Convergence of Modified Picard-Mann Hybrid Iteration Process For Nearly Nonexpansive Mappings

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Abstract - In this paper, we prove the strong convergence theorems for nearly nonexpansive mappings, using the modified Picard-Mann hybrid iteration process in the context of uniformly convex Banach space.

Keywords: Nonexpansive mapping, asymptotically nonexpansive mapping, nearly nonexpansive mapping, uniformly Lipschitzian mapping, fixed point, Mann iteration.

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I. INTRODUCTION

In 1965, Browder [3] proved that every nonexpansive mapping in a uniformly convex Banach space has a fixed point. Goebel and Kirk [5] extended Browder's result to the class of asymptotically nonexpansive mappings. In 2005, Sahu [13] introduced the class of nearly Lipschitzian mappings which is an important generalization of the class of Lipschitzian mappings. Later, Agarwal et.al [1] proposed a new iteration process for the iterative approximation of fixed points of nearly asymptotically nonexpansive mappings. In 2013, Khan [8] introduced a new iteration process for nonexpansive mappings, which is called the Picard-Mann hybrid iteration process and showed that the new process converges faster than Picard and Mann iteration process. Geethalakshmi and Hemavathy [4] proved strong convergence and stability results of the Picard-Mann hybrid iteration. Recently, Khan [9] proves the existence of fixed points of generalized nonexpansive mappings in CAT(0) spaces, and approximate them using Picard-Mann hybrid iterationprocesses. Akewe and Osilike [2] proved convergence and stability results for Picard-Mann hybrid iterative schemes for contractive-like operators in a real normed space.

Motived and inspired by this work, we introduce a modified Picard-Mann hybrid iteration process and prove some strong convergence theorems for nearly nonexpansive mappings in uniformly convex Banach space.

II. PRELIMINARIES

Let *C* be a nonempty subset of a real Banach space *X* and $T: C \to C$ be a mapping with the fixed point set F(T), i.e., $F(T) = \{p \in C : Tp = p\}$. Now, we recall some definitions and conclusions for our presentation.

Definition 2.1. ([5]) The function δ_X : [0,2] \rightarrow [0,1] is said to be the modulus of convexity of X if

$$\delta_X(\varepsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} \colon \|x\| \le 1, \|y\| \le 1, \|x-y\| \ge \varepsilon\right\}$$
(2.1)

X is said to be uniformly convex if $\delta_X(0) = 0$ and $\delta_X(\varepsilon) > 0$ for all $\varepsilon \in (0,2]$.

Definition 2.2. A mapping $T: C \rightarrow C$ is said to be nonexpansive if

$$|Tx - Ty|| \le ||x - y||$$
 for all $x, y \in C$. (2.2)

Definition 2.3. A mapping T: $C \rightarrow C$ is said to be asymptotically nonexpansive if there exists a sequence

 $k_n \subset [1, \infty)$ with $\lim_{n \to \infty} k_n = 1$ such that

$$||T^{n}x - T^{n}y|| \le k_{n}||x - y|| \quad \text{for all } x, y \in C, \ n \ge 1$$
(2.3)

Definition 2.4. A mapping T: $C \rightarrow C$ is said to be uniformly L-Lipschitzian if there exists a constant L > 0 such that

$$||T^{n}x - T^{n}y|| \le L||x - y|| \quad \text{for all } x, y \in C, \ n \ge 1.$$
(2.4)

Remark 2.1. It easy to see that every nonexpansive mapping T is asymptotically nonexpansive with sequence $k_n = 1$ and every asymptotically nonexpansive mapping is uniformly L-Lipschitzian with $L = \sup_{n \in \mathbb{N}} k_n$.

Definition 2.5. ([13]) Let $\{a_n\}$ be a sequence in [0,1) with $\lim_{n\to\infty} a_n = 0$. A mapping T: C \rightarrow C is said to be nearly nonexpansive with respect to $\{a_n\}$ if

$$||T^n x - T^n y|| \le ||x - y|| + a_n \quad \text{for all } x, y \in C, \ n \ge 1.$$
(2.5)

Remark 2.2. ([13]) If C is a bounded domain of an asymptotically nonexpansive mapping T, then T is nearly nonexpansive. In fact, we have

$$\begin{aligned} \|T^n x - T^n y\| &\leq k_n \|x - y\| \\ &\leq \|x - y\| + (k_n - 1)\|x - y\| \\ &\leq \|x - y\| + (k_n - 1) \cdot diam(C), \quad \text{for all } x, y \in C, \ n \geq 1. \end{aligned}$$

