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SHARP BOUNDS OF FEKETE-SZEGÖ FUNCTIONAL FOR QUASI-SUBORDINATION CLASS

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ABSTRACT. In this work, considering a special subclass of the family of holomorphic functions in an open unit disk, defined by means of quasi-subordination, we determine sharp bounds for Fekete-Szegö functional $|d_3 - \xi d_2^2|$ of functions in this class. Several results for new classes and connections to known classes are mentioned.
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1. INTRODUCTION, PRELIMINARIES AND DEFINITIONS

Let $\mathscr{A}$ be the family of normalized functions that have the form

$$s(z) = z + \sum_{k=2}^{\infty} d_k z^k, \tag{1.1}$$

which are holomorphic in $\mathfrak{D} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{S}$ be the collection of all members of $\mathscr{A}$ that are univalent in $\mathfrak{D}$. Let $\varsigma(z)$ be holomorphic function in $\mathfrak{D}$ with $|\varsigma(z)| \leq 1, z \in \mathfrak{D}$, such that

$$\varsigma(z) = R_0 + R_1 z + R_2 z^2 + ... \tag{1.2}$$

where $R_0, R_1, R_2, ...$ are real. Let $\mathfrak{h}(z)$ be holomorphic function in $\mathfrak{D}$, with $\mathfrak{h}(0) = 1, \mathfrak{h}'(0) > 0$, having the positive real part, such that

$$\mathfrak{h}(z) = 1 + Q_1 z + Q_2 z^2 + ... \tag{1.3}$$

where $Q_1, Q_2, Q_3, ...$ are real and $Q_1 > 0$. Through out this work we shall assume that the functions ς and $\mathfrak{h}$ follow the above conditions unless otherwise mentioned.

It is known that for $s \in \mathcal{S}$ given by (1.1), there holds upper bounds for $|d_3 - \xi d_2^2|$ when ξ is real, which are sharp (see [4]). Since then, the estimation of the sharp upper bounds for $|d_3 - \xi d_2^2|$ with ξ being an arbitrary real or complex number for any compact collection $\mathfrak{F}$ of elements in $\mathcal{S}$ is well- known as the Fekete- Szegö problem for $\mathfrak{F}$. Several researchers (including [2], [5], [9], [11], [13], [16]) have estimated sharp Fekete-Szegö bounds for many subclasses of $\mathcal{S}$.

We recall the principle of subordination and also the principle of majorization, between two holomorphic functions $s(z)$ and $\nu(z)$ in $\mathfrak{D}$. We say that $s(z)$ is subordinate to $\nu(z)$, written as $s(z) \prec \nu(z)$, $z \in \mathfrak{D}$, if there is a holomorphic function $u(z)$ in $\mathfrak{D}$, with $u(0) = 0$ and $|u(z)| < 1$, $z \in \mathfrak{D}$, such that $s(z) = \nu(u(z))$. Moreover $s(z) \prec \nu(z)$ is equivalent to $s(0) = \nu(0)$ and $s(\mathfrak{D}) \subset \nu(\mathfrak{D})$, if ν is univalent in $\mathfrak{D}$. We know that $s(z)$ is majorized by $\nu(z)$, written as $s(z) \prec\prec \nu(z), z \in \mathfrak{D}$, if there exists a holomorphic function $\varsigma(z)$, $z \in \mathfrak{D}$, with $|\varsigma(z)| \leq 1$, such that $s(z) = \varsigma(z)\nu(z)$, $z \in \mathfrak{D}$.

Robertson [15] introduced a new concept called quasi-subordination, which generalizes both the concepts of subordination and majorization. For two holomorphic functions $s(z)$ and $\nu(z)$, $s(z)$ is quasi-subordinate to $\nu(z)$, written as $s(z) \prec_q \nu(z)$, $z \in \mathfrak{D}$, if there exists two holomorphic functions ς and u with $|\varsigma(z)| \leq 1, u(0) = 0$ and $|u(z)| < 1$ such that $s(z) = \varsigma(z)\nu(u(z))$, $z \in \mathfrak{D}$.

