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Int-Normal Spaces

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Abstract: *In this paper we introduce and study a new class of spaces, namely int-normal spaces. The relationships between normal, ii-normal and int-normal spaces are investigated. Moreover, we introduced some function related with int-normal spaces, we obtain some preservation theorems of int-normal spaces.*

Keywords: *int-open, ii-open, int-normal, ii-normal spaces.*

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I. Introduction

In 1937, Stone **[7]** introduced the notion of regular-open sets. In 2019, Mohammed and Abdullah [**1**] introduced the concepts of ii-open sets, int-open sets and obtained their properties. In 2020, Hamant [**5**] introduced the concepts of ii-normal spaces and its properties.

II. Preliminaries

In what follows, spaces always mean topological spaces on which no separation axioms are assumed unless explicitly stated and $f : (X, \Im) \rightarrow (Y, \sigma)$ (or simply $f : X \rightarrow Y$) denotes a function f of a space $(X, \Im)$ into a space (Y, σ). Let A be a subset of a space X. The closure and the interior of A are denoted by CL(A) and INT(A), respectively.

2.1 Definition. A subset A of a space X is said to be:

(1) **regular open[7]** if A = INT(CL(A)).

(2) **int-open [1]** if there exist an open set $G \in \Im$, such that

(i)) $G \neq \phi$, X

(ii) INT(A) = G

(3) **ii-open [1]** set if there exists an open set $G \in \Im$, such that

(i) $G \neq \phi$, X

(ii) $A \subseteq CL(A \cap G)$

(iii) INT(A) = G

The complement of a regular open (resp. int-open, ii-open) set is called **regular closed** (resp. **int-closed**, **ii-closed**).

The intersection of all ii-closed (resp. int-closed) sets containing A is called the **ii-closure (resp. int-closure**) of A and is denoted by **ii-CL(A)** (**resp. intCL(A)**). Dually, the **ii-interior(resp. int-interior**) of A, denoted by **ii-INT(A) (resp. int-INT(A)**) is defined to be the union of all ii-open (resp. int-open) sets contained in A.

The family of all regular open(resp. regular closed, ii-open , ii-closed, int-open, int-closed) sets of a space X is denoted by R-O(X)(resp. R-C(X), ii-O(X), ii-C(X), int-O(X), int-C(X)).

We have the following implications for the properties of subsets:

open ⇒ ii-open ⇒ int -open

Where none of the implications is reversible as can be seen from the following examples:

2.2 Example Let X = {a, b, c, d} and ℑ = {φ, {a}, {b}, {a, b},{a,c}, {a, b, c}, X}. Then
(1) closed sets in (X, ℑ) are φ, X ,{d}, {b,d}, {c,d}, {a,c,d}, {b,c,d}.
(2) ii-open sets in (X, ℑ) are φ, X,{a}, {b}, {a,b}, {a,c}, {a,d}, {b,d}, {a,b,c},{a,b,d}, {a,c,d}.
(3)int-open sets in (X, ℑ) are φ, X,{a}, {b}, {a,b}, {a,c}, {a,d}, {b,c}, {b,d}, {a,b,c},{a,b,d}, {a,c,d},{b,c,d}.
Here {b,c} is int-open but not ii-open also not open.

2.3 Example. Let X = {a, b, c, d} and ℑ = {φ, {a}, {b}, {a, b}, {a, b, c}, X}. Then
(1) cosed sets in (X, ℑ) are φ, X , {d}, {c,d}, {a,c,d}, {b,c,d}.
(2) ii-open sets in (X, ℑ) are φ, X, {a}, {b}, {a, b}, {a, c}, {a, d}, {b, c}, {b, d}, {a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}.
(3) int -open sets in (X, ℑ) are φ, X, {a}, {b}, {a, b}, {a, c}, {a, d}, {b, c}, {b, d}, {a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}.
In this example we see that {b,c} is ii-open as well as int-open set but not open set.

III. int-normal spaces

3.1 Definition A topological space X is termed as **int-normal** if for any pair of disjoint closed sets A and B, there exist disjoint int-open sets G and H such that A ⊂ G and B ⊂ H.

3.2 Definition A topological space X is termed as **ii-normal [5]** if for any pair of disjoint closed sets A and B, there exist disjoint ii-open sets G and H such that A ⊂ G and B ⊂ H.

3.3 Remark. The following diagram holds for a topological space (X, ℑ) :

normal ⇒ ii-normal ⇒ int-normal

None of these implications is reversible as shown by the following examples.

3.3 Example. Let X = {a, b, c, d} and ℑ = {X, ϕ, {a}, {b}, {a, b}, {a, b, c}, {a, b, d}}. Let A = {c} and B = {d} be disjoints closed sets, there exist disjoint ii-open sets G = {a, c} and H = {b, d} such that A ⊂ G and B ⊂ H. Then the space (X, ℑ) is ii-normal and int-normal since every ii-open set is int-open. But it is not normal since G and H are not open set.

