International Journal of Mathematics Trends and Technology
ISSN: 2231 - 5373 /doi:10.14445/22315373/IJMTT-V66I12P502
Volume 66 Issue 12, 10-16, December 2020

Laceability in the Interleaver Graph of Brick Product Graph $C(2n, 1, n)$

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Abstract - *Interleavers are used as a tool in constructing good turbo codes which are the class of error correcting codes introduced by Berrou and Glavieux in 1993. Interleavers have been used in communication systems such as 3G/4G mobile communications and in satellite communications etc. The interleaver's function is to permute low weight code words in one encoder into high weight code words for the other encoder. In this paper we present the construction of interleaver graphs $IG_N, N \geq 6$ from the brick product graph $C(2n, 1, n)$, $n \geq 3$ and discuss its Hamiltonian laceability properties.*

Keywords: *Interleaver graphs, Brick product graphs, Hamiltonian connected, Hamiltonian laceability.*

1 Introduction

Graphs are important structures by using which many applications of the real world such as communication networks and social networks are described. In [1] Arya Mazumdar et.al have proposed a new construction of interleavers for turbo codes from 3-regular hamiltonian graphs. A good interleaver for turbo codes can be constructed from 3-regular Hamiltonian graphs having large girth. The girth of a graph is the shortest cycle contained in a graph. In topological point of view it is important for any interconnection network to have various graph theoretic properties which includes girth.

Using the concept of brick product Brain Alspach et.al. showed in [2] that all cubic Cayley graphs over dihedral groups are Hamiltonian. Also in [3] Brain Alspach et.al. showed that most $C(2n, 1, r)$ are Hamiltonian laceable for $3 \leq r \leq 9$. Inspired by their work we developed the construction of interleaver graph IG_N for $N \geq 6$ from the brick product graph $C(2n, 1, n)$ and explore its Hamiltonian t^* laceable properties. Further fault tolerance is an important property in the network performance which enables systems to continue to operate properly in the case of failure of some of the components. Sun-Yuan Hsieh et.al.in [4] proposed the edge fault tolerant Hamiltonicity property in the faulty network. Extenting this we also explore the edge fault tolerant Hamiltonian t^* laceability of interleaver graphs IG_N, $N \geq 6$ for all even N such that $1 \leq t \leq diam(G)$.

2 Interleavers from the brick product graph $C(2n, 1, n)$

Let $G = C(2n, 1, n)$. Let $2n = N$, $n \geq 3$ be the number of vertices labelled from 1 to N placed on a cycle. Initially the fixed cycle in G becomes the Hamiltonian cycle of the graph.

To construct IG_N from G we first draw two chains of N vertices, one in the upper part and the other in the lower part of the graph. Note that the chain in the upper and lower parts are of length N. Let i be the vertex in the chain of upper part and j be the vertex in the chain of lower part then i and j should be connected in IG_N if there is an edge which connects i and j in G and is not included in the Hamiltonian cycle of G.

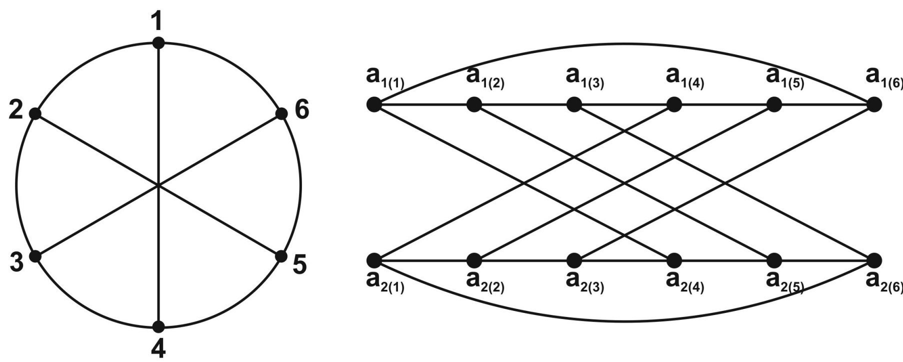

Figure 1 : Construction of IG_6 from $C(6, 1, 3)$

3 Terminologies

We need to first introduce the following terminologies to establish results.
For each vertex a_{1i} and a_{2i} of IG_N where $1 \leq i < N$, we shall write:

$$a_{1i}\phi[N] = a_{1i}(a_{1(i+1)})(a_{1(i+2)})(a_{1(i+3)})(a_{1(i+4)}).......(a_{1(i+N-1)})$$
$$a_{1i}\phi^{-1}[N] = a_{1i}(a_{1(i-1)})(a_{1(i-2)})(a_{1(i-3)})(a_{1(i-4)})......(a_{1(i-N+1)})$$
$$a_{2i}\Gamma[N] = a_{2i}(a_{2(i+1)})(a_{2(i+2)})(a_{2(i+3)})(a_{2(i+4)})......(a_{2(i+N-1)})$$
$$a_{2i}\Gamma^{-1}[N] = a_{2i}(a_{2(i-1)})(a_{2(i-2)})(a_{2(i-3)})(a_{2(i-4)})......(a_{2(i-N+1)})$$

Note that the symbols $\phi[N], \phi^{-1}[N], \Gamma[N], \Gamma^{-1}[N]$ are the paths of order N where as the symbols $R[x], R^{-1}[x], S[x], S^{-1}[x], T[x], T^{-1}[x], H[x], H^{-1}[x], L[x]$ are self explanatory.

For $x = 1$

$$R[x] = a_{11} \to a_{1N}$$
$$R^{-1}[x] = a_{1N} \to a_{11}$$
$$S[x] = a_{21} \to a_{2N}$$
$$S^{-1}[x] = a_{2N} \to a_{21}$$
$$L[x] = a_{12} \to a_{22}$$

For $1 \leq i \leq \dfrac{N}{2}$, $k = \dfrac{N}{2} + i$; $x = 1$

$$\begin{aligned} T[x] &= a_{1i} \rightarrow a_{2k} \\ T^{-1}[x] &= a_{2k} \rightarrow a_{1i} \\ H[x] &= a_{2i} \rightarrow a_{1k} \\ H^{-1}[x] &= a_{1k} \rightarrow a_{2i} \end{aligned}$$

3.1 Definition: A connected graph G is said to be Hamiltonian t laceable (Hamiltonian t^* laceable) if there exists a Hamiltonian path between every pair (at least one pair) of distinct vertices u and v in G with the property $d(u,v) = t$ such that $1 \leq t \leq diam(G)$ where $diam(G)$ is the diameter of G.

3.2 Definition: A connected graph G is termed as Hamiltonian t_o^* (t_e^*) laceable if there exists in G, a Hamiltonian path between atleast one pair of its vertices u and v with the property $d(u,v) = t$ for odd (even) t such that $1 \leq t \leq diam(G)$.

3.3 Definition: The graph $G^{'}$ is k-edge fault tolerant with respect to a graph G if every graph obtained by removing any k-edges from $G^{'}$ contains G. Further a graph $G^{'}$ is k-edge fault tolerant Hamiltonian laceable if $G^{'} - F$ remains Hamiltonian laceable for every $F \subseteq E(G^{'})$ with $\mid F \mid \leq k$.

3.4 Definition: Let P be a path between the vertices a_{v_i} and a_{v_j} in a graph G and $P^{'}$ be the path between the vertices a_{v_j} and a_{v_k}. Then the path $P \cup P^{'}$ is obtained by extending P from a_{v_i} to a_{v_j} and then from a_{v_j} to a_{v_k} through the common vertex a_{v_j}. That is if $P : a_{v_i} \rightarrow a_{v_j}$ and $P^{'} : a_{v_j} \rightarrow a_{v_k}$ then $P \cup P^{'} : a_{v_i} \rightarrow a_{v_j} \rightarrow a_{v_k}$.