Observe that if $\varsigma(z) = 1$, then $s(z) = \nu(u(z)), z \in \mathfrak{D}$, so that $s(z) \prec \nu(z)$ in $\mathfrak{D}$. Also note that if $u(z) = z$, then $s(z) = \varsigma(z)\nu(z)$, $z \in \mathfrak{D}$ and hence $s(z) \prec\prec \nu(z)$ in $\mathfrak{D}$. There are more studies related to quasi-subordination such as [1], [3], [5], [6], [7], [10], [11], [13], [14], [18].

Let Υ be the family of functions holomorphic in $\mathfrak{D}$ of the form

$$u(z) = u_1 z + u_2 z^2 + u_3 z^3 + ... \tag{1.4}$$

satisfying the condition $|u(z)| < 1$, $z \in \mathfrak{D}$. We now state a lemma of [8], which is used to prove our main result.

Lemma 1.1. *If $u \in \Upsilon$, then for any complex number ξ, we have $|u_1| \leq 1, |u_2 - \xi u_1^2| \leq 1 + (|\xi| - 1)|u_1|^2 \leq max\{1, |\xi|\}$. $u(z) = z$ or $u(z) = z^2$ exhibit the sharpness of the result.*

Inspired by recent trends on quasi-subordination, we define the following new subclasses of the family $\mathscr{A}$.

Definition 1.1. We say that $s(z)$ in $\mathscr{A}$ belongs to $\mathfrak{R}_q(\tau, \mu, \mathfrak{h})$, $\tau \geq 0$, $\mu \in \mathbb{C} - \{0\}$, if

$$\frac{1}{\mu}\left(\left(\frac{zs'(z)}{s(z)}\right)\left(1 + \tau\frac{zs''(z)}{s'(z)}\right) - 1\right) \prec_q (\mathfrak{h}(z) - 1),\ z \in \mathfrak{D},$$

where $\mathfrak{h}$ is as stated in (1.3).

Definition 1.2. We say that $s(z)$ in $\mathscr{A}$ belongs to $\mathfrak{H}_q(\tau, \mu, \mathfrak{h})$, $\tau \geq 1$, $\mu \in \mathbb{C} - \{0\}$, if

$$1 + \frac{1}{\mu}\left(\tau\frac{zs''(z)}{s'(z)}\right) \prec_q \mathfrak{h}(z), z \in \mathfrak{D},$$

where $\mathfrak{h}$ is as stated in (1.3).

Definition 1.3. We say that $s(z)$ in $\mathscr{A}$ belongs to $\mathfrak{K}_q(\mu, \mathfrak{h})$, $\mu \in \mathbb{C} - \{0\}$, if

$$\frac{1}{\mu}\left(\frac{(z^2 s'(z))'}{(zs(z))'} - 1\right) \prec_q (\mathfrak{h}(z) - 1), z \in \mathfrak{D},$$

where $\mathfrak{h}$ is as stated in (1.3).

Definition 1.4. We say that $s(z)$ in $\mathscr{A}$ belongs to $\mathfrak{J}_q(\mu, \mathfrak{h})$, $\mu \in \mathbb{C} - \{0\}$, if

$$\frac{1}{\mu}\left(\frac{(z^2 s'(z))'}{2s(z)} - 1\right) \prec_q (\mathfrak{h}(z) - 1),\ z \in \mathfrak{D},$$

where $\mathfrak{h}$ is as stated in (1.3).

Definition 1.5. We say that $s(z)$ in $\mathscr{A}$ belongs to $\mathfrak{L}_q(\mu, \mathfrak{h})$, $\mu \in \mathbb{C} - \{0\}$, if

$$\frac{1}{\mu}\left(\frac{2z(zs'(z))'}{(zs(z))'} - 1\right) \prec_q (\mathfrak{h}(z) - 1),\ z \in \mathfrak{D},$$

where $\mathfrak{h}$ is as stated in (1.3).

