International Journal of Mathematics Trends and Technology
ISSN: 2231 – 5373 /doi:10.14445/22315373/IJMTT-V66I12P514
Volume 66 Issue 12, 91-105, December, 2020

Stability and Hopf Bifurcation of Simplified Fluid Flow Model for Wireless Network with PD Control

Na Han [#1], Yanhui Zhai[*2]

[1,2]*School of mathematical science, TianGong University, Tianjin 300387, China*

***Abstract* — *In this paper, based on the control and bifurcation theory, a PD controller is proposed to control the Hopf bifurcation of the fluid flow model in the wireless network congestion control system. First, communication delay is selected as a bifurcation parameter to obtain the critical value of communication delay that keeps the original system and the controlled system stable. When the delay value exceeds the critical value, the system will lose stability at the equilibrium point and generate Hopf bifurcation. It is found that the addition of PD controller can effectively delay the generation of Hopf bifurcation, increase the critical value of bifurcation parameters, and expand the stability region. Besides, the direction of the Hopf bifurcation and the stability of the bifurcating periodic solutions are studied by using the center manifold theorem and the normal form theory. At last, some numerical simulation results with mathematical software are confirmed that the feasibility of the theoretical analysis.*

Keywords — *Center manifold theorem, Hopf bifurcation, Normal form theory, PD controller, Stability*

I. INTRODUCTION

In recent years, with the popularization and application of wireless network technology, people pay more and more attention to the research of wireless network congestion control. A large number of experiments show that the network congestion control system has nonlinear characteristics, and the system is in an unstable state under abnormal conditions with bifurcation, chaos and other dynamic characteristics [1]. And congestion may directly lead to higher packet loss rate the increase of end-to-end delay, and even the crash of the whole system [2-4]. At present, in order to ensure the stable operation of the network, using bifurcation theory to solve the Internet bifurcation problem has become a research hotspot for scholars [5-8]. Literature [9-11] studies the Hopf bifurcation problem of wireless network congestion control model, and selects communication delay as bifurcation parameter. When the delay exceeds the critical value, the system loses stability and Hopf bifurcation occurs, thus reducing the performance of the system. In the face of these bad behaviors, the wireless network must be controlled to delay Hopf bifurcation so as to improve the stability of the system. For this reason, many effective control methods have been proposed, such as delay feedback control [12-13], dynamic delay feedback control [14], state feedback control [15], mixed control [16-17], etc..In this paper, the proportional differential controller is applied to the wireless network congestion control system to control the Hopf bifurcation of the system.

II. MODEL BUILDING

According to literature [10], a simplified fluid flow model for wireless network congestion control is presented

Na Han, School of mathematical science, Tiangong University, Tianjin 300387, Tianjin, PR China

Yan-hui Zhai, School of mathematical science, Tiangong University, Tianjin 300387, Tianjin, PR China

$$\begin{cases}\dot{W}(t) = \frac{1}{\mathrm{R}} - \frac{(1+P_{dl})W^2(t) - 2P_{dl}W(t)}{2R}P(t-R),\\ \dot{q}(t) = \frac{N(1-P_{ul})}{R}W(t) - C,\end{cases} \tag{1}$$

where $W(t)$ represents the average TCP window size (packets), $q(t)$ is the average queue length (packets), $p(t)$ is the packet identification probability function, N is the number of TCP connections, C is the queue bandwidth (packet·s^{-1}), and R is the round trip time (seconds). When the queue delay is much less than the transmission delay, it can be assumed that $p(t)$=$Kq(t)$. P_{ul} and P_{dl} represent the packet loss probability of downlink and uplink channel respectively and are assumed to be constant.

First, let the equilibrium point of model (1) be (W_0, q_0), then it satisfies the following equation:

$$\begin{cases}\frac{1}{R} - \frac{(1+P_{dl})W_0 - 2P_{dl}W_0}{2R}Kq_0 = 0,\\ \frac{N(1-P_{ul})}{R}W_0 - C = 0.\end{cases} \tag{2}$$

Solve the above equation, and get

$$\begin{cases}W_0 = \frac{CR}{N(1-P_{ul})},\\ q_0 = \frac{2}{K(1+P_{dl}){W_0}^2 - 2KP_{dl}W}.\end{cases} \tag{3}$$

A conclusion can be drawn from the reference [10]: for the uncontrolled system, equation (1), when

$$\omega_0 = \sqrt{\frac{\sqrt{{a_c}^4 + 4b^2} - {a_c}^2}{2}},\ R_0 = \frac{1}{\omega_0}\arctan\left(\frac{a_c}{\omega_0}\right),$$

among them,

$$a_c = \frac{1}{R}(1+P_{dl})W_0Kq_0 - \frac{P_{dl}}{R}Kq_0, b_c = -\frac{N}{R}(1-P_{ul})K\left[-\frac{1}{2R}(1+P_{dl}){W_0}^2 + \frac{P_{dl}}{R}W_0\right].$$

(1) When $R < R_0$, the system is locally asymptotically stable at the equilibrium point.