From Remark 2.1 and 2.2 we have the following implications:

nonexpansive \Rightarrow asymptotically nonexpansive \Rightarrow nearly nonexpansive

Example 2.1. Let $X = \mathbb{R}$, C = [0,1] and $T: C \to C$ be a mapping defined by

$$Tx = \begin{cases} qx, & if \ x \in [0,1) \\ 0, & if \ x = 1. \end{cases}$$

where $q \in (0,1)$. It is clear that T is discontinuous mapping. However, it is nearly nonexpansive mapping with respect to the sequence $a_n = q^n, a_n \rightarrow 0$. Indeed,

$$||T^{n}x - T^{n}y|| \le q^{n}||x - y|| + q^{n}$$
$$\le ||x - y|| + a_{n} \text{ for all } x, y \in C, n \ge 1$$

Lemma 2.1([11]). Let $\{a_n\}, \{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality

$$a_{n+1} \leq (1 + \delta_n)a_n + b_n$$
, for all $n \in \mathbb{N}$

If $\sum_{n=0}^{\infty} \delta_n < \infty$ and $\sum_{n=0}^{\infty} b_n < \infty$ then $\lim_{n\to\infty} a_n$ exists. In particular, if $\{a_n\}$ has a subsequence which converges strongly to zero, then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.2 ([14]). Let X be a real uniformly convex Banach space and $0 < a \le t_n \le b < 1$, for all $n \in \mathbb{N}$. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X such that $\lim_{n\to\infty} \sup ||x_n|| \le r$, $\lim_{n\to\infty} \sup ||y_n|| \le r$, and $\lim_{n\to\infty} ||(1-t_n)x_n + t_ny_n|| = r$, hold for some $r \ge 0$. Then $\lim_{n\to\infty} ||x_n - y_n|| = 0$.

We will now consider some well-known iteration schemes. Let C be a nonempty convex subset of normed space X and $T: C \rightarrow C$ a self-map.

(a) The Picard iteration process is defined by

$$x_{n+1} = Tx_n$$

for all $n \ge 0$, (see for more information [12]).

(b) The Mann iteration process (see, for example [10]) is defined by

$$x_{n+1} = (1 - \alpha_n)x_n + Tx_n$$

for all $n \ge 0$ and $\{\alpha_n\}_{n\ge 0}$ is a real sequence in [0, 1] which satisfies the conditions: $0 \le \alpha_n < 1$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$.

(C) The Ishikawa iteration process (see, for example [7]) is defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T y_n$$
$$y_n = (1 - \beta_n)x_n + \beta_n T x_n$$

where $\{\alpha_n\}_{n\geq 0}$ and $\{\beta_n\}_{n\geq 0}$ be real sequences in [0, 1] for all $n\geq 0$.

(d) The modified Mann iteration process (see, for example [14]) is defined by

$$x_{n+1} = (1 - \alpha_n)x_n + T^n x_n$$

where $\{\alpha_n\}_{n\geq 0}$ is a real sequence in [0, 1] which satisfies condition $0 < a \le \alpha_n \le b < 1$ for all $n \ge 0$.

(e) The Picard-Mann hybrid iteration process (see, for example [8]) is defined by

$$x_{n+1} = Ty_n$$

$$y_n = (1 - \alpha_n)x_n + \alpha_n Tx_n$$

for all $n \ge 0$ and $\{\alpha_n\}_{n\ge 0}$ is a real sequence in [0, 1].

III. MAIN RESULTS

We introduce a modified Picard-Mann hybrid iteration process by

$$x_{n+1} = T^n y_n$$

$$y_n = (1 - \alpha_n) x_n + \alpha_n T^n x_n,$$
(3.1)

for all $n \ge 0$ and $\{\alpha_n\}_{n\ge 0}$ is a real sequence in [0, 1].

The purpose of this section is to prove some strong convergence theorems with respect to the iteration scheme (3.1) for nearly nonexpansive mapping in real uniformly convex Banach spaces.