4 Results

4.1 Theorem

Let IG_N be the interleaver graph constructed from $C(2n,1,n)$ for $n \geq 3$. Then IG_N, $N \geq 6$ is Hamiltonian t_o^* laceable for $1 \leq t \leq diam(G)$.

Proof: Let the vertex set of IG_N be $V = V_1 \cup V_2$ such that $V_1 = \{a_{i(j)} : i = 1, 1 \leq j \leq N\}$, $V_2 = \{a_{i(j)} : i = 2, 1 \leq j \leq N\}$ and the edge set of IG_N be $E = \{b_k : 1 \leq k \leq 3N\}$. Clearly the diameter of IG_N is $D = \dfrac{N}{2} + 1$.

Case (i)
In IG_N, $N = 4h + 2$ for $h \geq 1$, we have $d(a_{1(1)}, a_{1(2)}) = 1$ and the path $P : \{a_{1(1)} - T[x] - \Gamma^{-1}[2] - H[x] - \phi^{-1}[2] - H^{-1}[x] - \Gamma^{-1}[2] - H[x] - \phi^{-1}[2] - \{H^{-1}[x] - \Gamma^{-1}[2] - H[x] - \phi^{-1}[2]\}_{h-1} - T[x] - \Gamma^{-1}[2] - T^{-1}[x] - \{\phi^{-1}[2] - T[x] - \Gamma^{-1}[2] - T^{-1}[x]\}_{h-1} - a_{1(2)}\}$ is a Hamiltonian path between the vertices $a_{1(1)}$ and $a_{1(2)}$.

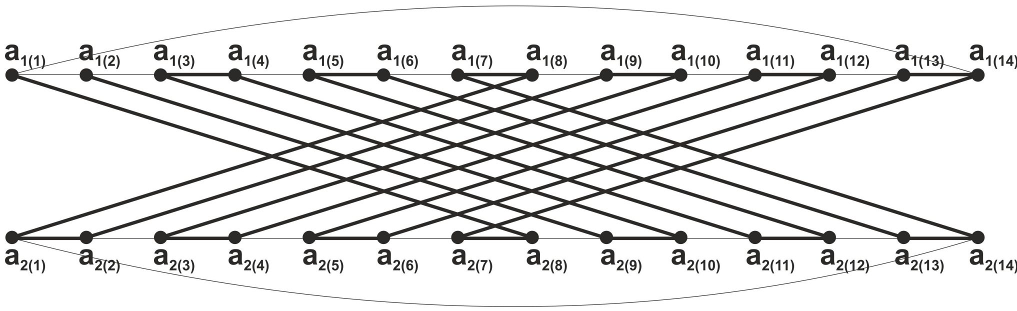

Figure 2 : Hamiltonian path between $a_{1(1)}$ and $a_{1(2)}$ IG_{14}.

Case (ii)
In IG_N, $N = 4h + 2$ for $h \geq 1$, we have $d(a_{1(1)}, a_{1(2h+2)}) = t$ such that $3 \leq t \leq 2h + 1$ for odd t and the path $P : \{a_{1(1)} - \phi[2] - T[x] - \Gamma^{-1}[3] - \{H[x] - \phi^{-1}[2] - H^{-1}[x] - \Gamma^{-1}[2]\}_h - S[x] - \Gamma^{-1}[2h - 1] - T^{-1}[x] - \phi[2h] - a_{1(2h+2)}\}$ is a Hamiltonian path between the vertices between $a_{1(1)}$ and $a_{1(2h+2)}$.

