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Concept of Modern Group Theory

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Abstract - *In mathematics and abstract algebra, Group theory studies the algebraic structures known as groups. The concept of group is central to abstract algebra. Other well-known algebraic structure, such as rings, fields and vector spaces, can all be seen as groups endowed with additional operations and axioms. Modern group theory an active mathematical discipline-studies groups in their own right. To explore groups, mathematicians have devised various notions to break group into smaller, better understandable pieces, such as subgroups, quotient groups and simple groups.*

Keywords - *Group, Abelian-group, Groupoid, Semigroup, Monoid, Finite and Infinite Group, Subgroup and their types, Trivial Subgroup, Symmetry Group, Permutations, Cycles, Cyclic Group, Dihedral Group, Homomorphism of Group, Kernel of Homomorphism, Isomorphism of Group, Coset, Center of Group, Quotient Group, Symmetry.*

I. INTRODUCTION

Modern group theory studies groups in their own right. To explore groups, mathematicians have devised various notions to break group into smaller, better-understandable pieces, such as subgroups, quotient group and simple group. In addition to their abstract properties, group theorists also study the different way in which a group can be expressed concretely, both from a point of view of representation theory and of computational group theory. A theory has been developed for finite groups, which culminated with the classification of finite simple groups, completed in 2004. Since the mid-1980's, geometric group theory, which studies finitely generated groups as geometric objects, has become an active area in group theory.

Definition and illustration-

First example, the integers one of the most familiar groups is the set of integers $\mathbb{Z}$ which consists of the numbers …. -4, -3, -2, -1, 0, 1, 2, 3, 4…together with addition.

The following properties of integer addition serve as a model for the group axioms given in the definition below.

- For any two integers a and b, the sum a + b is also an integer. That is, addition of integer always yields an integer. These properties are known as closure under addition.
- For all integer a, b and c, (a + b) +c = a+ (b + c). we also know that, adding a to b first, and then adding the result to c gives the same result as adding a to the sum of b and c, a property known as associativity.
- If a is any integer, then 0+a = a+0 = a. Zero is called the identity element of addition because adding it to any integer returns the same integer.
- For every integer a, there is an integer b such that a + b = b + a = 0. The integer b is called the inverse element of the integer a and is denoted –a.
- The integer, together with the operation +, from a mathematical object belonging to a broad class sharing similar structural aspects as a collective, the following definition is developed.

Set-

In mathematics, a set is a well-defined collection of distinct objects, considered as an object in its own right. For example, the number 2,4 and 6 are distinct objects when considered separately, but when they are considered collectively, they are from a single set of size three, written {2,4,6}.

Binary operation-

A binary operation is a calculation that combines two elements (called operands) to produce another element. more formally, a binary operation is an arity two.

Example include the familiar arithmetic operations of addition, subtraction, multiplication. Addition of all natural is a binary operation 2+3 = 5(2,3,5 all element is a natural number) but subtraction of all natural is not a binary operation 2-3 = -1 (2 and 3 is natural number but -1 is not natural number).

II. RESEARCH OBJECTIVE

In mathematics and abstract algebra, group theory studies the algebraic structures known as groups. Thus, group theory and the closely related representation theory have many important applications in physics, chemistry and materials science etc. Group theory is also control to public key cryptography. So, it is most important thing to know about the evolution of group theory and concept modern group theory.

III. HYPOTHESIS

A hypothesis is an unproven statement which is supported by all the available data and by much weaker result. It is a sequential process of research. It is an indicator of the work to be done for research. It describes the theoretical information in relation to how the research study will work. The hypothesis is a formal suggestion that express the outline of the study′s proposal. In this paper we will able to understand the evolution the group theory that describe the concept of modern group theory.

IV. RESEARCH METHOD

A research design is the set of methods and procedures used in collecting and analyzing measures of variables specified in the problem research. A research design is a frame work that has been created to find answers to research questions. There are numerous types of research design that are appropriate for the different types of research projects. The choice of which design to apply depends on the problems posed by the research aim.

Each type of research design has a range of research methods that are commonly used to collect and analyses the type of data, which is generated by the investigations. Here is a list of some of more common research design, which is used to find expected outcome of my proposed work.