Theorem3.1. Let C be a nonempty compact convex subset of a real uniformly convex Banach space X. Let T: $C \rightarrow C$ be a uniformly L-Lipschitzian, nearly nonexpansive mapping with respect to $\{a_n\}$ such that $\sum_{n=0}^{\infty} a_n < \infty$. Let $\{x_n\}$ be the modified Picard-Mann iteration defined by (3.1) where $\{\alpha_n\}$ is sequence in (0,1). Then the following hold:

- (i) $\lim_{n\to\infty} ||x_n p||$ exists for all $p \in F(T)$;
- (ii) $\lim_{n\to\infty} ||x_n T^n x_n|| = 0;$
- (iii) $\{x_n\}$ converges strongly to a fixed point of T.

Proof. (*i*) By Schauder's fixed point theorem, we obtain that $F(T) \neq \emptyset$. Let $p \in F(T)$, by (3.1) we have

 $||x_{n+1} - p|| = ||T^n x_n - p|| \le ||y_n - p|| + a_n$

$$\|y_n - p\| = (1 - \alpha_n) \|x_n - p\| + \alpha_n \|T^n x_n - p\|$$

$$\leq (1 - \alpha_n) \|x_n - p\| + \alpha_n (\|x_n - p\| + \alpha_n)$$

$$= \|x_n - p\| + (1 + \alpha_n) a_n$$
(3.2)

So, we have

$$\|x_{n+1} - p\| \le \|x_n - p\| + (1 + \alpha_n)a_n \tag{3.3}$$

It follows from $\sum_{n=0}^{\infty} a_n < \infty$ and Lemma 2.1 that $\lim_{n \to \infty} ||x_n - p||$ exists for $p \in F(T)$.

(*ii*) We set $\lim_{n\to\infty} ||x_n - p|| = c$, from (3.2) we have

$$\lim_{n \to \infty} \sup \|y_n - p\| \le c \tag{3.4}$$

Also,

$$||T^{n}y_{n} - p|| \le ||y_{n} - p|| + a_{n}$$

So, we have

$$\lim_{n \to \infty} \sup \|T^n y_n - p\| \le c \tag{3.5}$$

Similarly

 $||T^n x_n - p|| \le ||x_n - p|| + a_n$

and we have

$$\lim_{n \to \infty} \sup \|T^n x_n - p\| \le c \tag{3.6}$$

Now

$$||x_{n+1} - p|| \le ||y_n - p|| + a_n$$

Taking limit infimum, we have

$$\lim_{n \to \infty} \inf \|x_{n+1} - p\| \le \lim_{n \to \infty} \inf \|y_n - p\|$$

$$c \le \lim_{n \to \infty} \inf \|y_n - p\| \le c$$
(3.7)

From (3.4) and (3.7) we have

$$c = \lim_{n \to \infty} \|y_n - p\| \tag{3.8}$$

i.e.,

$$c = \lim_{n \to \infty} \|y_n - p\| = \lim_{n \to \infty} \|(1 - \alpha_n)(x_n - p) + \alpha_n(T^n x_n - p)\|$$

Therefore by Lemma 2.2, we obtain

$$\lim_{n \to \infty} \|x_n - T^n x_n\| = 0.$$
(3.9)

(iii) Finaly, since T is uniformly L-Lipschitzian mapping, we have

$$\begin{aligned} \|x_n - Tx_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - T^{n+1}x_{n+1}\| + \|T^{n+1}x_{n+1} - T^{n+1}x_n\| + \|T^{n+1}x_n - Tx_n\| \\ &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - T^{n+1}x_{n+1}\| + L\|x_{n+1} - x_n\| + L\|x_n - T^nx_n\| \end{aligned}$$

Since T is uniformly continuous, it follows from (3.9) that

$$\lim_{n \to \infty} \|x_n - Tx_n\| = 0.$$
(3.10)

By the compactness of *C*, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\lim_{n \to \infty} x_{n_k} = p^* \tag{3.11}$$

Since *T* is continuous, it follows from (3.10) that $p^* \in F(T)$. Since $\lim_{n\to\infty} ||x_n - p||$ exists for all $p \in F(T)$, and we conclude from (3.11) that $\lim_{n\to\infty} x_n = p^* \in F(T)$.

Theorem 3.2. Assume that all the conditions of Theorem 3.1 are satisfied. Then the sequence $\{x_n\}$ generated by (3.1) converges strongly to a fixed point of *T* if and only if

$$\lim_{n\to\infty}\inf d(x_n,F(T))=0,$$

where $d(x, F(T)) = \inf \{ d(x, p) : p \in F(T) \}$.