Motivated by the paper [17] and earlier works on quasi-subordination, we now define a new special class $\mathscr{S}_q(\tau, \gamma, \mu, \mathfrak{h})$.

Definition 1.6. We say that $s(z)$ in $\mathscr{A}$ belongs to $\mathscr{S}_q(\tau, \gamma, \mu, \mathfrak{h})$, $0 \leq \gamma \leq 1$, $\tau \geq 0$, $\tau \geq \gamma$, $\mu \in \mathbb{C} - \{0\}$, if

$$\frac{1}{\mu}\left(\frac{zs'(z) + \tau z^2 s''(z)}{(1-\gamma)s(z) + \gamma zs'(z)} - 1\right) \prec_q (\mathfrak{h}(z) - 1),\ z \in \mathfrak{D},$$

where $\mathfrak{h}$ is as stated in (1.3).

Clearly a function s is in the class $\mathscr{S}_q(\tau,\gamma,\mu,\mathfrak{h})$ if and only if there exits a holomorphic function $\varsigma(z)$ with $|\varsigma(z)|\leq 1$, such that

$$\frac{\frac{1}{\mu}\left(\frac{zs'(z)+\tau z^2 s''(z)}{(1-\gamma)s(z)+\gamma zs'(z)}-1\right)}{\varsigma(z)} \prec (\mathfrak{h}(z)-1),\ z\in\mathfrak{D},$$

where $\mathfrak{h}$ is as stated in (1.3).

If we set $\varsigma(z)\equiv 1$, $z\in\mathfrak{D}$, then the class $\mathscr{S}_q(\tau,\gamma,\mu,\mathfrak{h})$ is denoted by $\mathscr{S}(\tau,\gamma,\mu,\mathfrak{h})$ satisying the condition

$$1+\frac{1}{\mu}\left(\frac{zs'(z)+\tau z^2 s''(z)}{(1-\gamma)s(z)+\gamma zs'(z)}-1\right)\prec \mathfrak{h}(z),\ z\in\mathfrak{D}.$$

The family of $\mathscr{S}_q(\tau,\gamma,\mu,\mathfrak{h})$ is of special interest . In view of this, we deem it worth while to note the relevance of the class $\mathscr{S}_q(\tau,\gamma,\mu,\mathfrak{h})$ with classes defined above as well as some known ones. Indeed we have$i)\mathscr{S}_q(\tau,0,\mu,\mathfrak{h})=\mathfrak{R}_q(\tau,\mu,\mathfrak{h})$, $ii)\mathscr{S}_q(\tau,1,\mu,\mathfrak{h})=\mathfrak{H}_q(\tau,\mu,\mathfrak{h})$, $iii)\mathscr{S}_q(1/2,1/2,\mu,\mathfrak{h})=\mathfrak{K}_q(\mu,\mathfrak{h})$, $iv)\mathscr{S}_q(1/2,0,\mu,\mathfrak{h})=\mathfrak{J}_q(\mu,\mathfrak{h})$, $v)\mathscr{S}_q(1,1/2,\mu,\mathfrak{h})=\mathfrak{L}_q(\mu,\mathfrak{h})$ and $vi)\mathscr{S}_q(\gamma,\gamma,1,\mathfrak{h})$, $0\leq\gamma\leq 1$ is the class investigated by Panigrahi and Raina [13].

In the second section, we find Fekete-Szegö functional $|d_3-\xi d_2^2|$ for elements in the class $\mathscr{S}_q(\tau,\gamma,\mu,\mathfrak{h})$. Many new consequences of this result are pointed out.