1. ***Historical –*** This aim at systematic and objective evaluation and synthesis of evidence in order to establish facts and draw conclusion about past events. It uses primary historical data, such as archaeological remains as well as documentary sources of the past, it is usually necessary to carry out tests in order to check the authenticity of these sources.
2. ***Descriptive –*** This design relies on observation as a means of collecting data. It attempts to examine situations in order to establish. Observation can take many forms depending on the type of information sought; people can be interviewed questionnaires distributed.
3. ***Correlation –*** This design is used to examine a relationship between two concepts. There are two broad classifications of relational statements and association between two concepts.
4. ***Comparative –*** This design is used to compare past and present or different parallel situations. It can look at situation at different scales, macro (international, national) or micro (community, individual).

V. ANALYSIS

1. ***Group-*** A group is a set G, together with a binary operation that combines any two-element a and b to form another element, denoted a*b. To qualify as a group, the set and operation, (G, *), must satisfy four requirements known as the group axioms.

Closure
For all a, b in G, the result of the operation, a*b, is also in G.
Associativity
For all a, b and c in G, (a*b) *c = a*(b*c).
Identity element
There exists an element e in g such that, for every element in G, the equation e*a = a*e = a hold. Such an element unique and thus one speaks of the identity element.
Inverse element
For each a in G, there exists an element b in G, commonly denoted by a^{-1} (or –a, if the operation is denoted "+"), such that a*b = b*a =e, where e is the identity element.
The result of the group operation may depend on the order of the operands. In other words, the result of combining element a with element b need not yield the same result as combining element b with element a. The equation a*b =b*a may not be true for every two-element a and b. This equation always holds in the group of integers under addition, because a + b = b + a for any two integers (commutativity of addition).

2. ***Abelian group-*** A group G, is said to be Abelian (or commutative) if a*b = b*a for all element of a and b. The words 'Abelian' is honor of great mathematician Niels Henrik Abel.
3. ***Groupoid-*** A set G satisfied only closure axioms called groupoid.
4. ***Semigroup-*** A set G satisfied closure and associative axioms called semigroup.
5. ***Monoid-*** A set G satisfied closure, associative and existence of identity called monoid.
6. ***Finite and Infinite groups-*** If a group G consists of a finite number of elements, then it is called a finite group, otherwise it is called an infinite group.
7. ***Order of a group-*** The number of elements in a finite group is called the order of the group. An infinite group has an infinite element, said to be infinite order. Order of group is denoted by o(G) or |G|.
8. ***Subgroup-*** In group theory, a branch of mathematics, given a group G a binary operation *, a subset H of G is called a subgroup of G if H also forms a group under the operation *,
9. ***Proper subgroup-*** A proper subgroup of a group G is a subgroup H which is a proper subset of G (that is $H \neq G$). This is usually represented notation ally by $H < G$, read as 'H is proper subgroup of G'.
10. ***Improper subgroup-*** For a group G, the set {e} and G are always subgroup of G and are called improper subgroup of G. Any other subgroup [other than G and {e}] is called a proper subgroup.
11. ***Trivial subgroup-*** The trivial subgroup of any group is the subgroup consisting of just the identity {e} element.
12. ***Symmetry (Permutation)group-*** Two figures in the plane are congruent if one can be changed into the other using a combination of rotations, reflections, and translations. Any figure is congruent to itself. However, some figures are congruent to themselves in more than one way, and these extra congruences are called symmetries. A square has eight symmetries.
13. ***Permutations-*** A permutation of a set X is simply a re-arrangement of the elements, or more precisely a function p that maps each element of X to an element of X with no who distinct elements being mapped to the same element (and for infinite sets, we also need p(X) = X). Another way of saying this is that a permutation of X is simply a bijection $p: X \to X$. Normally we are interested only in permutations of finite sets, and we really only care how many elements there are to permute. Hence it is customary to consider permutations of the set {1, 2, 3, ..., n}. Since permutations are just functions, we can define them as we would any other function, by specifying the value that the function takes at each point in the domain. Unfortunately, unlike the usual functions you see in a calculus course, you usually can't specify permutations using a formula.