Proof. Necessity is obvious. Conversely, suppose that $\lim_{n\to\infty} \inf d(x_n, F(T)) = 0$. From Theorem 3.1 we know that $\lim_{n\to\infty} ||x_n - p||$ exists for all $p \in F(T)$, so $\lim_{n\to\infty} d(x_n, F(T))$ exists for all $p \in F(T)$. Thus by hypothesis

$$\lim_{n \to \infty} d(x_n, F(T)) = 0. \tag{3.12}$$

Now we show that $\{x_n\}$ is a Cauchy sequence in C. Indeed, from (3.3), we have

$$\|x_{n+1} - p\| \le \|x_n - p\| + (1 + \alpha_n)a_n$$

Now, we set $b_n \coloneqq (1 + \alpha_n)a_n$. For any $m, n \in \mathbb{N}$, $m > n \ge 1$, we have
 $\|x_m - p\| \le \|x_{m-1} - p\| + b_{m-1}$
 $\le \|x_{m-2} - p\| + b_{m-1} + b_{m-2}$

$$\vdots$$

$$\leq \|x_n - p\| + \sum_{i=0}^{m-1} b_i$$

$$\leq \|x_n - p\| + \sum_{i=0}^{\infty} b_i$$

Since $\lim_{n\to\infty} d(x_n, F(T)) = 0$ and $\sum_{i=0}^{\infty} b_i < \infty$ for any $\varepsilon > 0$ there exists a positive integer n_0 such that

$$d(x_n, F(T)) < \frac{\varepsilon}{4}, \quad \sum_{i=n_0}^{\infty} b_i < \frac{\varepsilon}{4}.$$

Therefore, there exists $\bar{p} \in F(T)$ such that

$$||x_{n_0} - \bar{p}|| < \frac{\varepsilon}{4}, \quad \sum_{i=n_0}^{\infty} b_i < \frac{\varepsilon}{4}$$

Thus, for all $m, n \ge n_0$ we get from the above inequality that

$$\|x_m - x_n\| \le \|x_m - \bar{p}\| + \|x_n - \bar{p}\|$$

$$\le \|x_{n_0} - \bar{p}\| + \sum_{i=n_0}^{\infty} b_i + \|x_{n_0} - \bar{p}\| + \sum_{i=n_0}^{\infty} b_i$$

$$= 2\left(\|x_{n_0} - \bar{p}\| + \sum_{i=n_0}^{\infty} b_i\right)$$

$$< 2\left(\frac{\varepsilon}{4} + \frac{\varepsilon}{4}\right) = \varepsilon.$$

Thus, it follows that $\{x_n\}$ is a Cauchy sequence. Since *C* is a closed subset of Banach space *X*, the sequence $\{x_n\}$ converges strongly to some $p^* \in C$. Since F(T) is a closed subset of *C* and $\lim_{n\to\infty} d(x_n, F(T)) = 0$ we have $p^* \in F(T)$. Thus, the sequence $\{x_n\}$ converges strongly to a fixed point of *T*. This completes the proof.

Senter and Dotson [15] introduced the notion of mapping satisfying Condition (I) which is defined as follows:

Definition 3.1. A mapping $T: C \rightarrow C$ is said to satisfy Condition (I), if there exists a non-decreasing function

 $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$ and $\varphi(t) > 0$, for all t > 0 such that

$$d(x,Tx) \geq \varphi(d(x,F(T)))$$

for all $x \in C$.

Theorem 3.3. Assume that all the conditions of Theorem 3.1 are satisfied and let *T* be a mapping satisfying Condition (I). Then the sequence $\{x_n\}$ generated by (3.1) converges strongly to a fixed point of *T*.

Proof. We proved in Theorem 3.2, that $\lim_{n\to\infty} d(x_n, F(T))$ exists. From Theorem 3.1 we have $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

It follows from Condition (I) that

$$\lim_{n \to \infty} \varphi\left(d(x_n, F(T))\right) \le \lim_{n \to \infty} d(x_n, Tx_n) = 0$$
$$\lim_{n \to \infty} \varphi\left(d(x_n, F(T))\right) = 0.$$

Since $\varphi : [0, \infty) \to [0, \infty)$ is a non-decreasing function with $\varphi(0) = 0$ and $\varphi(t) > 0$, for all t > 0, we obtain

$$\lim_{n\to\infty}d\bigl(x_n,F(T)\bigr)=0$$

Consequently, $\{x_n\}$ converges strongly to a fixed point of *T*.

i.e,

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