2. Main Results

Theorem 2.1. *Let $0\leq\gamma\leq 1$, $\tau\geq 0$, $\tau\geq\gamma$ and $\mu\in\mathbb{C}-\{0\}$. If $s(z)\in\mathscr{S}_q(\tau,\gamma,\mu,\mathfrak{h})$, then*

$$|d_2|\leq\frac{|\mu|Q_1}{(1-\gamma+2\tau)} \tag{2.1}$$

and for any complex number ξ,

$$|d_3-\xi d_2^2|\leq\frac{|\mu|Q_1}{2(1-\gamma+3\tau)}max\left(1,\left|\frac{Q_2}{Q_1}-MQ_1\right|\right), \tag{2.2}$$

where

$$M=\mu\left(\frac{2\xi(1-\gamma+3\tau)}{(1-\gamma+2\tau)^2}-\frac{1+\gamma}{1-\gamma+2\tau}\right). \tag{2.3}$$

The result is sharp.

Proof. Suppose that $s\in\mathscr{S}_q(\tau,\gamma,\mu,\mathfrak{h})$. Then there exists a Schwarz function $u(z)$ and a holomorphic function $\varsigma(z)$ such that

$$\frac{1}{\mu}\left(\frac{zs'(z)+\tau z^2 s''(z)}{(1-\gamma)s(z)+\gamma zs'(z)}-1\right)=\varsigma(z)(\mathfrak{h}(u(z))-1),\ z\in\mathfrak{D}. \tag{2.4}$$

Series expansions of s and its successive derivatives from (1.1) gives

$$\frac{1}{\mu}\left(\frac{zs'(z)+\tau z^2 s''(z)}{(1-\gamma)s(z)+\gamma zs'(z)}-1\right)=\frac{1}{\mu}\left[(1-\gamma+2\tau)d_2 z+\left((1-\gamma+3\tau)2d_3-(1+\gamma)(1-\gamma+2\tau)d_2^2\right)z^2+...\right]. \tag{2.5}$$

Similarly from (1.2), (1.3) and (1.4), we obtain

$$\mathfrak{h}(u(z))-1=Q_1u_1z+(Q_1u_2+Q_2u_1^2)z^2+...$$

and

$$\varsigma(z)(\mathfrak{h}(u(z))-1)=R_0Q_1u_1z+\left[R_1Q_1u_1+R_0(Q_1u_2+Q_2u_1^2)\right]z^2+... \tag{2.6}$$

Using (2.5) and (2.6) in (2.4), we get

$$d_2 = \frac{\mu R_0 Q_1 u_1}{1-\gamma+2\tau} \tag{2.7}$$

and

$$d_3 = \frac{\mu Q_1}{2(1-\gamma+3\tau)}\left[R_1 u_1 + R_0\left\{u_2 + \left(\frac{\mu(1+\gamma)R_0 Q_1}{1-\gamma+2\tau} + \frac{Q_2}{Q_1}\right)u_1^2\right\}\right]. \tag{2.8}$$

Thus for any $\xi \in \mathbb{C}$, we get

$$d_3 - \xi d_2^2 = \frac{\mu Q_1}{2(1-\gamma+3\tau)}\left[R_1 u_1 + \left(u_2 + \frac{Q_2}{Q_1}u_1^2\right)R_0 - MQ_1 R_0^2 u_1^2\right], \tag{2.9}$$

where M is as stated in (2.3).

Since $\varsigma(z) = R_0 + R_1 z + R_2 z^2 + ...$ is holomorphic and $|\varsigma(z)| \leq 1, z \in \mathfrak{D}$, we have (see [12])

$$|R_0| \leq 1 \quad \text{and} \quad R_1 = (1-R_0^2)y \qquad (y \leq 1). \tag{2.10}$$

The assertion (2.1) follows from (2.7) using (2.10) and Lemma 1.1. From (2.9) and (2.10), we obtain

$$d_3 - \xi d_2^2 = \frac{\mu Q_1}{2(1-\gamma+3\tau)}\left[y u_1 + \left(u_2 + \frac{Q_2}{Q_1}u_1^2\right)R_0 - (MQ_1 u_1^2 + y u_1)R_0^2\right]. \tag{2.11}$$