Example 13.1

The following function p is a permutation of the set {1, 2, 3, 4, 5, 6, 7, 8}:
p(1) = 2 p(2)= 4 p(3) = 6 p(4)= 8
p(5) = 7 p(6)= 5 p(7) = 3 p(8)= 1

A more compact way of writing down a permutation is to write it as an array of numbers, with 1, through n on the top row, and the respective image of each in the second row, like so:

$$p = \begin{pmatrix} 1 & 2 & 3 & \ldots\ldots & n \\ p(1) & p(2) & p(3) & \ldots\ldots & p(n) \end{pmatrix}$$

Example 13.2

The permutation p of the previous example can be written as follows:

$$p = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 4 & 6 & 8 & 7 & 5 & 3 & 1 \end{pmatrix}$$

We denote the set of all permutations of {1, 2, 3, . . . , n} by Sn.

Example 13.3

The set S3 is

$$\left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \right\}$$

Note that, as in the above example, the identity permutation p(k) = k is always a permutation. Since every permutation is a one-to-one and onto function, there is an inverse function p^{-1} associated with every permutation p. We can "multiply" two permutations by applying the first permutation, and then using the second permutation to permute the result. If p and q are permutations of the same set, pq(k) is the what you get from applying q to p(k), ie. pq(k) = q(p(k)), so pq = q *o* p (note the reversal of terms in the product versus the composition).

14. ***Cycles***

Even with the current notation, expressing and working with permutations can be cumbersome. There is another, alternative, notation which can speed up the process of working with permutations. This notation works by

looking at the cycles withing a permutation. If p is a permutation of the set X, the cycle of an element k of X in p is the sequence of elements (k, p(k), p^2(k), . . . , p^m(k)) (where p^l is the product of p with itself l times) such that m is the smallest number such that, p^{m+1} (k) = k. Note that the order of the elements in a cycle is important, but not where we start in the cycle. For example, we regard (k, p(k), p^2(k), . . . , p^m(k)), (p(k), p^2(k), . . . , p^m(k), k), (p^2(k), . . . , p^m(k), k, p(k)), etc. as representing the same cycle. If X is the set {1, 2, . . . n}, it is usual to write a cycle starting with the smallest number in the cycle. A cycle with m elements is called an m-cycle. A 2-cycle is sometimes called a transposition, since it transposes two elements.
Example 14.1
In the following permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 4 & 6 & 8 & 7 & 5 & 3 & 1 \end{pmatrix}$$

we have 1 → 2, 2 → 4, 4 → 8 and 8 → 1, so (1, 2, 4, 8) is a cycle. We could also write this cycle as (2, 4, 8, 1), (4, 8, 1, 2), or (8, 1, 2, 4). The smallest element not in this cycle is 3, and we have 3 → 6, 6 → 5,5 → 7 and 7 → 3, so (3, 6, 5, 7) is another cycle. Since every element is in one of these two cycles, these are the only cycles in this permutation. If we find all of the cycles of a permutation, we can represent the permutation as a whole as a product of its cycles. But to do that we need to understand how to multiply cycles. To work out how a product of cycles permutes a particular element k, all you need do is work from left to right until you find the element in a cycle, and then find the element which follows it in that cycle. You continue from left to right starting with the the next cycle looking for an occurrence of the new element.
If there is, then you find the element which follows it in the cycle. Continue on in this fashion until you run out of cycles. The final value of the element is the image of k under the product of cycles.

15. ***Cyclic group***- A group G is said to be cyclic if it's all elements can be generated by a single element.
Example14.1- G= {x:x = a^n} where a∈ G and n is any positive integer, G =(a), G is cyclic group, a is generator. G = {1, -1, i, - i} (fourth root of unity) is multiplicative group and is also a cyclic group. i^1 = i, i^2 = -1, i^3 = -i, i^4 = 1 so generator is i → G = (i).

(-i) = - i, $(-i)^2$ = -1, $(-i)^3$ = i, $(-i)^4$= 1 so generator is –i → G = (-i).

16. ***Dihedral group***- The set of all symmetries of a regular polygon of n-sides forms a group with respect to the composition(product) of symmetries and is known as n^{th} dihedral group and is denoted by D_n. The dihedral group D_n is of order 2n and it contains n rotations and n reflections. Product of two rotations is again rotation and product of two reflections is rotation, product of one rotation and one reflection is a reflection.