(2) When $R > R_0$, The system is unstable at the equilibrium point.

(3) When $R = R_0$, The system generates Hopf bifurcation near the equilibrium point, and periodic solution appears.

In recent years, many scholars have studied the Hopf bifurcation of wireless network congestion model. In literature [18], the author studied the Hopf bifurcation problem after adding a state feedback controller to the wireless network congestion model. In literature [21], the author studied the network congestion model by adding a delay feedback controller.

Inspired by the above studies, this paper aims to delay the generation of Hopf branch, proportional differential controller (PD) is added to the wireless network congestion model. In order to add PD controller (PD) to system (1), the general rule of PD control is first introduced. PD control refers to proportional and differential control, and its rule expression is as follows:

$$u(t) = k_p e(t) + k_d \frac{d}{dt}e(t), \tag{4}$$

where k_p represents the proportional control parameter, k_d belongs to the differential control parameter, and $e(t)$ is the difference between the real-time state variable and the state equilibrium point.

According to Equations (1) and (2), the controlled system with PD controller is obtained:

$$\begin{cases}\dot{W}(t)=\frac{1}{\mathrm{R}}-\frac{(1+P_{dl})W^2(t)-2P_{dl}W(t)}{2R}Kq(t-R)+k_p(W(t)-W_0)+k_d\frac{d}{dt}(W(t)-W_0),\\ \dot{q}(t)=\frac{N(1-P_{ul})}{R}W(t)-C.\end{cases}$$

For the convenience of writing, write down $\alpha=P_{dl},\beta=P_{ul},R=\tau$, then the above model is further rewritten as

$$\begin{cases}\dot{W}(t)=\frac{1}{1-k_d}\left[\frac{1}{\tau}-\frac{(1+\alpha)W^2(t)-2\alpha W(t)}{2\tau}Kq(t-\tau)\right]+\frac{k_p}{1-k_d}(W(t)-W_0),\\ \dot{q}(t)=\frac{N(1-\beta)}{R}W(t)-C.\end{cases} \quad (5)$$

III. STABPLPTY AND LOCAL HOPF BIFURCATION ANALYSIS

Obviously, the equilibrium point of the controlled system (5) is the same as that of the original system (1), which means that the structure of the original system will not be changed after PD controller is added.

Let $Y_1(t)=W(t)-W_0$, and $Y_2(t)=q(t)-q_0$. After linearizing the controlled system (3) at the equilibrium point, the linearization equation is

$$\begin{cases}\dot{Y}_1(t)=a_1Y_1(t)+a_2Y_2(t-\tau),\\ \dot{Y}_2(t)=b_1Y_1(t).\end{cases} \quad (6)$$

Among them

$$a_1=-\frac{K(1+\alpha)}{\tau(1-k_d)}W_0q_0+\frac{K\alpha}{\tau(1-k_d)}q_0+\frac{k_q}{1-k_d},\ a_1=-\frac{K(1+\alpha)}{2\tau(1-k_d)}{W_0}^2+\frac{K\alpha}{\tau(1-k_d)}W_0$$

The characteristic equation of the system (6) is

$$\lambda^2-a_1\lambda-a_2b_1e^{-\lambda\tau}=0. \quad (7)$$

Lemma 1. when $\tau<\tau_0$, the characteristic roots of equation (7) have a negative real part.

Proof. Using quadratic approximation $e^{-\lambda\tau}=1-\lambda d+\frac{\lambda^2d^2}{2}$, the above equation becomes

$$(1-\frac{a_1b_1\tau}{2})\lambda^2+(a_2b_1\tau-a_1)\lambda-a_2b_1=0. \quad (8)$$

Routh-Hurwitz stability criterion shows that the closed-loop system is stable if and only if all values are greater than zero, that is, the following coefficient conditions are satisfied

$$1-\frac{a_1b_1\tau}{2}>0, a_2b_1\tau-a_1>0, a_2b_1>0$$

that is

$$a_2b_1<0\ \ ,\ a_1<0.$$

As τ gradually approaches zero, equation (7) has no non-negative real part roots, and the system (5) is stable; When τ increases gradually, as long as τ is small enough, the system (5) is still stable with critical delay τ_0, so that when $\tau\in(0,\ \tau_0)$, $\mathrm{Re}(\lambda)<0$ in equation (7), the proof is done.