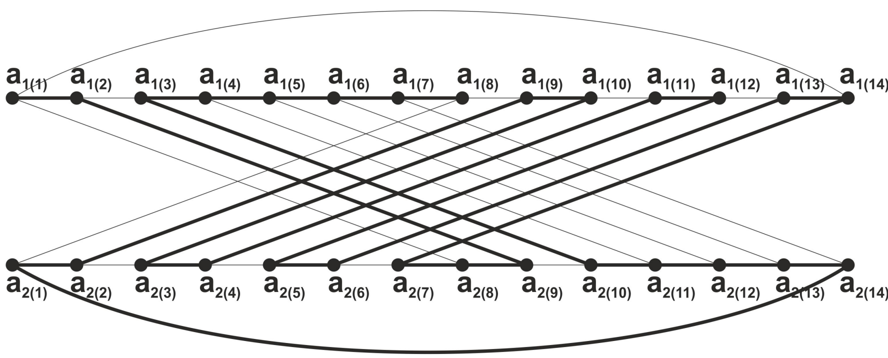

Figure 3 : Hamiltonian path between $a_{1(1)}$ and $a_{1(8)}$ IG_{14}.

Case (iii)
In IG_N, $N = 4h+6$ for $h \geq 1$, we have $d(a_{1(1)}, a_{1(2h+2)}) = t$ such that $3 \leq t \leq 2h+1$ for odd t and the path $P : \{a_{1(1)} - \phi[2] - T[x] - \Gamma^{-1}[3] - \{H[x] - \phi^{-1}[2] - H^{-1}[x] - \Gamma^{-1}[2]\}_{h+s} - H[x] - \phi^{-1}[2] - T[x] - \{\Gamma^{-1}[2] - T^{-1}[x] - \phi^{-1}[2] - T[x]\}_{h-1} - \Gamma[2h+1] - T^{-1}[x] - \phi[2h] - a_{1(2h+2)}\}$ is a Hamiltonian path between the vertices $a_{1(1)}$ and $a_{1(2h+2)}$ where $s = t - (p+1), p \geq 1$.

4.2 Theorem

Let IG_N be the interleaver graph constructed from $C(2n, 1, n)$ for $n \geq 3$. Then IG_N, $N \geq 6$ is 1-edge fault tolerant Hamiltonian t_e^* laceable for $2 \leq t \leq diam(G)$.

Proof: Let IG_N be an interleaver graph having cardinality N. Let $V = V_1 \cup V_2$ be the set of vertices such that $V_1 = \{a_{i(j)} : i = 1, 1 \leq j \leq N\}$,$V_2 = \{a_{i(j)} : i = 2, 1 \leq j \leq N\}$. Let E be the set of edges of IG_N such that $E = \{b_k : 1 \leq k \leq 3N\} \cup E'$ where $E' = \{a_{1(2)} - a_{2(2)}\}$ is the fault edge of IG_N.

Case (i)
In IG_N, $N = 4h+2$ for $h \geq 1$, we have $d(a_{1(1)}, a_{1(3)}) = 2$ and the path $P : \{P_1 : a_{1(1)} - \phi[2] - a_{1(2)}\} \cup \{P_2 : a_{1(3)} - \phi[2h+2] - \{\Gamma[2] - H[x] - \phi[2] - H^{-1}[x]\}_{h-1} - \Gamma[2h+2] - S^{-1}[x] - \Gamma[2] - a_{2(2)}\} \cup \{P_3 : a_{1(2)} - L[x] - a_{2(2)}\}$ is a Hamiltonian path between the vertices $a_{1(1)}$ and $a_{1(3)}$.

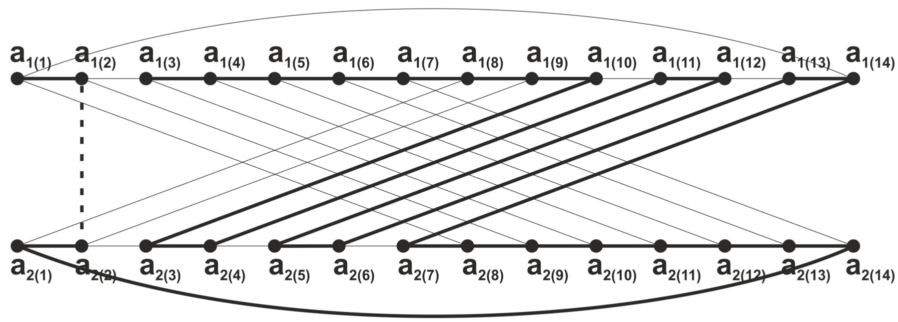

Figure 4 : Hamiltonian path between $a_{1(1)}$ and $a_{1(3)}$ in IG_{14}.

