International Journal of Mathematics Trends and Technology
ISSN: 2231 – 5373 /doi:10.14445/22315373/IJMTT-V66I12P507

Volume 66 Issue 12, 44-54, December, 2020
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On Minimum Hub D-distance Spectra of some graphs

P.R.Hampiholi[1], Anjana S Joshi[2], Vinaya V Deshpande[3]

[1]*Professor,KLS Gogte Institute Of Technology, Karnataka ,India*

[2]*Assistant Professor, KLS Gogte Institute Of Technology, Karnataka ,India*

[3] *Assistant Professor, KLS Gogte Institute Of Technology, Karnataka ,India*

Abstract: *In this paper, the concept of minimum hub D-distance matrix* $M_{HD^d}(G)$ *of a connected graph* G *is introduced and minimum hub D-distance spectra of some graphs are computed.*

Keywords: *Minimum hub number, D- distance, minimum hub D-distance matrix, minimum hub D-distance spectra.*

AMS Subject Classification: 05C50, 05C99.

I. INTRODUCTION

In this paper, by a graph G, we mean non-trivial, finite and undirected graph without multiple edges and loops. For graph theoretic terminologies we refer[4]

In graph G, the usual distance d(u,v) is the length of the minimum path connecting the vertices u and v of G.

The D-distance d^D(u,v) between two vertices of a connected graph G is defined as $d^D(u,v) = min\{d(u,v) + \deg(u) + \deg(v) + \sum \deg(w)\}$ where sum runs over all the intermediate vertices w in the path and minimum is taken over all u-v paths in G[1].

The D-eccentricity of any vertex v,$e^D(v)$ is defined as the maximum D-distancefrom v to any other vertex that is $e^D(v) = \max\{d^D(u,v): u \in V(G)\}$,where $V(G)$ is the vertex set of graph G [1].

Let $\beta_1 \geq \beta_2 \geq \beta_3 \ldots . \geq \beta_r$ denote different eigenvalues of the matrix Dε(G). Since, this matrix is symmetric , all the Dε eigen values are real $D\varepsilon$ spectrum is denoted by $spec\ D\varepsilon$ and denoted as,

$$spec\ D\varepsilon = \begin{Bmatrix} \beta_1 & \beta_2 & \beta_3 \ldots . \beta_r \\ m_1 & m_2 & m_3 \ldots . m_r \end{Bmatrix}$$

Where m_i is the algebraic multiplicity of the eigenvalues $\beta_i, for\ 1 \leq i \leq r$ [11].

For more details on mathematical aspects of the theory of graph spectra one can refer [2,3,6,7]

The theory of hub number was introduced by M. Walsh[9]. A set $H \subseteq V(G)$ is a hubset of G, if it has a property that , for any $x, y \in V(G) - H$, there is a $H-$path between x, y .The smallest size of a hub set is called a hub number of G .A $H-$path between vertices x, y of graph G is a path where all intermediate vertices are from H .

In this article ,motivated by the definition of minimum hub distance matix $A_{Hd}(G)$ of a connected graph G[10] we define minimum hub D-distance matix and find minimum hub D- distance spectraof some class of graphs.

II. The Minimum hub D–distance spectra of a graph

Let G be a graph of order n with vertex set $V=\{v_1,v_2,v_3....v_n\}$ and edge set E. Let H be a minimum hub set of a graph G. The minimum hub D-distance matrix of G is denoted as $M_{Hd^D}(G)$ and defined as

$$M_{Hd^D}(G)=1 \quad if \quad i=j \quad and \quad v_i \in H$$
$$= d^D(v_i,v_j), \quad otherwise$$

The minimum hub D-distance eigenvalues of the graph G are the eigenvalues of $M_{Hd^D}(G)$. Also,all the eigenvalues are real,since $M_{HD^d}(G)$ is real and symmetric.

We use the following Lemma to prove the Theorems.

Lemma 2.1[5]: If matrix A is an $n \times n$ matrix partitioned as $A=\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ and A_{11},A_{22} are square matrices. If A_{11} is non singular then, $\det(A)=\det(A_{11})\det(A_{22}-A_{21}A_{11}^{-1}A_{12})$. Also, if A_{22} is non singular then, $\det(A)=\det(A_{22})\det(A_{12}-A_{12}A_{22}^{-1}A_{21})$.

