

# Pointwise Approximation By $Q$ - Bernstein Type Operators

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**ABSTRACT :** We concern in this paper a new study of Pointwise approximation by  $q$ -Bernstein operators in the mobile interval  $x \in [-1, 1 - 1/n]$  with use of exponential operators and obtain the Direct theorem and Weighted approximation theorem for that operators with Rate of convergence.

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## I. INTRODUCTION AND AUXILIARY RESULTS

In the year 1912 S.N.Bernstein introduced famous proof of the Weierstrass approximation theorem and the defined operators called Bernstein operators.

Definition: Bernstein [12] : If  $f : [0, 1] \rightarrow \mathbf{R}$ . Then the Bernstein operators of  $f$  is

For each positive integer  $n = 1, 2, 3, \dots$

Bernstein proved that if  $f \in C[0,1]$  then the sequence  $B_n f(x)$  converges uniformly on  $[0,1]$ .

Definition: Phillips [5] : If  $f : [0, 1] \rightarrow \mathbb{C}$ ,  $q > 0$  then  $q$  - Bernstein operators of  $f$  is

$$B_{n,q}(f;x) = \sum_{i=0}^n f\left(\frac{[i]_q}{[n]_q}\right) \prod_{s=0}^{n-1-i} (1 - q^s x) \begin{pmatrix} n \\ i \end{pmatrix}_q \quad \dots \quad (2)$$

Where  $n = 1, 2, 3, \dots$

Here if  $q = 1$  the polynomials  $B_{n,1}(f;x)$  are classical Bernstein polynomials.

**Theorem 1.0.1.** (Bernstein): If  $f \in C[0, 1]$  then  $B_{n,1}(f; x) \Rightarrow f(x)$  for  $x \in [0, 1]$  as  $n \rightarrow \infty$  for  $q \in (0, 1)$  convergence of the sequence  $B_{n,q}(f; x)$  was introduced by A.Illinskii. It follows directly from the definition of  $q$  - Bernstein polynomials possess the end point interpolation property i.e.

$$Bn, q(f; 0) \equiv f(0)$$

$$Bn, q(f; 1) \equiv f(1)$$

for all  $q > 0$ , and here  $n = 1, 2, 3, \dots$

In 2019 Adrian Holhas gave the general exponential operators let  $I \subset \mathbf{R}$  be an open interval and let  $\alpha \geq 0$  be a real numbers consider a continuous function  $\Theta : (0, \infty) \rightarrow \mathbf{R}$  and we denote by  $\mathbf{C}_{\theta,\alpha}$ . the space of continuous functions  $f \in \mathbf{C}(I)$  with the property that exist  $M > 0$  such that  $|f(x)| \leq M e^{\alpha \theta |x|}$ , for every  $x \in I$ . Because of the symmetry and to simplify the notation we consider in the following that  $I \subset (0, \infty)$ .



**Lemma 1.0.2.** Consider a sequence of positive linear operators ( $L_n$ ) preserving the constants and having the property that for every  $f \in C_{\theta,\alpha}$  there exists an integer  $n_\alpha \in N$  such that  $L_n f$  exists for every  $n \geq n_\alpha$ . Suppose that  $L_n(e^{\alpha\theta(t)})$  converges pointwise on  $I$ . Then, for every  $x \in I$  and for every  $\alpha \geq 0$

$$L_n(\max(e^{\alpha\theta(t)}, e^{\alpha\theta(x)}); x) \leq M_\alpha(x), \quad n \geq n_\alpha,$$

Where  $M_\alpha(x) > 0$  depends on  $\alpha$  and  $x$  but not on  $n$ .

In 2018 Cai et al.[11] gave some lemma and corollary for  $\lambda$  Bernstein operators and uses Bezier bases.

**Lemma 1.0.3.** For  $\lambda$ -Bernstein operators, they have the following equalities.

$$B_{n,\lambda}(1; x) = 1; \quad \dots \quad (4)$$

$$B_{n,\lambda}(t; x) = x + \frac{1-2x+x^{n+1}+(1-x)^{n+1}}{n(n-1)} \lambda \quad \dots \quad (5)$$

$$B_{n,\lambda}(t^2; x) = x^2 + \frac{x(x-1)}{n} + \lambda \left[ \frac{2x-4x^2+2x^{n+1}}{n(n-1)} + \frac{(1-x)^{n+1}+x^{n+1}-1}{n^2(n-1)} \right]; \quad \dots \quad (6)$$