Case (ii)
In IG_N, $N = 4h+2$ for $h \geq 1$ we have $d(a_{1(1)}, a_{1(2h+3)}) = t$ where $4 \leq t \leq 2h+2$ for even t and the path $P : \{P_1 : a_{1(1)} - \phi[2] - a_{1(2)}\} \cup \{P_2 : a_{1(2h+3)} - \phi[2] - H^{-1}[x] - \{\Gamma[2] - H[x] - \phi[2] - H^{-1}[x]\}_{h-1} - \Gamma[4] - \{T^{-1}[x] - \phi[2] - T[x] - \Gamma[2]\}_{h-1} - T^{-1}[x] - \phi[2] - H^{-1}[x] - \Gamma[2] - a_{2(2)}\} \cup \{P_3 : a_{1(2)} - L[x] - a_{2(2)}\}$ is a Hamiltonian path between the vertices $a_{1(1)}$ and $a_{1(2h+3)}$.

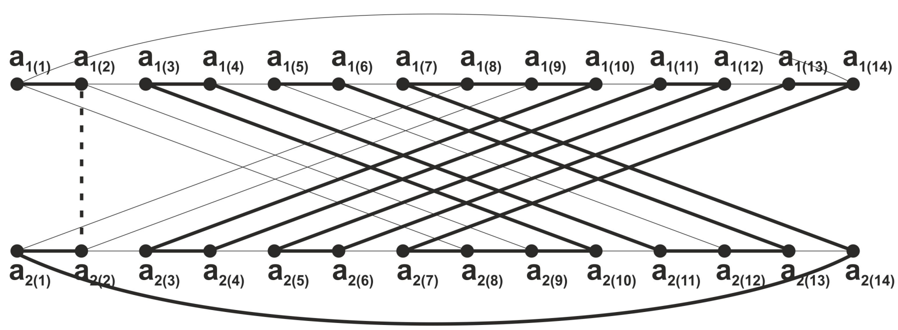

Figure 5 : Hamiltonian path between $a_{1(1)}$ and $a_{1(5)}$ in IG_{14}.

Case (iii)
In IG_N, $N = 4h+6$ for $h \geq 1$, we have $d(a_{1(1)}, a_{1(2h+3)}) = t$ such that $4 \leq t \leq 2h+2$ for even t and the path $P : \{P_1 : a_{1(1)} - \phi[2] - a_{1(2)}\} \cup \{P_2 : a_{1(2h+3)} - \phi[4] - \{H^{-1}[x] - \Gamma[2] - H[x] - \phi[2]\}_h - H^{-1}[x] - \Gamma[4] - \{T^{-1}[x] - \phi[2] - T[x] - \Gamma[2]\}_h - S^{-1}[x] - \Gamma[2] - a_{2(2)}\} \cup \{P_3 : a_{1(2)} - L[x] - a_{2(2)}\}$ is a Hamiltonian path between the vertices $a_{1(1)}$ and $a_{1(2h+3)}$.

