x_{(r-1)})*(x_{(r+1)}-x_r)}$$ will behave like following.

$s_r(x) = 0$ when $x \le r$,

$s_r(x) =$

$\frac{((x_{(r+1)}-x_r)^2-(x_r-x_{(r-1)})*(x_r-x_{(r-2)}))*(x-x_r)^3}{(x_{(r+1)}-x_r)^2*(x_{(r+1)}-x_{(r-2)})*(x_{(r+1)}-x_{(r-1)})*(x_{(r+1)}-x_r)}$ +

$\frac{(((x_{(r-1)}+x_{(r-2)}-2*x_r)*(x_{(r+1)}-x_r)^2+2*(x_r-x_{(r-1)})*(x_r-x_{(r-2)})*(x-x_r)^2}{(x_{(r+1)}-x_r)^2*(x_{(r+1)}-x_{(r-2)})*(x_{(r+1)}-x_{(r-1)})*(x_{(r+1)}-x_r)}$ when $x \geq r$ and $x \leq (r+1), s_r(x) =$

$\frac{(x-x_r)*(x-x_{(r-1)})*(x-x_{(r-2)})}{(x_{(r+1)}-x_r)*(x_{(r+1)}-x_{(r-1)})*(x_{(r+1)}-x_{(r-2)})}$ when $x \geq (r+1)$.

$s_r'(x) = 0$ when $x \leq r$,

$s_r'(x) =$

$\frac{((x_{(r+1)}-x_r)^2-(x_r-x_{(r-1)})*(x_r-x_{(r-2)}))*3*(x-x_r)^2}{(x_{(r+1)}-x_r)^2*(x_{(r+1)}-x_{(r-2)})*(x_{(r+1)}-x_{(r-1)})*(x_{(r+1)}-x_r)}$ +

$\frac{(((x_{(r-1)}+x_{(r-2)}-2*x_r)*(x_{(r+1)}-x_r)^2+2*(x_r-x_{(r-1)})*(x_r-x_{(r-2)})*2*(x-x_r)}{(x_{(r+1)}-x_r)^2*(x_{(r+1)}-x_{(r-2)})*(x_{(r+1)}-x_{(r-1)})*(x_{(r+1)}-x_r)}$ when $x \geq r$ and $x \leq (r+1)$,

$s_r'(x) = \frac{(x-x_r)*(x-x_{(r-1)})+(x-x_r)*(x-x_{(r-2)})+(x-x_{(r-1)})*(x-x_{(r-2)})}{(x_{(r+1)}-x_r)*(x_{(r+1)}-x_{(r-1)})*(x_{(r+1)}-x_{(r-2)})}$ when $x \geq (r+1)$.

This curve will have minimum turning points and monotonous incremental curve. Using this curve, we can write a discrete function if $f(x) \exists for\ x_0 < x_2 < x_3 <\ldots< x_r \forall 1 \leq r \leq (n-1)$ has values as following.

$$f(x) = f(x_0) + (f(x_1) - f(x_0)) * ((\frac{x-x_0}{x_1-x_0})) + \tag{435}$$

$$+(f(x_2) - (f(x_0) + (f(x_1) - f(x_0)) * ((\frac{x_2-x_0}{x_1-x_0}))))) * \frac{(x-x_0)*(x-x_1)}{(x_2-x_0)*(x_2-x_1)} +$$

$$+k * ((x - x_0) * (x - x_1) * (x - x_2)) + \sum_{r=2}^{n-2} a_r * s_r(x)$$

$$\text{Where } s_r(x) = \frac{((x_{(r+1)}-x_r)^2-(x_r-x_{(r-1)})*(x_r-x_{(r-2)}))*(T(x-x_r))^3}{8*(x_{(r+1)}-x_r)^2*(x_{(r+1)}-x_{(r-2)})*(x_{(r+1)}-x_{(r-1)})*(x_{(r+1)}-x_r)}$$

$$+\frac{(((x_{(r-1)}+x_{(r-2)}-2*x_r)*(x_{(r+1)}-x_r)^2+2*(x_r-x_{(r-1)})*(x_r-x_{(r-2)})*(T(x-x_r))^2}{8*(x_{(r+1)}-x_r)^2*(x_{(r+1)}-x_{(r-2)})*(x_{(r+1)}-x_{(r-1)})*(x_{(r+1)}-x_r)}$$