Lemma 2.2[5]: Let $B=\begin{bmatrix} B_0 & B_1 \\ B_1 & B_0 \end{bmatrix}$ be a symmetric 2×2 block matrix with B_0 *and* B_1 are square matrix of same order. Then spectrum of B is the union of spectra B_0+B_1 *and* B_0-B_1 .

Lemma 2.3[5]: If A and B are square matrices then, $|AI+B(J-I)|_{n\times n}=|A-B|^{n-1}|A+(n-1)B|$, where A and B are of same order.

Theorem 2.4 : For a complete graph K_n, $n\geq 2$ minimum hub D- distance spectra is,

$$spec_{Hd^D}(K_n)=\left\{\begin{Bmatrix} (n-1)(2n-1) & -(2n-1) \\ 1 & n-1 \end{Bmatrix}\right.$$

Proof: Let K_n be a complete graph with vertex set $\{v_1,v_2,v_3,......v_n\}$.The minimum hub number is $h(K_n)=0$.

$$M_{Hd^D}(K_n)=\begin{bmatrix} 0 & 2n-1 & 2n-1 & \cdots & 2n-1 \\ 2n-1 & 0 & 2n-1 & \cdots & 2n-1 \\ 2n-1 & 2n-1 & 0 & \cdots & 2n-1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2n-1 & 2n-1 & 2n-1 & \cdots & 0 \end{bmatrix}$$

The characteristic polynomial

$$|M_{Hd^D}(K_n)-\lambda I|=\begin{vmatrix} -\lambda & 2n-1 & 2n-1 & \cdots & 2n-1 \\ 2n-1 & -\lambda & 2n-1 & \cdots & 2n-1 \\ 2n-1 & 2n-1 & -\lambda & \cdots & 2n-1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2n-1 & 2n-1 & 2n-1 & \cdots & -\lambda \end{vmatrix}$$

$$=\left[\lambda+(2n-1)\right]^{n-1}\left[\lambda-(n-1)(2n-1\right]$$

Therefore,

$$spec_{Hd^D}(K_n)=\left\{\begin{Bmatrix}(n-1)(2n-1) & -(2n-1)\\ 1 & n-1\end{Bmatrix}\right\}$$

Example 1:

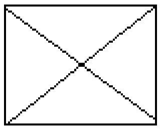

Figure 1 : K_4

$$spec_{Hd^D}(K_3)=\begin{Bmatrix}-7 & 21\\ 3 & 1\end{Bmatrix}$$

Theorem 2.5 : For $n\geq 2$,the minimum hub D-distance spectra of a star graph $K_{1,n-1}$ is,

$$spec_{Hd^D}(K_{1,n-1})=\begin{Bmatrix}-(n+3) & \dfrac{(n^2+n-5)\pm\sqrt{(n^2+n-5)^2+4\{(n+3)-(n-1)(2-n-n^2)\}}}{2}\\ n-2 & 1\end{Bmatrix}$$

Proof: Let $K_{1,n-1}$ be a star graph with vertex set $\{v_0,v_1,v_2,......v_n\}$, v_0 be the centre.The minimum hub set is $H=\{v_0\}$.

Then,

$$M_{Hd^D}(K_{1,n-1})=\begin{bmatrix}1 & n+1 & n+1 & \cdots & n+1\\ n+1 & 0 & n+3 & \cdots & n+3\\ n+1 & n+3 & 0 & \cdots & n+3\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ n+1 & n+3 & n+3 & \cdots & 0\end{bmatrix}$$

$$\left|M_{Hd^D}(K_{1.n-1})-\lambda I\right|=\begin{vmatrix}1-\lambda & n+1 & n+1 & \cdots & n+1\\ n+1 & -\lambda & n+3 & \cdots & n+3\\ n+1 & n+3 & -\lambda & \cdots & n+3\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ n+1 & n+3 & n+3 & \cdots & -\lambda\end{vmatrix}=\begin{vmatrix}1-\lambda & (n+1)J_{n-1\times 1}\\ (n+1)J_{1\times n-1} & (n+3)\{J_{n-1}-I_{n-1}\}-\lambda I_{n-1}\end{vmatrix}$$

