International Journal of Mathematics Trends and Technology Volume 66 Issue 12, 85-90, December, 2020
ISSN: 2231 – 5373 /doi:10.14445/22315373/IJMTT-V66I12P513

On Existence of Period Three Orbit and Chaotic Nature of a Family of Mappings

Kulkarni Pramod Ramakant

P. G. Department of Mathematics, N. E. S. Science College, Nanded. 431602 (Maharashtra State), India.

Abstract - *The occurrence of dynamical systems is observed in all branches of sciences like differential equations, Biological sciences, Physical sciences Mechanics, Economics and many more. On a broad spectrum, a dynamical system can be classified in to two categories viz. continuous and the discrete. As a discrete dynamical system, many mathematicians have extensively studied the one parameter family of functions; specially the logistic family* $F_\mu(x) = \mu x(1 - x)$, *the Tent family, Quadratic family, etc. In last few decades, the chaotic nature of many non-linear phenomenon has been a topic of great interest for the researchers all over the world. The occurrence of chaotic regime for certain values of the parameter in case of the family of mappings* $f_c(x) = x^2 - x + c$ *using the phenomenon of the period doubling has been proved by Kulkarni P. R. and Borkar V. C. In this paper, we have proved the existence of the period three orbit in the family of mappings* $f_c(x) = x^2 - x + c$ *for the parameter value* $c = -1.5$ *which proves that it is a chaotic map.*

Keywords — *chaotic behavior, critical points, one parameter family of functions, periodic points, Sarkovskii's theorem.*

I. INTRODUCTION

Dynamical systems have been attracting mathematicians, physicists, biologists, all over the world as it has a variety of strange properties and the applicability in all branches of sciences. The concept of chaos has became a topic of keen interest all over the world. Talking roughly, chaos, in mathematical concepts, means predictability along with randomness. Chaotic dynamics is the study of the phenomenon where there is certainty and periodicity for a particular values of the parameters, but as the parameter values go beyond a fixed range, the phenomenon shows a strange pattern which is quite unpredictable. Mathematicians have been trying to give more and more accurate mathematical model of a physical phenomenon and predict the changes that may take place over time.

II. DYNAMICAL SYSTEM AND ASSOCIATED TERMINOLOGY

A dynamical system consists of two parts: a state vector $x \in R^n$, which is a list of numbers and may change as the time passes, and a function $f : R^n \to R^n$, where the set R^n is called as the set of states or the state space. By means of the function f, one can determine the state of the vector at any position from the current state. This rule is described by the function f. Most of the mathematicians study two types of dynamical systems *viz.* discrete dynamical system and continuous dynamical system. The exact definitions and a number of examples are given by [1], [2], [3], [4] and [5]. In this paper, we will consider only one dimensional discrete dynamical systems.

Basic ideas in the field of dynamical systems have been introduced in the early text of the references provided in this paper; however, for the convenience of the reader, we introduce some of them in brief.

2.1 Iterations of a function

For a given dynamical system $f : S \to S$, $S \subseteq R$, iterations of the function f means the compositions of f with itself.

In general, the k^{th} iteration of f at a point x is the k times composition of f with itself at the point x, denoted by $f^k(x)$. For more details, refer [1] [4], [5], [7].

2.2 Fixed Points and Periodic Points

A point x is said to be a fixed point of a function f if $f(x) = x$. If x is a fixed point of f, then $f^n(x) = x$ for all $n \in Z^+$, where is Z^+ the set of positive integers.

A point x is said to be a periodic point with period n if there exists a positive integer n such that $f^n(x) = x$. Clearly, if x is periodic with period n, then it is periodic with period $2n$, $3n$, $4n$,...The smallest n, in this case, is called as the prime period of the orbit. Note that x is a periodic point with period n of f if it is a fixed point of f^n. Refer [1] [4], [5], [7] for more details.

2.3 Attracting and Repelling Fixed Points

Let p be a fixed point of a dynamical system $f : S \to S$, $S \subseteq R$.