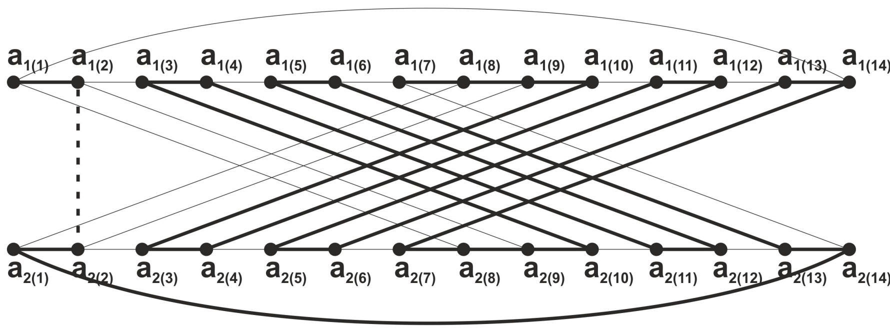

Figure 6 : Hamiltonian path between $a_{1(1)}$ and $a_{1(7)}$ in IG_{14}.

Case (iv)
In IG_N, $N = 4h + 10$ for $h \geq 1$, we have $d(a_{1(1)}, a_{1(2h+3)}) = t$ such that $4 \leq t \leq 2h + 2$ for even t and the path $P : \{P_1 : a_{1(1)} - \phi[2] - a_{1(2)}\} \cup \{P_2 : a_{1(2h+3)} - \phi[2h] - T[x] - \Gamma^{-1}[2h + 1] - \{T^{-1}[x] - \phi^{-1}[2] - T[x] - \Gamma^{-1}[2]\}_{h-1} - T^{-1}[x] - \phi^{-1}[2] - T[x] - \Gamma^{-1}[4] - \{H[x] - \phi^{-1}[2] - H^{-1}[x] - \Gamma^{-1}[2]\}_{h+s} - H[x] - \phi^{-1}[4] - T[x] - S^{-1}[x] - \Gamma[2] - a_{2(2)}\} \cup \{P_3 : a_{1(2)} - L[x] - a_{2(2)}\}$ is a Hamiltonian path between the vertices $a_{1(1)}$ and $a_{1(2h+3)}$ where $s = t - (p + 1), p \geq 1$.

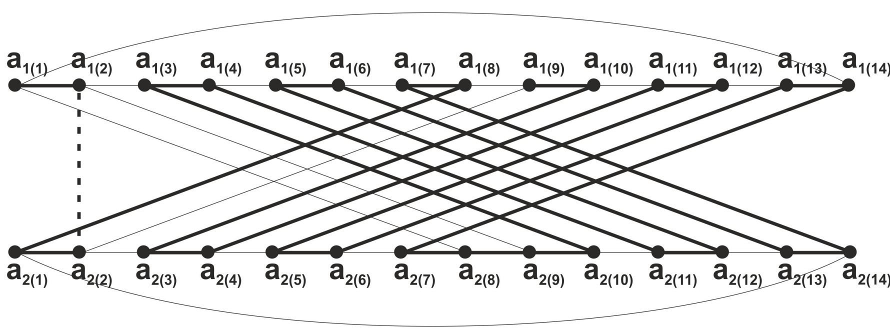

Figure 7 : Hamiltonian path between $a_{1(1)}$ and $a_{1(9)}$ in IG_{14}.

Conclusion

The study of Hamiltonicity and Hamiltonian laceability has lot of significance in computer networks. In this paper, we have studied the Hamiltonian laceability properties of the interleaver graph IG_N for $N \geq 6$, constructed from the brick product graph $C(2n, 1, n)$. In particular, we have shown that the interleaver graph IG_N, $N \geq 6$ is Hamiltonian t_0^* laceable for $1 \leq t \leq diam(G)$ and 1-edge fault tolerant Hamiltonian t_e^* laceable for $2 \leq t \leq diam(G)$. This concludes that the existence of Hamiltonian paths in such network suffice to solve many data communication problems.

Acknowledgement

The first author thankfully acknowledges the support and encouragement provided by the Management, Principal, HOD and Staff of St.Joseph's College Autonomous, Bengaluru and the second author is thankful to the Management, HOD and R & D centre, Department of Mathematics, B.N.M Institute of Technology, Bengaluru.

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