By ,Lemma 2.1

$$\left|M_{Hd^D}(K_{1.n-1})-\lambda I\right| = (1-\lambda)\det\left[\{(n+3)\{J_{n-1}-I_{n-1}\}-\lambda I_{n-1}\}-(n+1)J_{n-1\times 1}(1-\lambda)^{-1}(n+1)J_{1\times n-1}\right]$$

$$= (1-\lambda)\det\left[\{(n+3)(J_{n-1}-I_{n-1})-\lambda I_{n-1}\}-\frac{(n+1)^2}{1-\lambda}J_{n-1}\right]$$

$$= (1-\lambda)\det[\{(n+3)-\frac{(n+1)^2}{1-\lambda}\}J_{n-1}-\{(n+3)+\lambda I_{n-1}]$$

$$= [-\lambda-(n+3)]^{n-2}[\lambda^2-(n^2+n-5)\lambda-\{(n-3)-(n-1)[(n+3)-(n+1)^2]\}]$$

$$= [-\lambda-(n+3)]^{n-2}\left[\lambda-\frac{(n^2+n-5)\pm\sqrt{(n^2+n-5)^2+4\{(n+3)-(n-1)(2-n-n^2)\}}}{2}\right]$$

So,

$$spec_{Hd^D}(K_{1,n-1}) = \begin{Bmatrix} -(n+3) & \frac{(n^2+n-5)\pm\sqrt{(n^2+n-5)^2+4\{(n+3)-(n-1)(2-n-n^2)\}}}{2} \\ n-2 & 1 \end{Bmatrix}$$

Example 2:

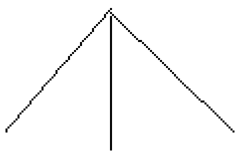

Figure 2: $K_{1,3}$

$$spec_{Hd^D}(K_{1,3}) = \begin{Bmatrix} -7 & -3.3282 & 18.3282 \\ 2 & 1 & 1 \end{Bmatrix}$$

Definition: The bistar $S_{n,m}$ is a graph obtained from $K_{1,n-1}$ *and* $K_{1,m-1}$ joining their centers .

We use the above definition to prove next theorem,

Theorem 2.6: For $n \geq 3$, the minimum hub D-distance spectra of double $S_{n,n}$ is

$spec_{Hd^D}(S_{n,n}) =$

$$\left\{\begin{matrix} -(n+4) & \dfrac{(3n^2+7n-11) \pm \sqrt{(3n^2+7n-11\ )^2 + 4\{(n+4)(2n+2)-(n-1)[(-3n^2-6n-7]}}{2} \\ 2(n-2) & 1 \\ \multicolumn{2}{l}{\dfrac{-(n^2+3n+3) \pm \sqrt{(n^2+3n+3)^2 - 4\{-(n-1)(-n-1)^2 + 2n^3 + 2n^2 + 6n\}}}{2}} \\ 1 & \end{matrix}\right\}$$

Proof: For double star $S_{n,n}$ whose vertex set $V = \{v_0, v_1, v_2.....v_{n-1}, u_0, u_1, u_2....u_{n-1}\}$,the minimum hub set is $H = \{u_0, v_0\}$

$$M_{Hd^D}(S_{n,n}) = \begin{bmatrix} 1 & n+2 & n+2 & \cdots & n+2 & 2n+1 & 2n+3 & 2n+3 & \cdots & 2n+3 \\ n+2 & 0 & n+4 & \cdots & n+4 & 2n+3 & 2n+5 & 2n+5 & \cdots & 2n+5 \\ n+2 & n+4 & 0 & \cdots & n+4 & 2n+3 & 2n+5 & 2n+5 & \cdots & 2n+5 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ n+2 & n+4 & n+4 & \cdots & 0 & 2n+3 & 2n+5 & 2n+5 & \cdots & 2n+5 \\ 2n+1 & 2n+3 & 2n+3 & \cdots & 2n+3 & 1 & n+2 & n+2 & \cdots & n+2 \\ 2n+3 & 2n+5 & 2n+5 & \cdots & 2n+5 & n+2 & 0 & n+4 & \cdots & n+4 \\ 2n+3 & 2n+5 & 2n+5 & \cdots & 2n+5 & n+2 & n+4 & 0 & \cdots & n+4 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 2n+3 & 2n+5 & 2n+5 & \cdots & 2n+5 & n+2 & n+4 & n+4 & \cdots & 0 \end{bmatrix}$$