$$+\frac{+(x_r-x_{(r-1)})*(x_r-x_{(r-2)})*((x_{(r+1)}-x_r)*(T(x-x_{(r+1)}))^2+}{8*(x_{(r+1)}-x_r)^2*(x_{(r+1)}-x_{(r-2)})*(x_{(r+1)}-x_{(r-1)})*(x_{(r+1)}-x_r)}$$

$$+\frac{+(x_r-x_{(r-1)})*(x_r-x_{(r-2)})*(T(x-x_{(r+1)}))^3}{8*(x_{(r+1)}-x_r)^2*(x_{(r+1)}-x_{(r-2)})*(x_{(r+1)}-x_{(r-1)})*(x_{(r+1)}-x_r)}$$

$$T(x) = x + |x|,\ 0 < a_r < b_r \leq x_{(r+1)} - x_r,\ a_r \leq \frac{2*b_r^2}{2*b_r+(x_r-x_{(r-1)})} \text{ and we get} \tag{436}$$

$$a_2 = f(x_3) - (f(x_0) + (f(x_1) - f(x_0)) * ((\frac{x-x_0}{x_1-x_0})) + \tag{437}$$

$$+(f(x_2) - (f(x_0) + (f(x_1) - f(x_0)) * ((\frac{x_2-x_0}{x_1-x_0}))))) * \frac{(x-x_0)*(x-x_1)}{(x_2-x_0)*(x_2-x_1)} +$$

$$+k * ((x - x_0) * (x - x_1) * (x - x_2)))$$

$$a_r = \frac{F(r-2,r-1,r,r+1)*f(x_{r-3})+F(r-3,r-1,r,r+1)*f(x_{r-2})+}{F(r-3,r-2,r-1,r)} \tag{438}$$

$$\frac{+F(r-3,r-2,r,r+1)*f(x_{r-1})+F(r-3,r-2,r-1,r+1)*f(x_r)+}{F(r-3,r-2,r-1,r)}$$

$$\frac{+F(r-3,r-2,r-1,r)*f(x_{r+1})}{F(r-3,r-2,r-1,r)} \text{ when } r > 2, \text{ where}$$

$$F(t,u,v,w) = ((f(x_t))^3 * G(u,v,w) - ((f(x_u))^3 * G(v,w,t) + \tag{439}$$

$$+((f(x_v))^3 * G(w,t,u) - ((f(x_w))^3 * G(w,t,u)$$

$$G(u,v,w) = ((f(x_u))^2 * f(x_v) + (f(x_v))^2 * f(x_w) + (f(x_w))^2 * f(x_u)) \tag{440}$$

$$-((f(x_v))^2 * f(x_u) + (f(x_w))^2 * f(x_v) + (f(x_u))^2 * f(x_w)),$$

$$k = \frac{\sum_{r=0}^{n-1}(f(x_r)-f(x_0)-(f(x_1)-f(x_0))*(x_r-x_0))*(X_r)}{\sum_{r=0}^{n-1}((x_r-x_0)*(x_r-x_1)*(x_r-x_2))^2} + \tag{441}$$

$$\frac{\sum_{r=0}^{n-1}(f(x_2)-(f(x_0)+(f(x_1)-f(x_0))*((\frac{x_2-x_0}{x_1-x_0}))))*\frac{(x-x_0)*(x-x_1)}{(x_2-x_0)*(x_2-x_1)}*(X_r)}{\sum_{r=0}^{n-1}((x_r-x_0)*(x_r-x_1)*(x_r-x_2))^2} \text{ where}$$

$$X_r = (x_r - x_0) * (x_r - x_1) * (x_r - x_2)$$

The same also can be extended to multidimensional as well as continuous functions. Thus we could transform to a polynomial curve with high smoothness and at the same time, less computations even for unequal distant points.

E. Derivation of Faster Smooth Geometrical Curve

Same concept can be extended to Geometric series also, For having 2 geometric powers $p^x - 1$ and $q^x - 1$ or $p^x * cos(u * x) - 1$ and $p^x * sin(u * x)$, Transformation curve is given below.