If $R_0 = 0$ in (2.11), we at once get

$$|d_3 - \xi d_2^2| \leq \frac{|\mu| Q_1}{2(1-\gamma+3\tau)}. \tag{2.12}$$

If $R_0 \neq 0$, then define a function

$$G(R_0) = y u_1 + \left(u_2 + \frac{Q_2}{Q_1}u_1^2\right)R_0 - (MQ_1 u_1^2 + y u_1)R_0^2,$$

which is quadratic in R_0 and hence holomorphic in $|R_0| \leq 1$. Clearly $|G(R_0)|$ attains its maximum value at $R_0 = e^{i\theta} \quad (0 \leq \theta \leq 2\pi)$. Thus

$$max|G(R_0)| = \max_{0\leq\theta\leq 2\pi}|G(e^{i\theta})| = |G(1)| = |u_2 - \left(MQ_1 - \frac{Q_2}{Q_1}\right)u_1^2|.$$

Therefore, it follows from (2.11) that

$$|d_3 - \xi d_2^2| \leq \frac{|\mu| Q_1}{2(1-\gamma+3\tau)}\left|u_2 - \left(MQ_1 - \frac{Q_2}{Q_1}\right)u_1^2\right|,$$

which on using Lemma 1.1, show that

$$|d_3 - \xi d_2^2| \leq \frac{|\mu| Q_1}{2(1-\gamma+3\tau)} max\left(1, \left|\frac{Q_2}{Q_1} - MQ_1\right|\right). \tag{2.13}$$

The assertion (2.2) now follows from (2.12) and (2.13). We exhibit the sharpness by defining $s(z)$ as

$$\frac{zs'(z) + \tau z^2 s''(z)}{(1-\gamma)s(z) + \gamma z s'(z)} = \mathfrak{h}(z),$$

or

$$\frac{zs'(z) + \tau z^2 s''(z)}{(1-\gamma)s(z) + \gamma z s'(z)} = \mathfrak{h}(z^2),$$

or

$$\frac{zs'(z) + \tau z^2 s''(z)}{(1-\gamma)s(z) + \gamma z s'(z)} = z(\mathfrak{h}(z) - 1).$$

This ends the proof. □

Remark 2.1. If we take $\tau = \nu$, $0 \le \nu \le 1$ and $\mu = 1$ in Theorem 2.1, we get Theorem 2.1 of [13].

We conclude the below sharp result for the class $\mathfrak{R}_q(\tau, \mu, \mathfrak{h})$ by putting $\gamma = 0$ in Theorem 2.1.

Corollary 2.1. *Let $\mu \in \mathbb{C} - \{0\}$ and $\tau \ge 0$. If $s \in \mathfrak{R}_q(\tau, \mu, \mathfrak{h})$, then $|d_2| \le \frac{|\mu|Q_1}{(1+2\tau)}$ and for some $\xi \in \mathbb{C}$, $|d_3 - \xi d_2^2| \le \frac{|\mu|Q_1}{2(1+2\tau)} max\left(1, \left|\frac{Q_2}{Q_1} - \mu\left(\frac{2\xi(1+3\tau)}{(1+2\tau)^2} - \frac{1}{1+2\tau}\right)Q_1\right|\right)$.*

Remark 2.2. For $\mu = 1$ and $\tau = 0$ the above corollary reduces to Corollary 2.2 of [13].

We conclude the following sharp result for the class $\mathfrak{H}_q(\tau, \mu, \mathfrak{h})$, on putting $\gamma = 1$ in Theorem 2.1.

Corollary 2.2. *Let $\mu \in \mathbb{C} - \{0\}$ and $\tau \ge 1$. If $s \in \mathfrak{H}_q(\tau, \mu, \mathfrak{h})$, then $|d_2| \le \frac{|\mu|Q_1}{2\tau}$ and for some $\xi \in \mathbb{C}$, $|d_3 - \xi d_2^2| \le \frac{|\mu|Q_1}{6\tau} max\left(1, \left|\frac{Q_2}{Q_1} + \mu\left(\frac{2-3\xi}{2\tau}\right)Q_1\right|\right)$.*

Remark 2.3. If $\mu = \tau = 1$ in Corollary 2.2, we get Corollary 2.3 of [13].