Since, this is of the form $B = \begin{bmatrix} B_0 & B_1 \\ B_1 & B_0 \end{bmatrix}$

By Lemma 2.2,

$spec_{Hd^D}(S_{n,n})$ is union of $spectra\ \ of\ \ [B_0 + B_1]\ and\ \ spectra\ \ of\ \ [B_0 - B_1]$

$$|(B_0 + B_1) - \lambda I_n| = \begin{vmatrix} (2n+2)-\lambda & (3n+5)J_{n-1\times 1} \\ (3n+5)J_{1\times n-1} & (3n+9)J_{n-1} - ((n+4)+\lambda)I_{n-1} \end{vmatrix}$$

Simplifying,

$$|(B_0 + B_1) - \lambda I_n| = \\ [-\lambda - (n+4)]^{n-2}\left[\lambda^2 - (3n^2+7n-11)\lambda - \{(n+4)(2n+2) - (n-1)[-3n^2-6n-7\right]$$

$spec_{Hd^D}\,(B_0 + B_1) =$

So, $\left\{\begin{matrix} -(n+4) & \dfrac{(3n^2+7n-11) \pm \sqrt{(3n^2+7n-11)^2 + 4\{(n+4)(2n+2)-(n-1)[-3n^2-6n-7]}}{2} \\ n-2 & 1 \end{matrix}\right\}$

and

$$\left|(B_0 - B_1) - \lambda I_n\right| = \begin{vmatrix} -2n-\lambda & (-n-1)J_{n-1\times 1} \\ (-n-1)J_{1\times n-1} & (-n-1)J_{n-1} - ((n+4)+\lambda)I_{n-1} \end{vmatrix}$$

Simplifying,

$\left|(B_0 - B_1) - \lambda I_n\right| =$

$[-\lambda-(n+4)]^{n-2}[\lambda^2+(3n^2+3n+3)\lambda+\{-(n-1)(-n-1)^2+2n(n+4)-(n^2-1)(-2n)\}]$

So,

$spec_{Hd^D}\,(B_0 - B_1) =$

$\left\{\begin{matrix} -(n+4) & \dfrac{-(n^2+3n+3) \pm \sqrt{(n^2+3n+3)^2 - 4\{-(n-1)(-n-1)^2+2n^3+2n^2+6n\}}}{2} \\ n-2 & 1 \end{matrix}\right\}$

Therefore,

$spec_{Hd^D}\,(S_{n,n}) =$

$$\left\{\begin{matrix} -(n+4) & \dfrac{(3n^2+7n-11) \pm \sqrt{(3n^2+7n-11\ \ )^2 + 4\{(n+4)(2n+2)-(n-1)[(-3n^2-6n-7]}}{2} \\ 2(n-2) & 1 \\ \dfrac{-(n^2+3n+3) \pm \sqrt{(n^2+3n+3)^2 - 4\{-(n-1)(-n-1)^2+2n^3+2n^2+6n\}}}{2} & \\ 1 & \end{matrix}\right\}$$

Example 3:

Figure:3 $S_{4,4}$

$$spec_{Hd^D}(S_{4,4}) = \begin{Bmatrix} -8 & -26.9564 & -4.0436 & 69.5574 & -4.5574 \\ 4 & 1 & 1 & 1 & 1 \end{Bmatrix}$$