Lemma 2. If $\omega_0\tau_0<\frac{\pi}{2}$ is true, then when $\tau=\tau_0$, equation (7) has a pair of pure virtual roots $\lambda=\pm i\omega_0$, where

$$\omega_0=\sqrt{\frac{-{a_1}^2+\sqrt{{a_1}^4+4{a_2}^2{b_1}^2}}{2}}, \quad (9)$$

$$\tau_0 = \frac{1}{\omega_0}\arctan(-\frac{a_1}{\omega_0}). \tag{10}$$

Proof. First we assume that $\lambda = i\omega(\omega > 0)$ is a root of the characteristic equation (6), then it satisfies the following equation

$$-\omega^2 - i\omega a_1 - b_1 a_2 e^{-i\omega\tau} = 0, \tag{11}$$

that is

$$-\omega^2 - i\omega a_1 - b_1 a_2(\cos\omega\tau - i\sin\omega\tau) = 0. \tag{12}$$

The separation of the real and imaginary parts, it follows

$$\begin{cases} -\omega^2 - b_1 a_2 \cos\omega\tau = 0, \\ -a_1\omega + b_1 a_2 \sin\omega\tau = 0. \end{cases} \tag{13}$$

From (10) we obtain

$$\omega^4 + a_1{}^2\omega^2 = b_1{}^2 a_2{}^2. \tag{14}$$

So, we can get

$$\omega = \sqrt{\frac{-a_1{}^2 + \sqrt{a_1{}^4 + 4a_2{}^2 b_1{}^2}}{2}},$$

$$\tau_0 = \frac{1}{\omega_0}\arctan(-\frac{a_1}{\omega_0}).$$

For $k = 0$, then

$$\omega_0 = \sqrt{\frac{-a_1{}^2 + \sqrt{a_1{}^4 + 4a_2{}^2 b_1{}^2}}{2}},$$

$$\tau_0 = \frac{1}{\omega_0}\arctan(-\frac{a_1}{\omega_0}).$$

That is, when $\tau = \tau_0$, $\lambda = \pm i\omega_0$ is a pair of pure virtual roots of the characteristic equation (6), the proof is done.

Lemma 3. If $\lambda(\tau) = \alpha(\tau) + i\omega(\tau)$ is the root of the characteristic equation (8) and the conditions $\alpha(\tau_0) = 0$ and $\omega(\tau_0) = \omega_0$ are satisfied, then the transversely condition $\mathrm{Re}(\frac{d\lambda}{d\tau_0})^{-1}|_{\tau=\tau_0} > 0$ is true.

Proof. By differentiating both sides of equation (7) with regard to τ and applying the implicit function theorem, we have :

$$\begin{aligned} \frac{d\lambda}{dR}|_{\tau=\tau_0} &= \frac{-\lambda a_2 b_1 e^{-\lambda\tau}}{2\lambda - a_1 + a_2 b_1 \tau e^{-\lambda\tau}}|_{\tau=\tau_0} \\ &= \frac{-ia_2 b_1 \omega_0(\cos\omega_0\tau - i\sin\omega_0\tau)}{2i\omega_0 - a_1 + a_2 b_1 \tau(\cos\omega_0\tau - i\sin\omega_0\tau)} \\ &= \frac{-ia_2 b_1\omega_0\cos\omega_0\tau - a_2 b_1\omega_0\sin\omega_0\tau}{i(2\omega_0 - \tau\sin\omega_0\tau) + (a_2 b_1\tau\cos\omega_0\tau - a_1)}, \end{aligned}$$

So

$$\mathrm{Re}(\frac{d\lambda}{dR})|_{\tau=\tau_0} = \frac{-2\omega_0{}^2 a_2 b_1\cos\omega_0\tau + a_1 a_2 b_1\omega_0\sin\omega_0\tau}{(2\omega_0 - \tau\sin\omega_0\tau)^2 + (a_2 b_1\tau\cos\omega_0\tau - a_1)^2}.$$

Because $a_2 b_1 < 0$, $a_1 < 0$, we know

$$\mathrm{Re}(\frac{d\lambda}{d\tau})|_{\tau=\tau_0} > 0.$$

Obvious

$$sign\,\mathrm{Re}(\frac{d\lambda}{d\tau_0})^{-1}|_{\tau=\tau_0} = sign\,\mathrm{Re}(\frac{d\lambda}{d\tau_0})|_{\tau=\tau_0} > 0.$$

The proof is done.

Lemma 4. When $\tau > \tau_0$, equation (7) has at least one root with a positive real part.

According to the above lemma and the Hopf branch theorem of delay differential equation in reference [22], we can reach the following conclusion.