(1) We say that p is an attracting fixed point or a sink of f if there is some neighborhood of p such that all points in this neighborhood are attracted towards p. In other words, p is a sink if there exists an epsilon neighborhood $N_\epsilon(p) = \{x \in S : |x - p| < \epsilon\}$ such that $\lim_{n\to\infty} f^n(x) = p$ for all $x \in N_\epsilon(p)$.

(2) We say that p is a repelling fixed point or a source of f if there is some neighborhood $N_\epsilon(p)$ of p such that each x in $N_\epsilon(p)$ except for p maps outside of $N_\epsilon(p)$. In other words, p is a source if there exists an epsilon neighborhood such that $|f^n(x) - p| > \epsilon$ for infinitely many values of positive integers n.

2.4 Hyperbolic Periodic Points

A periodic point p of a mapping f with prime period n is said to be hyperbolic if $|(f^n)'(p)| \neq 1$.

2.5 Attracting and Repelling Periodic Point

Let p be a periodic point of period n of a function f. Then p is said to be an attracting periodic point or a repelling periodic point according as it is an attracting or a repelling fixed point of the n^{th} iterate f^n.

2.6 Neutral Fixed Point

A fixed point p of a differentiable function f is said to be a neutral fixed point if $|f'(p)| = 1$.

2.7 A review of the results

Let us recall some of the theorems that have been already proved. For the proof, one can look in to the references provided.

2.8 Theorem

Let $f : [a, b] \to R$ be a differentiable function, where f' be continuous and p be a periodic point of f with period n. Then the periodic orbit of p is attracting or repelling according as $|(f^n)'(p)| < 1$ or $|(f^n)'(p)| > 1$.

2.9 Theorem

Let $f : [a, b] \to R$ be a differentiable function, where f' be continuous and p be a hyperbolic fixed point of f. If $|f'(p)| < 1$, then p is an attracting fixed point of f.

2.10 Theorem

Let $f : [a, b] \to R$ be a differentiable function, where f' be continuous and p be a hyperbolic fixed point of f. If $|f'(p)| > 1$, then p is a repelling fixed point of f.

2.11 Theorem

Let p be a neutral fixed point of a function f.

(i) If $f''(p) > 0$, then p is weakly attracting from the left and weakly repelling from the right.

(ii) If $f''(p) < 0$, then p is weakly repelling from the left and weakly attracting from the right.

Let p be a neutral fixed point of f with If $f''(p) = 0$.

(iii) If $f'''(p) > 0$ then p is weakly repelling.

(iv) If $f'''(p) < 0$, then p is weakly attracting.

III. CHAOTIC DYNAMICAL SYSTEMS

We use the word *chaos* in our daily life as an indication of confusion or ambiguity or randomness or uncertainty. Chaotic dynamics have been the topic of key interest amongst the scientists as the chaos is observed in nature most of the times. While defining chaos, many approaches have been made by different authors including measure theoretic notions, topological concepts, etc. We will use the most widely used and accepted definition of chaos as given by Devaney, R. L. [4].

3.1 Chaos

Let V be a set. A mapping $F : V \to V$ is said to be chaotic on V if

1. F has sensitive dependence on initial conditions.
2. F is topologically transitive.
3. periodic orbits are dense in V.

For details of the notions used in this definition, the reader is advised to refer [4], [7], [9].

3.2 Examples of chaotic mappings

Some examples of the mappings showing chaotic behavior are as follows. The chaotic nature of the first three mappings has been proved in the book by Devaney[4].

1. Let S^1 denote the unit circle in plane. The mapping $f : S^1 \to S^1$ defined by $f(\theta) = 2\theta$ is chaotic, where a point in S^1 is denoted by its angle θ measured in radians.

2. The logistic family $F_\mu(x) = \mu x(1 - x)$ is chaotic on the Cantor set Λ for $\mu > 2 + \sqrt{5}$. On a comparatively larger set $I = [0, 1]$, the mapping $F_\mu(x) = \mu x(1 - x)$ is chaotic for $\mu = 4$.