Theorem 2.1 reduces to the corollary given below, when $\gamma = \tau = \frac{1}{2}$.

Corollary 2.3. *Let $\mu \in \mathbb{C} - \{0\}$ and . If $s \in \mathfrak{K}_q(\mu, \mathfrak{h})$, then $|d_2| \le \frac{2|\mu|Q_1}{3}$ and for some $\xi \in \mathbb{C}$, $|d_3 - \xi d_2^2| \le \frac{|\mu|Q_1}{4} max\left(1, \left|\frac{Q_2}{Q_1} + \mu\left(1 - \frac{16}{9}\xi\right)Q_1\right|\right)$. which is a sharp result.*

Putting $\gamma = 0$ and $\tau = \frac{1}{2}$ in Theorem 2.1, we conclude the corollary given below:

Corollary 2.4. *Let $\mu \in \mathbb{C} - \{0\}$. If $s \in \mathfrak{I}_q(\mu, \mathfrak{h})$, then $|d_2| \le \frac{|\mu|Q_1}{2}$ and for some $\xi \in \mathbb{C}$, $|d_3 - \xi d_2^2| \le \frac{|\mu|Q_1}{5} max\left(1, \left|\frac{Q_2}{Q_1} + \mu\left(\frac{2-5\xi}{4}\right)Q_1\right|\right)$, which is a sharp result.*

Letting $\gamma = \frac{1}{2}$ and $\tau = 1$ in Theorem 2.1, we get the following:

Corollary 2.5. *Let $\mu \in \mathbb{C} - \{0\}$. If $s \in \mathfrak{L}_q(\mu, \mathfrak{h})$, then*

$$|d_2| \le \frac{2|\mu|Q_1}{5} \text{ and } |d_3 - \xi d_2^2| \le \frac{|\mu|Q_1}{7} max\left(1, \left|\frac{Q_2}{Q_1} + \mu\left(\frac{15 - 28\xi}{25}\right)Q_1\right|\right),$$

for some $\xi \in \mathbb{C}$ and the result is sharp.

The next result is based on majorization.

Theorem 2.2. *Let $0 \le \gamma \le 1$, $\tau \ge 0$, $\tau \ge \gamma$ and $\mu \in \mathbb{C} - \{0\}$. If an element $s(z) \in \mathscr{A}$ satisfies*

$$\frac{1}{\mu}\left(\frac{zs'(z) + \tau z^2 s''(z)}{(1-\gamma)s(z) + \gamma z s'(z)} - 1\right) \prec\prec (\mathfrak{h}(z) - 1),\ z \in \mathfrak{D}, \tag{2.14}$$

then

$$|d_2| = \frac{|\mu|Q_1}{1 - \gamma + 2\tau} \tag{2.15}$$

and for some $\xi \in \mathbb{C}$,

$$|d_3 - \xi d_2^2| \le \frac{|\mu|Q_1}{2(1-\gamma+3\tau)} max\left(1, \left|\frac{Q_2}{Q_1} - MQ_1\right|\right), \tag{2.16}$$

where M is as stated in (2.3). The result is sharp.

Proof. Assume that (2.14) holds. From the principle of majorization, there exists a holomorphic function $\varsigma(z)$ such that

$$\frac{1}{\mu}\left(\frac{zs'(z)+\tau z^2 s''(z)}{(1-\gamma)s(z)+\gamma z s'(z)}-1\right)=\varsigma(z)(\mathfrak{h}(z)-1),\ z\in\mathfrak{D}. \tag{2.17}$$

Taking $u(z)=z$, so that $u_1=1, u_n=0, n\geq 2$, we obtain (2.15) and (2.16), following the proof of Theorem 2.1. The function $s(z)$ given by

$$1+\frac{1}{\mu}\left(\frac{zs'(z)+\tau z^2 s''(z)}{(1-\gamma)s(z)+\gamma z s'(z)}-1\right)=\mathfrak{h}(z),\ z\in\mathfrak{D},$$

exhibit the sharpness of the result, which completes the proof of Theorem 2.2. □

Our next sharp result is associated with $\mathscr{S}(\tau,\gamma,\mu,\mathfrak{h})$.