$log(q) * p^x - log(p) * q^x - log(q) + log(p)$ or $u * p^x * cos(u * x) - log(p) * p^x * sin(u * x) - u.$

And to get monotonous incremental curve with turning points, Let me explain with the concept of values at unequal intervals which can be applied to equal intervals also. Let us take locations as x_{r-1}, x_r, x_{r+1} , If $T(x) = \frac{x+|x|}{2}$, and $S(x) = log(q) * p^x - log(p) * q^x - log(q) + log(p)$ or $u * p^x * cos(u * x) - log(p) * p^x * sin(u * x) - u$, then we need to solve k, l from 2 equations namely, $k * S(-a_r) + l * S(-b_r) = 0$ and $S(x_{r-1} - x_r) + k * S(x_{r-1} - x_r - a_r) + l * S(x_{r-1} - x_r - b_r) = 0$, subject to $0 < a_r < b_r \le x_{(r+1)} - x_r$.

Then if $V_r(x) = S(x - x_r) + k * S(x - x_r - a_r) + l * S(x_1 - x_0 - b_r)$, then $U_r(x) = V_r(T(x))$ is the curve with monotonous incremental turning points. To find co-efficient, we need to solve these 3 equations, $V_0(x_{r+n}) * k_0 + V_0(x_{r+n+1}) * k_1 + V_0(x_{r+n+2}) * k_2 + V_0(x_{r+n+3}) = 0$, by substituting $-1 \le n \le 1$.

Then our solution will be $f(x) = f(x_0) + (f(x_1) - f(x_0)) * (\frac{U_0(x) - U_0(x_0)}{U_0(x_1) - U_0(x_0)}) + \sum_{r=1}^{n-2} a_r * \frac{U_r(x)}{U_r(x_{(r+1)})}$ where

$$a_r = f(x_{(r-2)}) * k_0 + f(x_{(r-1)}) * k_1 + f(x_{(r)}) * k_2 + f(x_{(r+1)}).$$

This kind of smoothing with having 2 geometric powers curve can be used to find find approximate turning points of maximum and minimum because its derivative will be form $a * p^x - b * q^x = 0$ or $a * p^x * cos(u * x) - b * p^x * sin(u * x) = 0$, which will have solution as $x = \frac{log(a) - log(b)}{log(p) - log(q)}$ or $x = \frac{tan^{-1}(\frac{a}{b})}{u}$

Similarly for geometric series with 3 geometric powers, $p^x - 1, q^x - 1$ and $v^x - 1$,or $p^x * cos(u * x) - 1, p^x * sin(u * x)$ and $q^x - 1$, Transformation curve is given below.

$S(x, K_1, K_2) = K_1 * (log(v) * p^x - log(p) * v^x - log(v) + log(p)) + K_2 * (log(v) * q^x - log(q) * v^x - log(v) + log(q))$ or

$$S(x, K_1, K_2) = K_1 * (log(q) * p^x * cos(u * x) - log(p) * q^x - log(q) + log(p)) + K_2 * (log(q) * sin(u * x) - u * q^x + u).$$

And to get monotonous incremental curve with turning points, Let me explain with the concept of values at unequal intervals which can be applied to equal intervals also. Let us take locations as $x_{r-2}, x_{r-1}, x_r, x_{r+1}$,If $T(x) = \frac{x+|x|}{2}$, and Then, we need to solve K_1, K_2, L_1, L_2 from 3 equations namely, $S(x_r - x_{r+1}, L_1, L_2) = 0, S(x_{r-1} - x_r, K_1, K_2) + S(x_{r-1} - x_{r+1}, L_1, L_2) = 0$ and $S(x_{r-2} - x_r, K_1, K_2) + S(x_{r-2} - x_{r+1}, L_1, L_2) = 0$, assuming any one of the variable, say $K_1 = 1$. Then if $V_r(x) = S(x - x_r, K_1, K_2) + S(x - x_{r+1}, L_1, L_2)$, then $U_r(x) = V_r(T(x))$ is the curve with minimal monotonous incremental turning points. To find co-efficient we need to solve these 4 equations, $V_0(x_{r+n}) * k_0 + V_0(x_{r+n+1}) * k_1 + V_0(x_{r+n+2}) * k_2 + V_0(x_{r+n+3}) * k_3 + V_0(x_{r+n+4}) = 0$, by substituting $-2 \le n \le 1$. Then our solution will be