3. The tent mapping T_2 defined by

$$T_2(x) = \begin{cases} 2x & \text{for } 0 \le x \le 1/2 \\ 2(1-x) & \text{for } 1/2 < x \le 1 \end{cases}$$ is chaotic on the interval [0, 1].

4. The chaos in the family of mappings $f_c(x) = x^2 - x + c$ for the values of $c > -½$ has been explored through a period doubling phenomenon by Kulkarni P. R. and Borkar V. C. [7] Moreover, considering the topological aspects of the dynamical systems, Kulkarni P. R. and Borkar V. C. [9] has proved the chaos in the same mapping by proving that it is topologically conjugate to the to the shift map σ on Σ_2 via the itinerary S.

3.3 Period three and chaos

Now we will study the importance of the existence of period three in a dynamical system. The chaotic nature of a dynamical system is also guaranteed by the existence of the period three point. First we will state the period three theorem which was proved by Tien-Yien Li and James A. Yorke [6] in 1975.

3.4 Theorem (The Period Three Theorem)

Let $f : R \to R$ be a continuous function. If f has a periodic point of period three, then f has periodic points of all other periods.

For the proof of this theorem the reader can also refer the book by Devaney [4]. A generalization of this theorem is known as the Sarkovskii's theorem. Sarkovskii's theorem was proved by Sarkovskii in 1964. This theorem explains the order of the existence of the periodic points *i.e.* which period imply which other periods for continuous dynamical systems. Before stating the Sarkovskii's theorem, we need to understand the Sarkovskii ordering of natural numbers. This ordering is as follows.

3, 5, 7, 9,..., (list of all odd numbers except 1)
2.3, 2.5, 2.7,..., (two times the odd naturals)
$2^2.3, 2^2.5, 2^2.7,...,$ (2^2 times the odd naturals)
$2^3.3, 2^3.5, 2^3.7,...,$ (2^3 times the odd naturals)
.
.
.
$2^n, 2^{n-1}, 2^{n-2},..., 2^3, 2^2, 2, 1.$

This ordering by Sarkovskii exhausts the natural numbers. Now we state the Sarkovskii's theorem as follows.

3.5 Theorem (Sarkovskii's Theorem)

If a continuous function $f : R \to R$ has a point of period n, where n precedes k in Sarkovskii's ordering of natural numbers, then f has a periodic point of period k.

The proof of this theorem can be studied from [4], [5] and [6]. As the number 3 appears at the first in the Sarkovskii's ordering of natural numbers, if a function has a point of period 3, then there exists periodic points of all periods. This is nothing but the Period Three Theorem as we stated earlier. Since the powers of 2 appear in the last line of the Sarkovskii's ordering of natural numbers, we conclude that if a function f has finitely many periodic points, then they all must be some powers of 2. On the contrary, if f has a periodic point of period which is not a power of 2, then f must have infinitely many period points. Sarkovskii's has been proved only for one dimensional dynamical systems. The theorem does not hold for higher dimensional systems. The converse of Sarkovskii's theorem is also true, which can be stated as follows

3.6 Theorem (Converse of Sarkovskii's Theorem)

Let k and n be natural numbers. If k precedes n in the Sarkovskii's ordering of natural numbers, then there exists a continuous mapping $f : R \to R$ which has a periodic point of period n but no periodic point of period k.

IV. TOPOLOGICAL CONJUGACY

The topological properties like existence of dense periodic orbit and topological transitivity [4] are inherited from one dynamical system to another through a bijective mapping called topological conjugacy. It plays an important role in studying important properties of an unknown dynamical system by comparing those with a *known* dynamical system. We define topological conjugacy as follows.