**Corollary 1.0.4.** For fixed  $x \in [0,1]$  and  $\lambda \in [-1,1]$ , using above lemma (1.0.3) and by some easy computation, we have

$$B_{n,\lambda}(t-x; x) = \frac{1-2x+x^{n+1}+(1-x)^{n+1}}{n(n-1)} \lambda \leq \frac{1-2x+x^{n+1}+(1-x)^{n+1}}{n(n-1)} \quad \dots \quad (7)$$

$$B_{n,\lambda}((t-x^2); x) = \frac{x(1-x)}{n} + \lambda \left[ \frac{2x(1-x)^{n+1}-2x^{n+1}+2x^{n+2}}{n(n-1)} + \frac{(1-x)^{n+1}+x^{n+1}-1}{n^2(n-1)} \right];$$

$$\leq \frac{x(1-x)}{n} + \left[ \frac{2x(1-x)^{n+1}-2x^{n+1}+2x^{n+2}}{n(n-1)} + \frac{(1-x)^{n+1}+x^{n+1}-1}{n^2(n-1)} \right]; \quad \dots \quad (8)$$

$$\lim_{n \rightarrow \infty} n B_{n,\lambda}(t-x; x) = 0; \quad \dots \quad (9)$$

$$\lim_{n \rightarrow \infty} n B_{n,\lambda}(t-x^2; x) = x(1-x), \quad x \in (0, 1). \quad \dots \quad (10)$$

**Lemma 1.0.5.** Assuming a sequence of new Bernstein operators  $B_{n,\lambda}(f; x)$  preserve the constants and having the property that for all

$f \in C_{\theta,\alpha}$  then there exists an integers  $n_\alpha \in N$ . Suppose that  $B_{n,\lambda}(f; x)$  converges

pointwise on  $I$  then for  $x \in C[I]$  and for every  $\alpha \geq 0$

$$B_{n,\lambda}(f; x) \leq M_\alpha(x), \quad \forall n \geq n_\alpha.$$

Where  $M_\alpha(x) > 0$  depends on  $\alpha$  and  $x$  not on  $n$ .

## II. Direct Estimate

In this section we give Rate of convergence for  $q$  - Bernstein operators.

**Theorem 2.0.6.** Let  $f \in C[-1, 1 - 1/n]$  with new mobile interval then we have

$$|B_{n,q}(f; x) - f(x)| \leq M_\alpha(x).$$

Proof. Since  $B_{n,q}(f; x)$  when  $q = 1$  then we give classical Bernstein operators in the mobile interval

$C[-1, 1 - 1/n]$  as  $n \rightarrow \infty$  and  $f \in C[I]$  we have

$$|B_{n,q}(f; x) - f(x)| = |B_n(e^{\alpha\theta|t|}, e^{\alpha\theta|x|}; x) - f(x)|$$

$$\leq B_{n,}(\max e^{\alpha\theta|t+x|}; x) \leq B_{n,q}(f; x) \leq M_\alpha(x).$$

Hence proved that theorem by using above lemma (1.05).

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**Theorem 2.0.7.** Let  $f \in C[-1, 1 - 1/n]$  then we have

$$|B_{n,q}(f; x) - f(x)| \leq (1 + \beta)(f_+ + \sqrt{\lambda_\alpha})$$

Proof. By Exponential operators technique

Where  $M_\alpha(x) > 0$  depends on  $\alpha$  and  $x$  but not on  $n$ . Now we use linearity and positivity

property of  $\{B_{n,q}(f; x)\}$  and for all  $n \in N$  and  $x \in C[-1, 1 - 1/n]$  we get

By equation (11) and (12)

$$|B_{n,q}(f; x) - f(x)| \leq (f, \delta) \left( \frac{M\alpha(x)}{\delta} + 1; x \right) \leq (f, \delta) \left( \frac{M\alpha(x)}{\delta}; 1+x \right)$$

Using Cauchy-Schwartz inequality and above corollary (1.0.4) equation (8) .

$$|B_{n,q}(f; x) - f(x)| \leq (f, \delta) \left( \left( \left( \frac{|M\alpha(x)|}{\delta} \right)^2 \right)^{1/2} + 1 \right) \leq (f, \delta) \left( \frac{\sqrt{\lambda}\alpha^x}{\delta} + 1 \right) \quad \dots \quad (13)$$

If we take  $\delta = \sqrt{\lambda_\alpha}$  in equation (13) hence proved desired result.

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**Theorem 3.0.8. Direct theorem:**

Let  $f \in C[-1, 1 - 1/n]$  and  $B_{n,q}(f; x)$  definition of  $q$  - Bernstein operator such that  $0 < q \leq 1$  and  $q \rightarrow 1$  as  $n \rightarrow \infty$  then exist such that

$$\| B_{n,q}(f; x) - f(x) \| \leq C n^{-1} M_\alpha^2(x)$$

Proof: We know that corollary (1.0.4)

$$\lim_{n \rightarrow \infty} n B_{n,q}((t-x)^2, x) \leq C n^{-1} M_\alpha^2(x) \quad \dots \dots \dots (14)$$

Let us

$$M_{n,q}(f; x) = f(x) - f\left(\frac{1-2x+x^{n+1}+(1-x)^{n+1}}{n(n-1)}\right) \leq M_\alpha(x) \quad \dots \dots \dots (15)$$