$f(x) = f(x_0) + (f(x_1) - f(x_0)) * (\frac{U_0(x) - U_0(x_0)}{U_0(x_1) - U_0(x_0)}) + (f(x_2) - f(x_0) - (f(x_1) - f(x_0)) * (\frac{U_0(x_2) - U_0(x_0)}{U_0(x_1) - U_0(x_0)})) * (\frac{U_1(x) - U_1(x_1)}{U_1(x_2) - U_1(x_1)}) + \sum_{r=2}^{n-2} a_r * \frac{U_r(x)}{U_r(x_{(r+1)})}$ where $a_r = f(x_{(r-3)}) * k_0 + f(x_{(r-2)}) * k_1 + f(x_{(r-1)}) * k_2 + f(x_{(r)}) * k_3 + f(x_{(r+1)})$.

The same also can be extended to multiple geometric powers or combination of polynomial and geometric powers and also for multidimensional discrete as well as continuous functions. To find appropriate geometric powers of the data in continuous or discrete functions, we need to go with approach of finding roots as explained in (198) to (203). Thus we could transform to a geometrical curve with high smoothness and at the same time, less computations even for unequal distant points. This might be useful where there are high fluctuations such as stock market or gold rates. Since most of the cubical equation has one real root and 2 complex roots, cubical geometric power will suit high fluctuation functions such as stock market or gold rates.

F. Derivation of Decorative Polygonal Curve

I have found one method to get a decorative curve with out much computation needed to find a_r. Let us have the following identity.Let $a_1 < a_2 < a_3 < \ldots < a_e, 0 \le (a_e - a_1) \le 1$ for having e edges and Let,

$$T(x) = \sum_{s=1}^{e} k_s * (|x - a_s|) \text{ and } \sum_{s=1}^{e} k_s = 0, \sum_{s=1}^{e} k_s * a_s = 0, T(m) \ne 0, S(x, r) = \frac{T(x-r)}{T(m)} \quad (442)$$

then this curve will have the following property

$S(x, r) = 0$ when $x \le r + a_1$ and $S(x, r) = 0$ when $x \ge r + a_e$. (443)

for example when $e = 3$ it will have triangular edges and we get following identity (444)

$$T(x) = (a_3 - a_2) * (|x - a_1|) + (a_1 - a_3) * (|x - a_2|) + (a_2 - a_1) * (|x - a_3|) \quad (445)$$

Since we need to solve only 2 equations to find k_s when $e > 3, k_s$ has multiple solutions which will define the degree of polygons with positive and negative edges. Then we can write a discrete function if $f(x) \exists \forall 0 \le x \le (n - 1)$ has values as following.

$f(x) = \sum_{r=0}^{n-1} (f(r) * S(x, r))$ and one of the example as triangular polygon is (446)

$$T(x) = (a_3 - a_2) * (|x - a_1|) + (a_1 - a_3) * (|x - a_2|) + (a_2 - a_1) * (|x - a_3|) \quad (447)$$

for symmetric, we can consider $a_1 = -0.5, a_2 = 0.0, a_3 = 0.5$

$$T(x) = 0.5 * (|x + 0.5|) - (|x|) + (0.5) * (|x - 0.5|), T(0) = 0.5, \text{ which leads to} \quad (448)$$

$$S(x,r) = \frac{(|2*x-2*r+1|)-(|2*x-2*r|)+(|2*x-2*r-1|)}{2} \text{ and hence} \quad (449)$$

$$f(x) = \sum_{r=0}^{n-1} \left(f(r) * \frac{(|2*x-2*r+1|)-(|2*x-2*r|)+(|2*x-2*r-1|)}{2} \right) \quad (450)$$