4.1 Topological conjugacy

Let $f : A \to A$ and $g : B \to B$ be two mappings. Then f is said to be topologically conjugate to g if there exists a homeomorphism $h : A \to B$ such that $h \circ f \circ h^{-1} = g$ or what amounts to the same thing, if $h \circ f = g \circ h$.

As h is a homeomorphism, it follows that if f is topologically conjugate to g, then g is also topologically conjugate to f. In this case, the homeomorphism h is called as a topological conjugacy between f and g, or we say that f and g are conjugate via the mapping h. The following results concerning the topological conjugacy are proved by Kulkarni P. R. and Borkar V. C.[9] We state these results here for reference.

4.2 Theorem

If f and g are topologically conjugate via mapping h, and if p is a fixed point of f, then $h(p)$ is a fixed point of g.

4.3 Theorem

If f and g are topologically conjugate via mapping h, then $h \circ f^n = g^n \circ h$. Thus, if f and g are conjugate, the periodic points are carried into periodic points of the same period under conjugacy.

4.4 Theorem

Let $f : A \to A$ and $g : B \to B$ be topologically conjugate via a mapping $h : A \to B$. Then f is transitive if and only if g is transitive. That is, topological conjugacy preserves transitivity.

4.5 Theorem

If $h : A \to B$ is an onto continuous mapping, then the image under h of a set dense in A is a set dense in B.

4.6 Theorem

Let $Per_n(f)$ denote the set of periodic points of period n of the mapping f. Let f and g be topologically conjugate. Then $Per_n(f)$ is dense if and only if $Per_n(g)$ is dense.

From these theorems and the definition of chaos, it follows that if two mb appings are topologically conjugate, then the number and nature of their fixed and periodic points is the same. We summarize this as the following.

4.7 Theorem

Let f and g be topologically conjugate. Then f is chaotic if and only if g is chaotic.

V. EXISTENCE OF PERIOD THREE AND CHAOS IN $f_c(x) = x^2 - x + c$

Now we will obtain a period three point for the mapping $f_c(x) = x^2 - x + c$ and thereby prove that $f_c(x) = x^2 - x + c$ is chaotic on R for $c = -1.25$. As a first attempt in this direction, we will find period three points for the mapping T_2 defined by

$$T_2(x) = \begin{cases} 2x & \text{for } 0 \le x \le 1/2 \\ 2(1-x) & \text{for } 1/2 < x \le 1 \end{cases}$$

It is clear that {2/7, 4/7, 6/7} is a 3-cycle for T_2 since

$T_2\left(\frac{2}{7}\right) = \frac{4}{7},\ T_2^2\left(\frac{2}{7}\right) = T_2\left(\frac{4}{7}\right) = \frac{6}{7},\ T_2^3\left(\frac{2}{7}\right) = \frac{2}{7}$

Now we will prove that the mapping T_2 is topologically conjugate to the mapping $F_4(x) = 4x(1-x)$ via the mapping $h(x) = sin^2\frac{\pi x}{2}$. We obtain the composition $F_4 \circ$ h as

$$(F_4 \circ h)(x) = F_4(h(x)) = F_4\left(sin^2\frac{\pi x}{2}\right) = 4.sin^2\frac{\pi x}{2}\left(1 - sin^2\frac{\pi x}{2}\right) = sin^2(\pi x) \quad \text{. . . (1)}$$

If $0 \le x \le \frac{1}{2}$, then $\left(h \circ T_2\right)(x) = h(T_2(x)) = h(2x) = sin^2(\pi x)$ . . . (2)

and if $\frac{1}{2} < x \le 1$, then $\left(h \circ T_2\right)(x) = h(2(1-x)) = sin^2(\pi - \pi x) = sin^2(\pi x)$ . . . (3)

Thus from Eq.(1), Eq.(2), and Eq.(3), we have $F_4 \circ h = h \circ T_2$ so that T_2 is topologically conjugate to the mapping F_4 via the mapping $h(x) = sin^2\frac{\pi x}{2}$. As {2/7, 4/7, 6/7} is a 3-cycle for T_2, by theorem 3.2, it follows that $\{ sin^2\frac{\pi}{7},\ sin^2\frac{2\pi}{7},\ sin^2\frac{3\pi}{7}\}$ is a 3-cycle for F_4. Similarly we can prove that the mapping $F_4(x) = 4x(1-x)$ is conjugate to the mapping $f_{-1.25}(x) = x^2 - x - 1.25$ via the mapping $H(x) = -4x + 2.5$.