Theorem 2.7: For the complete bipartite graph $K_{n,n}, n \geq 3$, the minimum hub D-distance spectra is,

$$spec_{Hd^D}(K_{n,n}) =$$

$$\begin{cases} -(3n+2) & \dfrac{\{5n^2-3n-3\} \pm \sqrt{\{5n^2-3n-3\}^2 + 4\{(6n^2+10n+4)+(5n+3)^2(n-1)-(n-1)(10n^2+16n+16)\}}}{2} \\ 2(n-2) & 1 \\ \dfrac{-(n^2+5n+3) \pm \sqrt{(n^2+5n+3)^2 + 4\{(-2n)(3n+2)-(n^2-1)(3n+1)\}}}{2} & \\ 1 & \end{cases}$$

Proof: For a complete bipartite graph $K_{n,n}, n \geq 3$, with vertex set $V = \{u_1, u_2, u_3, ...u_n; v_1, v_2, v_3...v_n\}$, consider a minimum hub set to be $H = \{u_1, v_1\}$.

$$M_{Hd^D}(K_{n,n}) = \begin{bmatrix} 1 & 3n+2 & 3n+2 & \cdots & 3n+2 & 2n+1 & 2n+1 & 2n+1 & \cdots & 2n+1 \\ 3n+2 & 0 & 3n+2 & \cdots & 3n+2 & 2n+1 & 2n+1 & 2n+1 & \cdots & 2n+1 \\ 3n+2 & 3n+2 & 0 & \cdots & 3n+2 & 2n+1 & 2n+1 & 2n+1 & \cdots & 2n+1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 3n+2 & 3n+2 & 3n+2 & \cdots & 0 & 2n+1 & 2n+1 & 2n+1 & \cdots & 2n+1 \\ 2n+1 & 2n+1 & 2n+1 & \cdots & 2n+1 & 1 & 3n+2 & 3n+2 & \cdots & 3n+2 \\ 2n+1 & 2n+1 & 2n+1 & \cdots & 2n+1 & 3n+2 & 0 & 3n+2 & \cdots & 3n+2 \\ 2n+1 & 2n+1 & 2n+1 & \cdots & 2n+1 & 3n+2 & 3n+2 & 0 & \cdots & 3n+2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 2n+1 & 2n+1 & 2n+1 & \cdots & 2n+1 & 3n+2 & 3n+2 & 3n+2 & \cdots & 0 \end{bmatrix}$$

Since, this is of the form $B = \begin{bmatrix} B_0 & B_1 \\ B_1 & B_0 \end{bmatrix}$

By Lemma 2.2,

$spec_{Hd^D}(K_{n,n})$ is union of $spectra\ \ of\ \ [B_0 + B_1]\ \ and\ \ spectra\ \ of\ \ [B_0 - B_1]$

$$\left|(B_0+B_1)-\lambda I_n\right| = \begin{vmatrix} (2n+2)-\lambda & (5n+3)J_{n-1\times 1} \\ (5n+3)J_{1\times n-1} & (5n+3)J_{n-1}-((3n+2)+\lambda)I_{n-1} \end{vmatrix}$$

Simplifying,

$$\left|(B_0+B_1)-\lambda I_n\right| =$$
$$\left[-\lambda-(3n+2)\right]^{n-2}\left[\lambda^2-\{5n^2-3n-3\}\lambda-\{(6n^2+10n+4)+(5n+3)^2(n-1)-(n-1)(10n^2+16n+6)\}\right]$$

So,

$$spec_{Hd^D}(B_0+B_1)=$$
$$\begin{Bmatrix} -(3n+2) & \frac{\{5n^2-3n-3\}\pm\sqrt{\{5n^2-3n-3\}^2+4\{(6n^2+10n+4)+(5n+3)^2(n-1)-(n-1)(10n^2+16n+6)\}}}{2} \\ n-2 & 1 \end{Bmatrix}$$

and

$$\left|(B_0-B_1)-\lambda I_n\right| = \begin{vmatrix} -2n-\lambda & (n+1)J_{n-1\times 1} \\ (n+1)J_{1\times n-1} & (n+1)J_{n-1}-((3n+2)+\lambda)I_{n-1} \end{vmatrix}$$