Here we get straight forward the co-efficient as $f(r)$ and no computation needed at all. But this curve is of polygonal nature and decorative in between integer intervals because at equal distance, it falls to 0.i.e) at $-r - 0.5, -r + 0.5$ In similar way, For Multidimensional discrete functions, if $f(x_1, x_2, x_3, \dots, x_d) \exists \forall 0 \le x_s \le (n_s - 1) \forall 1 \le s \le d$ has values for d dimensions, same can be written as following.

$$f(x_1, x_2, x_3, \dots, x_d) = \sum_{r_d=0}^{n_d-1} \left(\dots \sum_{r_3=0}^{n_3-1} \left(\sum_{r_2=0}^{n_2-1} \left(\sum_{r_1=0}^{n_1-1} \left(f(r_1, r_2, r_3, \dots, r_d) * \prod_{s=1}^{d} S(x_d, r_d) \right) \right) \right) \right) \text{ where} \quad (451)$$

$$S(x,r) = \frac{(|2*x-2*r+1|)-(|2*x-2*r|)+(|2*x-2*r-1|)}{2}. \quad (452)$$

Here there is no computation needed at all. But this curve is of polygonal nature in between integer intervals and also decorative in between integer intervals because at equal distance, it falls to 0 at $-r - 0.5$, and $-r + 0.5$

G. Derivation of Decorative Smooth Quartic Curve

Let us have a function, $T(x) = \frac{(2*x+1)*(2*x-1)*(((2*x+1)*(|(2*x-1)|))-((2*x-1)*(|(2*x+1)|)))}{2}, S(x,r) = T(x - r)$.

Here $S(x,r) = S'(x,r) = 0$ when $x \le r$ and $S(x,r) = S'(x,r) = 0$ when $x \ge (r + 1)$ .Then we can write a discrete function if $f(x) \exists \forall 0 \le x \le (n - 1)$ has values as following.

$$f(x) = \sum_{r=0}^{n-1} (f(r) * T(x - r)) \text{ where} \quad (453)$$

$$T(x) = \frac{(2*x+1)*(2*x-1)*(((2*x+1)*(|(2*x-1)|))-((2*x-1)*(|(2*x+1)|)))}{2} \quad (454)$$

Here, there is no computation needed at all. It will be smooth, Since it is quartic curve in between integer intervals and at integer interval, the slope is same since $T'(x) = 0$ at both the edges and hence, it will be a decorative curve. For continuous functions,if $f(x) \exists \forall 0 \le x \le (n - 1)$ we can compute as following.

$$f(x) = \sum_{r=0}^{n-1} (a_r * T(x - r)) \text{ where} \quad (455)$$

$$T(x) = \frac{(2*x+1)*(2*x-1)*(((2*x+1)*(|(2*x-1)|))-((2*x-1)*(|(2*x+1)|)))}{2} \quad (456)$$

$$a_r = \frac{\int_{x=0}^{n-1} ((f(x)*T(x-r))*dx)}{\int_{x=0}^{n-1} ((T(x-r))^2*dx)} \text{ which reduces to} \quad (457)$$

$$a_r = \frac{15}{4} * \int_{x=r}^{r+1} \left((f(x) * T(x - r)) * dx \right) \quad (458)$$

In similar way, For Multidimensional discrete functions, if $f(x_1, x_2, x_3, \dots, x_d) \exists \forall 0 \le x_s \le (n_s - 1) \forall 1 \le s \le d$ has values for d dimensions, same can be written as following.

$$f(x_1, x_2, x_3, \dots, x_d) = \sum_{r_d=0}^{n_d-1} \left(\dots \sum_{r_3=0}^{n_3-1} \left(\sum_{r_2=0}^{n_2-1} \left(\sum_{r_1=0}^{n_1-1} \left(f(r_1, r_2, r_3, \dots, r_d) * \prod_{s=1}^{d} T(x_d - r_d) \right) \right) \right) \right) \text{ where} \quad (459)$$

$$T(x) = \frac{(2*x+1)*(2*x-1)*(((2*x+1)*(|(2*x-1)|))-((2*x-1)*(|(2*x+1)|)))}{2} \quad (460)$$

and there is no computation needed at all. It will be smooth, Since it is quartic curve in between integer intervals and at integer interval, the slope is same since $T'(x) = 0$ at both the edges and hence, it will be a decorative curve. For continuous functions, Like we did in single dimension, we need to take integral to compute at the interval.