Hence a 3-cycle for $f_{-1.25}(x) = x^2 - x - 1.25$ is
$\{2.5 - 4\,sin^2\frac{\pi}{7},\ 2.5 - 4\ sin^2\frac{2\pi}{7},\ 2.5 - 4\,sin^2\frac{3\pi}{7}\}$.
An approximate value of this 3-cycle is $\{1.746979603717467,\ 0.054958132087371, -1.301937735804839\}$.

The iterates of a function can be studied by means of the graphical analysis. The orbit diagram for the function $f_{-1.25}(x) = x^2 - x - 1.25$ is as shown in the following Figure1.

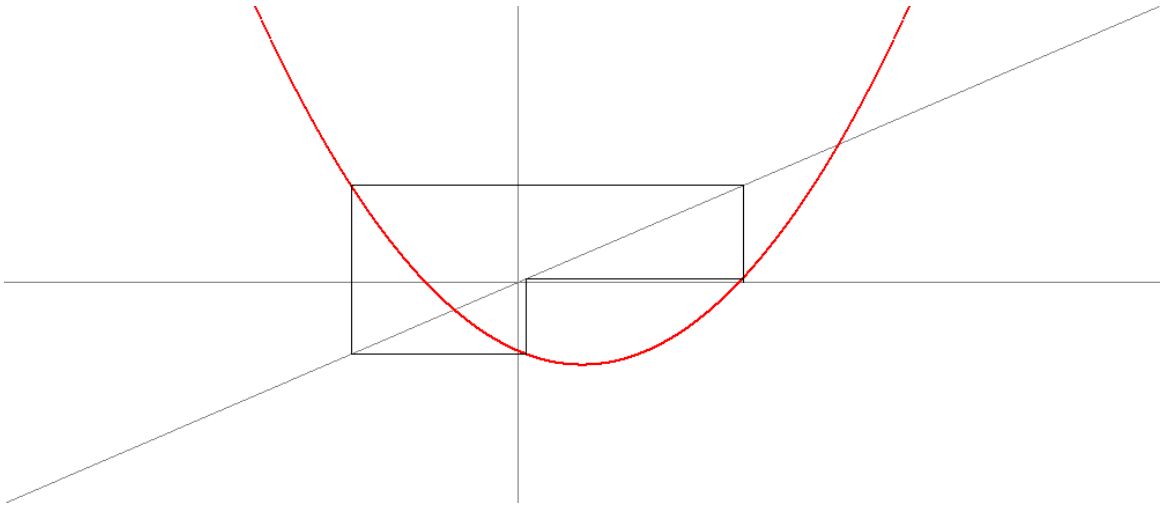

Fig. 1 The orbit diagram for the function $f_{-1.25}(x) = x^2 - x - 1.25$

A bifurcation diagram (Refer [4], [5], [7], [9]) exhibits the transition to chaos through successive period-doubling bifurcations as the parameter value is varied in one parameter family of mappings. For the one parameter family if mappings $f_c(x) = x^2 - x + c$, all the interesting dynamics occur in the interval $-2 \le c \le 1$. We divide the parameter range [–2, 2] into a number of specified subdivisions and the orbits are computed and plotted using the initial condition $x_0 = 0$ and obtain the bifurcation diagram as shown in the Figure 2.