Simplifying,

$$\left|(B_0-B_1)-\lambda I_n\right| =$$
$$\left[-\lambda-(3n+2)\right]^{n-2}\left[\lambda^2+\{-n^2+5n+3\}\lambda-\{(-2n)(3n+2)-(n^2-1)(3n+1)\}\right]$$

$$spec_{Hd^D}(B_0-B_1)=$$
$$\begin{Bmatrix} -(3n+2) & \frac{-(n^2+5n+3)\pm\sqrt{(n^2+5n+3)^2+4\{(-2n)(3n+2)-(n^2-1)(3n+1)\}}}{2} \\ n-2 & 1 \end{Bmatrix}$$

Therefore,

$$spec_{Hd^D}(K_{n,n})=$$
$$\left\{\begin{matrix} -(3n+2) & \frac{\{5n^2-3n-3\}\pm\sqrt{\{5n^2-3n-3\}^2+4\{(6n^2+10n+4)+(5n+3)^2(n-1)-(n-1)(10n^2+16n+16)\}}}{2} \\ 2(n-2) & 1 \\ \frac{-(n^2+5n+3)\pm\sqrt{(n^2+5n+3)^2+4\{(-2n)(3n+2)-(n^2-1)(3n+1)\}}}{2} & \\ 1 & \end{matrix}\right.$$

Example 4:

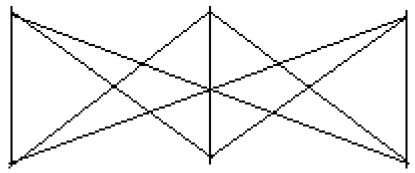

Figure:4 $K_{3,3}$

$$spec_{Hd^D}(K_{3,3}) = \begin{Bmatrix} -11 & 1.3523 & -10.3523 & -10.3375 & 43.3375 \\ 2 & 1 & 1 & 1 & 1 \end{Bmatrix}$$

Theorem 2.8 : For the Friendship graph $F_n, n \geq 2$, the minimum hub D-distance spectra is $|X_1||X_2||X_3| = 0$, where

$$|X_1| = \left[(-\lambda - (2n+6))^2 - 5^2\right]^{n-1}$$

$$|X_2| = [(-\lambda + (2n+6))^2 - (n-1)((2n+6)^2 + 5)]$$

$$|X_3| = [(1-\lambda) - \left(\frac{(2n)(2n+3)^2}{-\lambda + 5 + (2n+6)(2n-2)}\right)]$$

Proof: Let $F_n, n \geq 2$,be the Friendship graph, having $V = \{v_0, v_1, v_2 .....v_{2n}\}$ and v_0 is the centre and the hub number is $h(F_n) = 1$

$$M_{Hd^D} = \begin{bmatrix} 1 & 2n+3 & 2n+3 & 2n+3 & 2n+3 & \cdots & 2n+3 & 2n+3 \\ 2n+3 & 0 & 5 & 2n+6 & 2n+6 & \cdots & 2n+6 & 2n+6 \\ 2n+3 & 5 & 0 & 2n+6 & 2n+6 & \cdots & 2n+6 & 2n+6 \\ 2n+3 & 2n+6 & 2n+6 & 0 & 5 & \cdots & 2n+6 & 2n+6 \\ 2n+3 & 2n+6 & 2n+6 & 5 & 0 & \cdots & 2n+6 & 2n+6 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2n+3 & 2n+6 & 2n+6 & 2n+6 & 2n+6 & \cdots & 0 & 5 \\ 2n+3 & 2n+6 & 2n+6 & 2n+6 & 2n+6 & \cdots & 5 & 0 \end{bmatrix}$$

$$\left|M_{Hd^D}(F_n)-\lambda I\right| = \begin{vmatrix} 1-\lambda & 2n+3 & 2n+3 & 2n+3 & 2n+3 & \cdots & 2n+3 & 2n+3 \\ 2n+3 & -\lambda & 5 & 2n+6 & 2n+6 & \cdots & 2n+6 & 2n+6 \\ 2n+3 & 5 & -\lambda & 2n+6 & 2n+6 & \cdots & 2n+6 & 2n+6 \\ 2n+3 & 2n+6 & 2n+6 & -\lambda & 5 & \cdots & 2n+6 & 2n+6 \\ 2n+3 & 2n+6 & 2n+6 & 5 & -\lambda & \cdots & 2n+6 & 2n+6 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2n+3 & 2n+6 & 2n+6 & 2n+6 & 2n+6 & \cdots & -\lambda & 5 \\ 2n+3 & 2n+6 & 2n+6 & 2n+6 & 2n+6 & \cdots & 5 & -\lambda \end{vmatrix}_{(2n+1)\times(2n+1)}$$