3.4 Example. Let X={a,b,c,d} and ℑ={ φ, {a}, {c}, {a,b}, {a,c}, {a,d}, {a,b,c}, {a,b,d}, {a,c,d}, X}.Take A={b}, B={d} disjoints closed sets ,there exist disjoint int-open sets G={a,d} and H={b,c}such that A ⊂ G and B ⊂ H. Then the space (X, ℑ) is int-normal. But it is not ii-normal as there does not exist any disjoint ii-open sets containing A and B. Also space (X, ℑ) is not normal space.

3.5 Example. Let X = {a, b, c, d} and ℑ = {X, ϕ, {b, d}, {a, b, d}, {b, c, d}}. Let A = {a} and B = {c} be disjoints closed sets, there does not exist any pair of disjoint open sets G and H such that A ⊂ G and B ⊂ H. Then the space (X, ℑ) is not normal, also it is neither ii-normal nor int-normal space.

3.6 Example. Let X = {a, b, c} and ℑ = {X, ϕ, {a}, {b}, {a, b}, {b, c}, X}. Then the space (X, ℑ) is normal as well as int-normal.

3.7 Theorem. For a space X, the following are equivalent:

(1) X is int-normal,

(2) For every pair of open sets G and H whose union is X, there exist int-closed sets A and B such that A ⊂ G, B ⊂ H and A ∪ B = X,

(3) For every closed set P and every open set Q containing P, there exists an int-open set G such that $P \subset G \subset$ int-CL(G) $\subset Q$.

Proof: (1) ⇒ (2) : Let G and H be a pair of open sets in an int-normal space X such that $X = G \cup H$. Then X – G and X − H are disjoint closed sets. Since X is int-normal, there exist disjoint int-open sets G_1 and H_1 such that $X - G \subset G_1$ and $X - H \subset H_1$. Let $A = X - G_1$, $B = X - H_1$. Then A and B are int-closed sets such that $A \subset G$, $B \subset H$ and $A \cup B = X$.

(2) ⇒ (3) : Let P is a closed set and Q is an open set containing P. Then X − P and Q be open sets whose union is X. Then by (2), there exist int-closed sets C_1 and C_2 such that $C_1 \subset X - P$ and $C_2 \subset Q$ and $C_1 \cup C_2 = X$. Then $P \subset X - C_1$, $X - Q \subset X - C_2$ and $(X - C_1) \cap (X - C_2) = \phi$. Let $G = X - C_1$ and $H = X - C_2$. Then G and H are disjoint int-open sets such that $P \subset G \subset X - H \subset Q$. As X − H is int-closed set, we have int-CL(U) $\subset X - H$ and $P \subset G \subset$ int-CL(G) $\subset Q$.

(3) ⇒ (1) : Let P_1 and P_2 are any two disjoint closed sets of X. Put $Q = X - P_2$, then $P_2 \cap Q = \phi$. $P_1 \subset Q$, where Q is an open set. Then by (3), there exists an int-open set G of X such that $P_1 \subset G \subset$ int-CL(G) $\subset Q$. It follows that $P_2 \subset X$ – int-CL(G) = H , say, then H is int-open and $G \cap H = \phi$. Hence P_1 and P_2 are separated by int-open sets G and H. Therefore X is int-normal.

IV. Some functions with int-normal spaces

4.1 Definition . Let X be a topological space. A subset $N \subset X$ is called an **int-neighbourhood** (briefly **int-nhd**) of a point $x \in X$ if there exist an int-open set M such that $x \in M \subset N$.

4.2 Definition . A function $f : X \to Y$ is called

(1) **R-map [3]** if $f^{-1}(A)$ be a regular open in X for every regular open set A of Y ,

(2) **completely continuous [2]** if $f^{-1}(A)$ be a regular open in X for every open set A of Y ,

(3) **rc-continuous [4]** if for each regular closed set B in Y, $f^{-1}(B)$ be a regular closed in X.

4.3 Definition . A function $f : X \to Y$ is called

(1) **softly int-open** if $f(U) \in$ int-O(Y) for each $U \in$ int-O(X),

(2) **softly int-closed** if $f(U) \in$ int-C(Y) for each $U \in$ int-C(X),

(3) **almost int-irresolute** if for every $a \in X$ and each int-neighbourhood V of f(x), int-CL(f^{-1}(V)) is an int-neighbourhood of a.

4.4 Theorem . A function $f : X \to Y$ be a softly int-closed if and only if for each subset A in Y and for each int-open set U in X containing $f^{-1}(A)$, there exists an int-open set V containing A such that $f^{-1}(V) \subset U$.