And let

$$L_{n,q}(f; x) = B_{n,q}(f; x) + M_{n,q}(f; x) \quad \dots \dots \dots \quad (16)$$

Then we have

Now

$$L_{n,q}(1; x) = 1$$

$$L_{n,q}((t-x); x) = 0 \text{ And } L_{n,q}((t-x)^2; x) \leq C n^{-1} M_\alpha^2(x) \quad \dots \dots \dots \quad (18)$$

So that

$$|L_{n,q}(f, x) - f(x)| \leq |L_n((f-g), x)|_q + |f(x) - g(x)|_q + |L_n(g, x) - g(x)|_q \\ \leq M_\alpha(x) + M_\alpha(x) + |L_n(g, x) - g(x)|_q \quad \dots \dots \dots (19)$$

Now using Taylors expansion with integral remainder

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u) g''(u) du$$

then by P.687[7].

At  $q = 1$

By equation (19) and (20 ) gives us

$$| L_n(f, x) - f(x) |_q \leq 2 M_\alpha(x) + C_n^{-1} M_\alpha^2(x)$$

Hence  $f \in C[-1, 1-1/n]$  so

$$\| B_{n,q}(f; x) - f(x) \| \leq \| L_n(f, x) - f(x) \|_q + \| L_n(f, x) \|_q$$

Then by equation (15) and (20)

$$\leq 2 M_\alpha(x) + M_\alpha(x) + C_n^{-1} M_\alpha^2(x) \leq C_n^{-1} M_\alpha^2(x).$$

Hence proved the famous Direct theorem.

#### **IV. Weighted Approximation with exponential functions defined as :**

We give another type of modification of Mishra [14].

1 -  $B_x[-1, 1-1/n]$  is a space of functions  $f$  defined  $[-1, 1-1/n]$  satisfying

$$|f(x)| \leq M e^{\alpha\theta|x|}, M > 0.$$

2 -  $C_x[-1, 1-1/n]$  be the subspace of all continuous function in  $B_x[-1, 1-1/n]$ .

3 -  $C_x^* [-1, 1-1/n]$  is the subspace of functions  $f \in [-1, 1-1/n]$  for which

$\lim_{n \rightarrow \infty} \frac{f(x)}{e^{\alpha\theta|x|}}$  is finite.

Note that the space  $B_x [-1, 1-1/n]$  is a normal linear space with the norm

$$\|f\|_q = \sup_{x \geq 0} \frac{|f(x)|}{e^{\alpha\theta|x|}}.$$

## V. Weighted Theorem

Theorem 5.0.9. Consider  $q = q_n$  such that  $0 < q_n \leq 1$  and as  $n \rightarrow \infty$  we get  $q_n \rightarrow 1$  for any

$f \in C_x^* [-1, 1-1/n]$  we have

$$\lim_{n \rightarrow \infty} \sup_{x \in [-1, 1-1/n]} \frac{|B_{n,\alpha}^{q_n}(f,x) - f(x)|}{e^{\alpha\theta|x|}} = 0.$$

Proof: We take L.H.S.

$$\begin{aligned} \sup_{x \in [-1, 1-1/n]} \frac{|B_{n,\alpha}^{q_n}(f,x) - f(x)|}{e^{\alpha\theta|x|}} &\leq \sup_{x \leq x_0} \frac{|B_{n,\alpha}^{q_n}(f,x) - f(x)|}{e^{\alpha\theta|x|}} + \sup_{x \geq x_0} \frac{|B_{n,\alpha}^{q_n}(f,x) - f(x)|}{e^{\alpha\theta|x|}} \\ &\leq \|B_{n,\alpha}^{q_n}(f,x) - f(x)\|_{C[-1, x_0]} + \|f(x)\|_x \sup_{x \geq x_0} \frac{|B_{n,\alpha}^{q_n}(e^{\alpha\theta|x|}; x)|}{e^{\alpha\theta|x|}} \\ &\quad + \sup_{x \geq x_0} \frac{|f(x)|}{e^{\alpha\theta|x|}} \end{aligned}$$

Where  $|f(x)| \leq M e^{\alpha\theta|x|}$  for arbitrary  $\epsilon > 0$  large value of  $x_0$  such that

$$\frac{\|f(x)\|_x}{e^{\alpha\theta|x|}} < \frac{\epsilon}{3}$$

By using lemma (0.0.2) and theorem of rate of convergence

$$\leq \frac{\epsilon}{3} + M_\alpha(x) + M_\alpha(x) \leq \frac{\epsilon}{3} + 2M_\alpha(x) \leq \epsilon.$$

So we get if  $\epsilon \rightarrow 0$  then we have

$$\lim_{n \rightarrow \infty} \sup_{x \in [-1, 1-1/n]} \frac{|B_{n,\alpha}^{q_n}(f,x) - f(x)|}{e^{\alpha\theta|x|}} = 0.$$

L.H.S. = R.H.S.



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