Theorem 2.3. *Let $0\leq\gamma\leq 1$, $\tau\geq 0$, $\tau\geq\gamma$ and $\mu\in\mathbb{C}-\{0\}$. If $s(z)\in\mathscr{S}(\tau,\gamma,\mu,\mathfrak{h})$, then*

$$|d_2|\leq\frac{|\mu|Q_1}{1-\gamma+2\tau}$$

and for any $\xi\in\mathbb{C}$,

$$|d_3-\xi d_2^2|\leq\frac{|\mu|Q_1}{2(1-\gamma+3\tau)}max\left(1,\left|\frac{Q_2}{Q_1}-MQ_1\right|\right),$$

where M is as stated in (2.3).

Proof. Let $s\in\mathscr{S}(\tau,\gamma,\mu,\mathfrak{h})$. Taking $\varsigma(z)=1$, $z\in\mathfrak{D}$, we get $R_0=1$, $R_n=0$, $n\in N$ and by following the proof of Theorem 2.1, we attain the required result. We exhibit the sharpness by defining $s(z)$ as

$$1+\frac{1}{\mu}\left(\frac{zs'(z)+\tau z^2 s''(z)}{(1-\gamma)s(z)+\gamma z s'(z)}-1\right)=\mathfrak{h}(z),$$

or

$$1+\frac{1}{\mu}\left(\frac{zs'(z)+\tau z^2 s''(z)}{(1-\gamma)s(z)+\gamma z s'(z)}-1\right)=\mathfrak{h}(z^2).$$

Which ends the proof. □

We now settle sharp bounds of $|d_3-\xi d_2^2|$ for real ξ.

Theorem 2.4. *Let $0\leq\gamma\leq 1$, $\tau\geq 0$ and $\tau\geq\gamma$. If $s(z)\in\mathscr{S}_q(\tau,\gamma,\mu,\mathfrak{h})$, then for real ξ and μ, we have*

$$|d_3-\xi d_2^2|\leq\begin{cases}\frac{|\mu|Q_1}{2(1-\gamma+3\tau)}\left[\frac{Q_2}{Q_1}+\mu Q_1\left(\frac{1+\gamma}{1-\gamma+2\tau}-\frac{2\xi(1-\gamma+3\tau)}{(1-\gamma+2\tau)^2}\right)\right] & (\xi\leq\rho)\\ \frac{|\mu|Q_1}{2(1-\gamma+3\tau)} & (\rho\leq\xi\leq\rho+2\sigma)\\ -\frac{|\mu|Q_1}{2(1-\gamma+3\tau)}\left[\frac{Q_1}{Q_2}+\mu Q_1\left(\frac{1+\gamma}{1-\gamma+2\tau}-\frac{2\xi(1-\gamma+3\tau)}{(1-\gamma+2\tau)^2}\right)\right] & (\xi\geq\rho+2\sigma)\end{cases} \tag{2.18}$$

where

$$\rho=\frac{(1+\gamma)(1-\gamma+2\tau)}{2(1-\gamma+3\tau)}-\frac{(1-\gamma+2\tau)^2}{2\mu(1-\gamma+3\tau)}\left(\frac{1}{Q_1}-\frac{Q_2}{Q_1}\right) \tag{2.19}$$

and

$$\sigma=\frac{(1-\gamma+2\tau)^2}{2\mu(1-\gamma+3\tau)}. \tag{2.20}$$

Proof. Let ξ and μ be real values. Then (2.18) can be obtained from (2.2) and (2.3) respectively, under the cases:

$$Q_1M - \frac{Q_2}{Q_1} \leq -1, -1 \leq Q_1M - \frac{Q_2}{Q_1} \leq 1 \,\text{and}\, Q_1M - \frac{Q_2}{Q_1} \geq 1.$$

We also note the following:

(i) Equality holds for $\xi < \rho$ or $\xi > \rho + 2\sigma$ if and only if $\varsigma(z) \equiv 1$ and $u(z) = z$ or one of its rotations.