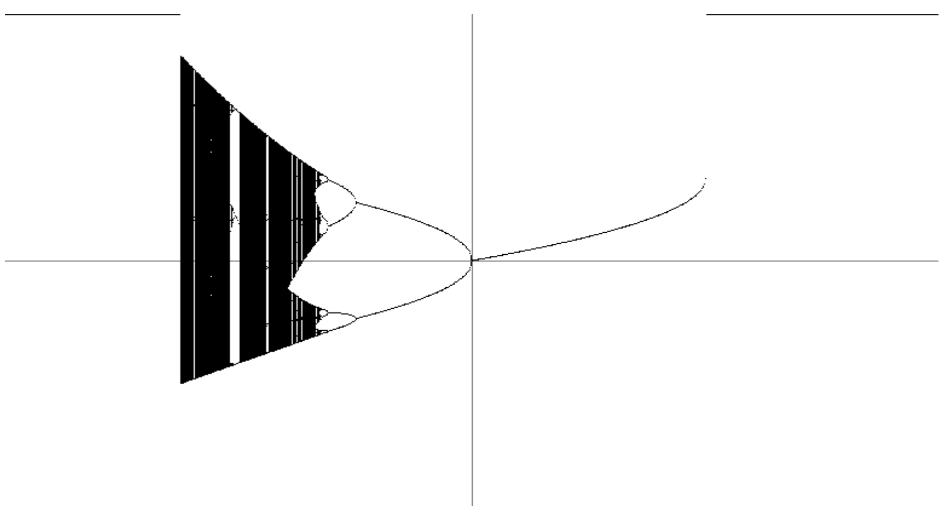

Fig. 2 A bifurcation diagram

Hence by Sarkovskii's theorem, the mapping $f_c(x) = x^2 - x + c$ is chaotic on R for $c = -1.25$.

VI. CONCLUSIONS

The chaos in the behavior of a one parameter family of mappings can be noted by observing the bifurcation diagram only however, it becomes very difficult to find out at what value of the parameter, the periodic orbit of a particular period comes in to picture of the bifurcation diagram. We have obtained the value of c for which the family of mappings $f_c(x) = x^2 - x + c$ has a period-3 orbit and thus proved that it is chaotic.

REFERENCES

[1]. Cook P. A., *Nonlinear Dynamical Systems* (Prentice-Hall International (UK) Ltd.1986.)

[2]. J. Banks, J. Brooks, G. Cairns, G. Davis, P. Stacey, *On Devaney's Definition of Chaos*, American Mathematical Monthly, Vol. 99,No.4 (Apr.,1992), 332-334.

[3]. Kathleen T Alligood, Tim D. Sauer, James A. Yorke, *Chaos an Introduction to Dynamical Systems* (Springer-Verlag New York, Inc.)

[4]. Devaney, Robert L., *An introduction to Chaotic Dynamical System* (Cambridge, M A : Persuse Books Publishing, 1988).

[5]. Edward R. Scheinerman, *Invitation to Dynamical Systems*.

[6]. Tien-Yien Li and James A. Yorke, *Period three implies chaos* American Mathematical Monthly, Vol. 82, No. 10, (Dec. 1975) pp 985-992.

[7]. Mr. Kulkarni P. R., Dr. Borkar V. C., *Chaos in the Dynamics of the Family of Mappings* $f_c(x) = x^2 - x + c$, IOSR Journal of Mathematics(IOSR-JM) Volume 10, Issue 4, Version IV (Jul.-Aug. 2014), pp 108-116.

[8]. Chyi-Lung Lin and Mon-Ling Shei, *Logistic Map* $f(x)=\mu x(1-x)$ *is Topologically Conjugate to the Map* $f(x)=(2-\mu)x(1-x)$, Tamkang Journal of Science and Engineering, Vol. 10, No. 1, pp 89-94 (2007).

[9]. Kulkarni P. R. and Borkar V. C., *Topological conjugacy and the Chaotic Nature of the Family of Mappings* $f_c(x) = x^2 - x + c$, International Journal of Scientific and Innovative Mathematical Research (IJSIMR), Volume 2, Issue 11, November 2014 PP 868-875.