V. COMPARISON WITH STANDARD ORTHOGONAL TRANSFORM FUNCTIONS

Fourier series of periodic function with L periodicity and obtaining $L * (r + 1)$ terms have error magnitude of $\frac{1}{L*(r+1)}$ where as polynomial function with degree r having same $L * (r + 1)$ terms have error magnitude of $\frac{1}{L^{(r+1)}}$ and sign functions have error magnitude of $\frac{1}{2*L*(r+1)}$. Also Exponential geometric functions have error magnitude of $\frac{1}{c*L*(r+1)}$ where $1 \le c$ and Harmonic functions have error magnitude of $\frac{1}{L^2*(r+1)}$. Also we could get series of variable amplitude at different intervals which is not possible with normal periodic functions.

VI. COMPARISON WITH STANDARD CURVE FUNCTIONS OF HIGH PRECISION OR SMOOTHING AND DECORATIVE FUNCTIONS

This series will be able to transform multidimensional discrete functions to a continuous coiled or smoothness rich function and coiled function will always be within the maximum and minimum value of any range. Instead of cubical curve which is currently in use with matching nearby slopes, we could get alternate high precision and smooth curve at random points with parabolic function itself without needing of near by points of $f'(x_m)$ slope to match current slope. Also we are able to get a solution even if $f'(x_m) = \pm\infty$.

VII. REAL-WORLD APPLICATIONS

A. Application on data reduction techniques

Sign functions can be used to compress photos which gives 20% more than Jpeg with similar noise ratio. Exponential geometric functions and harmonic functions can be used to compress audio. Combination of Polynomial, Exponential, harmonic and Sign functions can be used to compress multimedia. Combination of Polynomial, Exponential, harmonic and Sign functions can be used to encrypt Digital data. Combination of Polynomial, Exponential and Sign functions can be used to transmit multiple format of waves with same digital band and frequency. In smooth techniques, harmonic conversion can be used to compress multimedia since color, sound and animation will not exceed the ranges and Quadratic or polynomial conversion would be able to scale freely since this is a high smooth curve.

B. Application on prediction analysis with Extension to Fourier series

Exponential functions and harmonic functions can be used to predict Stock market, Gold rates, Economic growth. Combination of Polynomial and Sign functions can be used to predict any repetition functions like weather report, malaria, dengue outbreak etc.

C. Application on prediction analysis with Extension to High Precision Smooth Techniques

It is enough to take first few highest maximum and lowest minimum value to get more optimized continuous function with less error rate with harmonic function. Harmonic conversion can be used to compress multimedia since color, sound and animation will not exceed the ranges and Quadratic or polynomial conversion would be able to scale freely since this is a high smooth curve.Exponential functions and harmonic functions can be used to predict Stock market, Gold rates, Economic growth. This method also can be useful in finding roots of the equation and maximum, minimum turning points of the curve.

VIII. CONCLUSION

Thus I have extended the Fourier series to a combination of polynomial, geometric, sign and harmonic functions. Consequently, the nature of these functions finds application in data reduction techniques without compromising the fidelity and integrity of the function. Apart from extension to Fourier series, the high precision techniques written by me here also allow to transform any multidimensional discrete or continuous function to a continuous coiled or high smoothness or decorative function as well as help to find roots and maximum, minimum turning points. I have demonstrated the efficiency of the extension in terms of precision error, compression ratio and implementation complexity while applying it to real-world problems such as faster live streaming, prediction of stock market data, and storage of medical imaging data.

ACKNOWLEDGMENT

I heartfully thank my wife Mangai, my son Navinkumar, my brothers S. Mari Rajan, Ramraj, and especially Thiruppathi Rajan for suggestions, proof reading and corrections.

REFERENCES

[1] This paper is an original illustration consisting of 24 concepts, all of which are discovered by me (10 extensions to Fourier Series, 7 varieties of high precision functions, 5 categories of smoothing curves, and 2 types of decorating paths). Since I have neither referred to nor copied from any articles, there are no reference links mentioned here.