(ii) Equality holds for $\rho < \xi < \rho + 2\sigma$ if and only if $\varsigma(z) \equiv 1$ and $u(z) = z^2$ or one of its rotations.

(iii) Equality holds for $\xi = \rho$ if and only if $\varsigma(z) \equiv 1$ and $u(z) = \frac{z(z+\theta)}{1+\theta z}$, $0 \leq \theta \leq 1$, or one of its rotations, while for $\xi = \rho + 2\sigma$, the equality holds if and only if $\varsigma(z) \equiv 1$ and $u(z) = -\frac{z(z+\theta)}{1+\theta z}$, $0 \leq \theta \leq 1$, or one of its rotations.

□

The second part of assertion in (2.18) for real values of ξ and μ can be improved further as follows:

Theorem 2.5. *Let $0 \leq \gamma \leq 1$, $\tau \geq 0$ and $\tau \geq \gamma$. If $s(z) \in \mathscr{S}_q(\tau, \gamma, \mu, \mathfrak{h})$, then for real ξ and μ, we have*

$$|d_3 - \xi d_2^2| + (\xi - \rho)|d_2|^2 \leq \frac{|\mu| Q_1}{2(1-\gamma+3\tau)} \quad (\rho \leq \xi \leq \rho + \sigma) \tag{2.21}$$

and

$$|d_3 - \xi d_2^2| + (\rho + 2\sigma - \xi)|a_2|^2 \leq \frac{|\mu| Q_1}{2(1-\gamma+3\tau)} \quad (\rho + \sigma \leq \xi \leq \rho + 2\sigma) \tag{2.22}$$

where ρ and σ are given by (2.19) and (2.20), respectively.

Proof. Let $s \in \mathscr{S}_q(\tau, \gamma, \mu, \mathfrak{h})$. For real μ and ξ satisfying $\rho \leq \xi \leq \rho + \sigma$, and using (2.7) and (2.13), we get

$$\begin{aligned} |d_3 - \xi d_2^2| + (\xi - \rho)|d_2|^2 \leq & \frac{|\mu| Q_1}{2(1-\gamma+3\tau)} \left[|u_2| - \frac{2Q_1(1-\gamma+3\tau)}{(1-\gamma+2\tau)^2} (\xi - \rho - \sigma)|u_1|^2 \right. \\ & \left. + \frac{2Q_1(1-\gamma+3\tau)}{(1-\gamma+2\tau)^2} (\xi - \rho)|u_1|^2 \right]. \end{aligned}$$

Therefore, by using Lemma 1.1, we obtain

$$|d_3 - \xi d_2^2| + (\xi - \rho)|d_2|^2 \leq \frac{|\mu| Q_1}{2(1-\gamma+3\tau)} \left[1 - |u_1|^2 + |u_1|^2 \right],$$

which yields the assertion (2.21),

If $\rho + \sigma \leq \xi \leq \rho + 2\sigma$, then again from (2.7), (2.13) and Lemma 1.1, we obtain

$$\begin{aligned} |d_3 - \xi d_2^2| + (\xi - \rho)|d_2|^2 \leq & \frac{|\mu| Q_1}{2(1-\gamma+3\tau)} \left[|u_2| - \frac{2Q_1(1-\gamma+3\tau)}{(1-\gamma+2\tau)^2} (\xi - \rho - \sigma)|u_1|^2 \right. \\ & \left. + \frac{2Q_1(1-\gamma+3\tau)}{(1-\gamma+2\tau)^2} (\rho + 2\sigma - \xi)|u_1|^2 \right] \\ \leq & \frac{|\mu| Q_1}{2(1-\gamma+3\tau)} \left[1 - |u_1|^2 + |u_1|^2 \right], \end{aligned}$$

which estimates (2.22). □

Remark 2.4. Numerous consequences of Theorem 2.2 to Theorem 2.5 can be obtained for different choices of γ and τ.

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