Proof: (⇒) : Suppose that f is softly int-closed. Let A be a subset of Y and $U \in$ int-O(X) containing $f^{-1}(A)$. Put $V = Y - f(X - U)$, then V is an int-open set of Y such that $A \subset V$ and $f^{-1}(V) \subset U$.

(⇐) : Let K be any int-closed set of X. Then $f^{-1}(Y - f(K)) \subset X - K$ and $X - K \in$ int-O(X). There exists an int-open set V of Y such that $Y - f(K) \subset V$ and $f^{-1}(V) \subset X - K$. Therefore, we have $f(K) \supset Y - V$ and $K \subset f^{-1}(Y - V)$. Hence, we obtain $f(K) = Y - V$ and f(K) is int-closed in Y . This shows that f is softly int-closed.

4.5 Lemma. For a function $f : X \to Y$, the following are equivalent:

(1) f is almost int-irresolute,

(2) $f^{-1}(V) \subset$ int-INT(int -CL(f^{-1} (V))) for every $V \in$ int-O(Y).

4.6 Theorem. A function $f : X \to Y$ is almost int-irresolute if and only if f(int-CL(U)) ⊂ int-CL(f(U)) for every U ∈ int-O(X).

Proof: (⇒) : Let U ∈ int-O(X). Suppose y ∉ int-CL(f(U)). Then there exists V ∈ int-O(Y) such that V ∩ f(U) = ϕ. Hence, $f^{-1}(V) \cap U = \phi$. Since U ∈ int-O(X), we have int-INT(int-CL($f^{-1}(V)$)) ∩ int-CL(U) = ϕ. Then by **Lemma 4.5**, $f^{-1}(V)$ ∩ int-CL(U) = ϕ and hence V ∩ f(int-CL(U)) = ϕ. This implies that y ∉ f(int-CL(U)).

(⇐) : If V ∈ int-O(Y), then M = X – int-CLl($f^{-1}(V)$) ∈ int-O(X). By hypothesis, f(int-CL(M)) ⊂ int-CL(f(M)) and hence X − int-INT(int-CL($f^{-1}(V)$)) = int-CL(M) ⊂ f^{-1}(int-CL(f(M))) ⊂ f^{-1}(int-CL(f(X − $f^{-1}(V)$))) ⊂ f^{-1}(int-CL(Y − V)) = $f^{-1}(Y - V) = X - f^{-1}(V)$. Therefore, $f^{-1}(V)$ ⊂ int-INT(int-CL($f^{-1}(V)$)). By **Lemma 4.5**, f is almost int-irresolute.

4.7 Theorem. If $f : X \to Y$ is a softly int-open continuous almost int-irresolute function from an int-normal space X onto a space Y , then Y is int-normal.

Proof: Let A be a closed subset of Y and B be an open set containing A. Then by continuity of f, $f^{-1}(A)$ is closed and $f^{-1}(B)$ is an open set of X such that $f^{-1}(A) \subset f^{-1}(B)$. As X is int-normal, there exists an int-open set U in X such that $f^{-1}(A)$ ⊂ U ⊂ int-CL(U) ⊂ $f^{-1}(B)$ by **Theorem 3.7**. Then, f($f^{-1}(A)$) ⊂ f(U) ⊂ f(int-CL(U)) ⊂ f($f^{-1}(B)$). Since f is softly int -open almost int-irresolute surjection, we obtain A ⊂ f(U) ⊂ int-CL(f(U)) ⊂ B. Then again by **Theorem 3.7,** the space Y is int-normal.

4.8 Theorem. If $f : X \to Y$ is a softly int-closed continuous function from an int-normal space X onto a space Y , then Y is int-normal.

Proof: Let M_1 and M_2 be disjoint closed sets. Then $f^{-1}(M_1)$ and $f^{-1}(M_2)$ are closed sets. Since X is int-normal, then there exist disjoint int-open sets U and V such that $f^{-1}(M_1) \subset U$ and $f^{-1}(M_2) \subset V$. By **Theorem 4.4**, there exist int-open sets A and B such that $M_1 \subset A$, $M_2 \subset B$, $f^{-1}(A) \subset U$ and $f^{-1}(B) \subset V$. Also, A and B are disjoint. Thus, Y is int-normal.

4.9 Definition. A function $f : X \to Y$ is called **α-closed [6]** if for each closed set in X, f(U) is α-closed set in Y.

4.10 Theorem. If $f : X \to Y$ is an α-closed continuous surjection and X is normal, then Y is int-normal.

Proof: Let A and B be disjoint closed sets of Y. Then $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint closed sets of X by continuity of f. Since X is normal, there exist disjoint open sets U and V in X such that $f^{-1}(A) \subset U$ and $f^{-1}(B) \subset V$. By theorem 4.3 in [6] there exist disjoint α-open sets G and H in Y such that A ⊂ G and B ⊂ H. Since every α-open set is int-open, G and H are disjoint int-open sets containing A and B, respectively. Therefore, Y is int-normal.

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