Example of Dihedral group- Q_8 The group quaternions is given as Q_8 = {1, -1, i, -i ,j , -j , k , -k } where i^2= j^2 = k^2 = -1 and I . j = k, j . k = I, k . i = j, j .i = -k, k . j = - i, i.k = -j,

.	1	-1	i	-i	j	-j	k	-k
1	1	-1	i	-i	j	-j	k	-k
-1	-1	1	-i	i	-j	j	-k	k
i	i	-i	-1	1	k	-k	-j	j
-i	i	-i	1	-1	-k	k	j	-j
j	j	-j	-k	k	-1	1	i	-i
-j	-j	j	k	-k	1	-1	-i	i
k	k	-k	j	-j	-i	i	-1	1
-k	-k	k	-j	j	i	-i	1	-1

17. ***Normal subgroup***- Let H is the subgroup of group G if for all h ∈ H and for all x ∈ G, x h x^{-1} ∈ H Then H is called Normal subgroup of G.
18. ***Homomorphism of group***- - Let (G, *) and (G', *') are two groups. A mapping f: G→G' is said to be homomorphism if f (a o b) = f(a) o' f(b) for all a,b ∈ G and f(a) , f(b) ∈ G'.
19. ***Kernel of Homomorphism***- Let (G,*) and (G',*') are two groups with binary operation * and *' and also mapping f:G→G' is said to be homomorphism from G to G' then kernel of 'f' is denoted by k or ker(f) or k(f) and defined as-
Ker(f) = {x ∈ G: f(x) = e' ∈ G' where e' is identity element of G'}.
20. ***Isomorphism of group***- Let (G, *) and (G', *') are two groups. A mapping f: G→G' is called Isomorphism if –

f is one-one mapping.
f is on-to mapping.
f is homomorphism, f (a o b) = f(a) o' f(b) for all a,b ∈ G.

Example20.1

f: G→G' where G = additive group of integer and G' = additive group of even integer and f defined as f(x) = 2x for all x ∈ G.

Solution- (1) f is one-one mapping
Let f (x_1) and f (x_2) ∈ G', s.t. f (x_1) = f (x_2)
2 x_1 = 2 x_2 ⇒ $x_1 = x_2$ so f is one-one mapping.
(2) f is on-to mapping.
Let y ∈ G' (y is even integer), now ∋ an element such that x ∈ G
x = y /2 since f(y /2) = 2(y /2) = y so f(x) = y
for all y ∈ G' we have x ∈ G such that f(x) = y.
(1) f is homomorphism
x_1, x_2 ∈ G ⇒ f $(x_1 + x_2)$ = f 2 (x_1+x_2)
= 2 x_1 + 2 x_2 ⇒ f (x_1) + f (x_2) f is homomorphism
So f is isomorphism.

21. ***Coset-*** Let G be group and H be its subgroup. For any element a ∈ G, the set Ha = {ha: h ∈ H} is called right coset of H in G, and the set a H = {ah: h ∈ H} is called left coset of H in G. Not that if the binary operation in G is additive, then the right coset of H in g is H+a = {h+a: h ∈ H}. similarly, the left coset is a+H = {a+h: h ∈ H}.

22. ***Center of a group-*** Let G be a group then center of group G is defined to be the subset of all element of G which commute every element of G and its denoted by Z(G). in symbols.Z(G) = { a ∈ G : ax = xa for all x in G }.

23. ***Quotient group-*** Let G be a multiplicative group and N is normal subgroup of G then G/N is a set of all right coset of N in G, also G/N is a multiplicative group. Thus, G/N is called quotient group w.r.t. multiplication of coset.

24. Symmetry

You are familiar, at least in an informal way, with the idea of symmetry from Euclidean geometry and calculus. For example, the letter "A" has reflective symmetry in its vertical axis, "E" has reflective symmetry in its horizontal axis, "N" has rotational symmetry of π radians about its centre, "H" has all three types of symmetry, and the letter "F" has none of these symmetries. Symmetry is also important in understanding real world phenomena. As some examples: • The symmetries of molecules can affect possible chemical reactions. For example, many proteins and amino acids (the basic building blocks of life) have "left-handed" and "right-handed" versions which are reflections of one-another. Life on earth uses the "left-handed" versions almost exclusively.