Applying the column operations, $C_1-(2n+3)\left[\dfrac{C_2+C_3+........+C_{2n}}{-\lambda+5+(2n+6)(2n-2)}\right]$

We get,

$$\left|M_{Hd^D}(F_n)-\lambda I\right| = \begin{vmatrix} 1-\lambda-X & 2n+3 & 2n+3 & 2n+3 & 2n+3 & \cdots & 2n+3 & 2n+3 \\ 0 & -\lambda & 5 & 2n+6 & 2n+6 & \cdots & 2n+6 & 2n+6 \\ 0 & 5 & -\lambda & 2n+6 & 2n+6 & \cdots & 2n+6 & 2n+6 \\ 0 & 2n+6 & 2n+6 & -\lambda & 5 & \cdots & 2n+6 & 2n+6 \\ 0 & 2n+6 & 2n+6 & 5 & -\lambda & \cdots & 2n+6 & 2n+6 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 2n+6 & 2n+6 & 2n+6 & 2n+6 & \cdots & -\lambda & 5 \\ 0 & 2n+6 & 2n+6 & 2n+6 & 2n+6 & \cdots & 5 & -\lambda \end{vmatrix}$$

Where $X = \dfrac{(2n)(2n+3)^2}{-\lambda+5+(2n+6)(2n-2)}$

Hence

$$\left|M_{Hd^D}(F_n)-\lambda I\right| = \left|1-\lambda-X\right| \begin{vmatrix} -\lambda & 5 & 2n+6 & 2n+6 & \vdots & 2n+6 & 2n+6 \\ 5 & -\lambda & 2n+6 & 2n+6 & \vdots & 2n+6 & 2n+6 \\ 2n+6 & 2n+6 & -\lambda & 2n+6 & \vdots & 2n+6 & 2n+6 \\ 2n+6 & 2n+6 & 5 & -\lambda & \vdots & 2n+6 & 2n+6 \\ \cdots & \cdots & \cdots & \cdots & \ddots & \cdots & \cdots \\ 2n+6 & 2n+6 & 2n+6 & 2n+6 & \vdots & -\lambda & 2n+6 \\ 2n+6 & 2n+6 & 2n+6 & 2n+6 & \vdots & 5 & -\lambda \end{vmatrix}$$

We can observe that , the second determinant is of the form,

$$A = \begin{vmatrix} a & b & b & \dots & b \\ b & a & b & \cdots & b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & b & b & \cdots & b \\ b & b & b & \ddots & a \end{vmatrix}$$

Where $a = \begin{vmatrix} -\lambda & 5 \\ 5 & -\lambda \end{vmatrix}_{2\times 2}$ and $b = \begin{vmatrix} 2n+6 & 2n+6 \\ 2n+6 & 2n+6 \end{vmatrix}_{2\times 2}$

Hence, using the Lemma 2.3,

we obtain the spectra of Friendship graph by solving,

$$\left[(-\lambda-(2n+6))^2-5^2\right]^{n-1}[(-\lambda+(2n+6))^2-(n-1)((2n+6)^2+5)][(1-\lambda)-\left(\frac{(2n)(2n+3)^2}{-\lambda+5+(2n+6)(2n-2)}\right)]=0$$

Example 5:

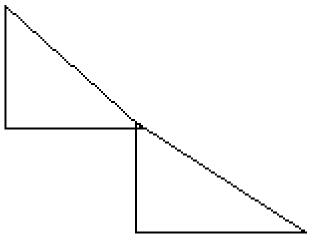

Figure:5 F₂

$$spec_{Hd^D}(F_2)=\begin{Bmatrix} -5 & -15 & 31.4391 & -5.4391 \\ 2 & 1 & 1 & 1 \end{Bmatrix}$$

Conclusion: In this paper, we obtain the minimum hub D-distance spectra of complete graph, star gaph, double star graph, complete bipartite graph ,friendship graph.

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