- Crystals have very strong symmetries, largely determined by the symmetries of the atoms or molecules of which the crystal is built.
- Most animals and plants have some sort of symmetry in their body-shapes, although they are never perfectly symmetrical. Most animals have bilateral symmetry, while plants often have five-fold or six-fold rotational symmetry.
- In art and design, symmetrical patterns are often found to be more pleasing to the eye than asymmetrical patterns, or simply more practical.
- Waves in fluids, and the vibrations of a drumhead or string are often symmetrical, or built out of symmetric components. These symmetries are usually inherent in the underlying equations that we use to model such systems, and understanding the symmetry can be crucial in finding solutions to these equations.

But what, precisely, do we mean by symmetry?

Definition 24.1

Let Ω be a subset of R^n. A symmetry of is a function T : $R^n \rightarrow R^n$ such that
(i) {T(x) : x ∈ Ω } = , and
(ii) T preserves distances between points: if d(x, y) is the distance between the points x and y, then d(T(x), T(y)) = d(x, y).
We denote the set of all symmetries of Ω by Sym(Ω). Every set has at least one symmetry, the identity symmetry I(x) = x.
Functions which preserve distance are called isometries, so every symmetry is an isometry.

Proposition 24.2

Let be a subset of R^n, and let S and T be symmetries of . Then
(i) T is one-to-one and onto.

(ii) the inverse function T^{-1} is a symmetry of
(iii) the composition T ∘ S is a symmetry of
(iv) the compositions T ∘ T^{-1} and T^{-1} ∘ T always equal the identity symmetry I.

Example 24.3

Let $\Omega \subseteq R^2$ be the H-shaped set illustrated in Figure 1.1. Then has percisely the following symmetries:

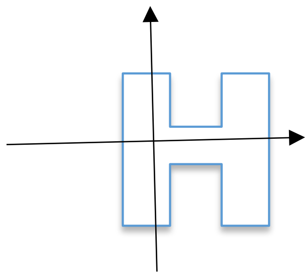

Figure24.4

I(x, y) = (x, y) (Identity)
H(x, y) = (x,−y) (Reflection in the x-axis)
V (x, y) = (−x, y) (Reflection in the y-axis)
R(x, y) = (−x,−y) (Rotation by _ radians about the origin)
We can confirm by direct calculation that I^{-1} = I, H^{-1} = H, V^{-1} = V and R^{-1} = R. In other words, each of these transformations is its own inverse.
These symmetries compose in the following ways:

H *o* H = I H *o* V = R H *o* R = V
V *o* H = R V *o* V = I V *o* R = H
R *o* H = V R *o* V = H R *o* R = I

In fact we can summarize this using a "multiplication table":

∘	*I*	*H*	*V*	*R*
I	I	H	V	R
H	H	I	R	V
V	V	R	I	H
R	R	V	H	I

This sort of "multiplication table" is called a Cayley table for the operation.

VI. CONCLUSION

In this paper we have learnt that modern algebra is a study of sets with operations defined on them. As the main example we have started a systematic study of groups. Group theory is one of the most important areas of contemporary mathematics, with applications ranging from physics and chemistry to coding and cryptography.

REFERENCES

[1] ISRAEL KLEINER, "The Evolution of Group Theory A Brief Survey" (York University North York, Ontario Canada M3J1P3 VOL 59 NO 4 OCTOBER 1986)
[2] Ferdi Arya Setiawan, "Group Theory" Dept of Theoretical physics university of Lund, Solvegatan 14A,223 62 Lund-Sweden
[3] Alistar Savage, "Modern Group Theory" Dept of Mathematics and Statistics university of Ottawa.
[4] Smith, David Eugene "History of Modern Mathematics" (1906), Mathematical Monographs,
[5] Debabrata Basu , "Introduction to Classical and Modern Analysis and Their Application to Group Representation Theory" (Indian Institute of Technology, India)
[6] Wussing, Hans, "The Genesis of the Abstract Group Concept: A Contribution to the History of the Origin of Abstract Group Theory" (2007)
[7] Judson, Thomas W., "Abstract Algebra: Theory and Applications" (1997)
[8] La Harpe, Pierre de, "Topics in geometric group theory" (2000) University of Chicago Press
[9] Scott, W. R. "Group Theory" (1987) [1964], Inexpensive and fairly readable, but somewhat dated in emphasis, style, and notation
[10] Rotman, Joseph, "An introduction to the theory of groups" (1994)