Theorem 1. For the controlled system (5), the following conclusion holds:

(1) When $\tau < \tau_0$, the controlled system is asymptotically and uniformly stable near the equilibrium point (W_0, q_0);

(2) When $\tau = \tau_0$, the controlled system generates Hopf branch at the equilibrium point (W_0, q_0);

(3) When $\tau > \tau_0$, the controlled system is unstable at the equilibrium point (W_0, q_0).

IV. DIRECTION AND STABILITY OF THE HOPF BIFURCATION

In the analysis in the previous section, we have obtained the conditions for the system to generate Hopf branch. In this section, we will use the normative theory and the central manifold theorem in literature [19-20] to study the characteristics of model (5), such as the direction of generating Hopf branch and stability of periodic solution of branch.

First we consider the Taylor expansion of model (5) at equilibrium (W_0, q_0):

$$\begin{cases}\dot{Y}_1(t) = a_1Y_1(t) + a_2Y_2(t-\tau) + a_3{Y_1}^2(t) + a_4Y_1(t)Y_2(t-\tau) + a_5{Y_1}^2(t)Y_2(t-\tau), \\ \dot{Y}_2(t) = b_1Y_1(t).\end{cases} \tag{15}$$

Among them

$$a_1 = -\frac{K(1+\alpha)}{\tau(1-k_d)}W_0q_0 + \frac{K\alpha}{\tau(1-k_d)}q_0 + \frac{k_q}{1-k_d},\ a_2 = -\frac{K(1+\alpha)}{2\tau(1-k_d)}{W_0}^2 + \frac{K\alpha}{\tau(1-k_d)}W_0,\ a_3 = -\frac{K(1+\alpha)}{2\tau(1-k_d)}q_0,$$

$$a_4 = -\frac{K(1+\alpha)\omega_0 - 2K\alpha}{2\tau(1-k_d)},\ a_5 = -\frac{K(1+\alpha)}{6\tau(1-k_d)},\ b_1 = \frac{N}{\tau}(1-\beta),$$

For the sake of research, let $\tau = \tau_0 + \mu$, $u(t) = (Y_1(t), Y_2(t))^T$ and $u_t(\theta) = u(t+\theta)$, $\theta \in [-\tau, 0)$, then $\mu = 0$ means that the model (5) generates Hopf branch at τ_0. Model (5) can be expressed as the following functional differential equation:

$$\dot{u}(t) = L_\mu + F(u_t, \mu), \tag{16}$$

there

$$L_\mu\varphi = B_1\varphi(0) + B_2\varphi(-\tau). \tag{17}$$

And

$$F(\mu,\varphi) = \begin{pmatrix} a_3{\varphi_1}^2(t) + a_4\varphi_1(t)\varphi_2(t-\tau) + a_5{\varphi_1}^2(t)\varphi_2(t-\tau) \\ 0 \end{pmatrix}. \tag{18}$$

Where L_μ is a bounded operator of $C([-\tau,0],\tau^2) \to \tau^2$ and $\varphi(\theta) = (\varphi_1(\theta), \varphi_2(\theta))^T \in C[-\tau, 0]$.

$$B_1 = \begin{pmatrix} a_1 & 0 \\ b_1 & 0 \end{pmatrix}, B_2 = \begin{pmatrix} 0 & a_2 \\ 0 & 0 \end{pmatrix},$$

By the Riesz representation theorem, there exists a bounded variation function $\eta(\theta,\mu):[-\tau,0]\to R^{2\times 2}$, such that

$$L_\mu\varphi=\int_{-\tau}^{0}d\eta(\theta,\mu)\varphi(\theta),\varphi\in C. \tag{19}$$

In fact, we can choose

$$\eta(\theta,\mu)=B_1\delta(\theta)+B_2\delta(\theta+\tau). \tag{20}$$

Here $\delta(\theta)$ is a Delta function. The operators A and R are defined as follows:

$$A(\mu)\varphi(\theta)=\begin{cases}\dfrac{d(\varphi(\theta))}{d\theta}, & \theta\in[-\tau,0),\\ \displaystyle\int_{-\tau}^{0}d(\eta(\theta,\mu)\varphi(\theta)), & \theta=0.\end{cases} \tag{21}$$

$$R(\mu)\varphi(\theta)=\begin{cases}0, & \theta\in[-\tau,0),\\ F(\mu,\varphi), & \theta=0.\end{cases} \tag{22}$$

Then equation (16) can be rewritten into the following form:

$$\dot{u}_t=A(\mu)u_t+R(\mu)u_t. \tag{23}$$

For $\psi\in C'[0,1]$, we define the adjoint operator $A^*(0)$ of $A(0)$ as

$$A^*(\mu)\psi(s)=\begin{cases}-\dfrac{d\psi(s)}{ds}, & s\in(0,\tau],\\ \displaystyle\int_{-\tau}^{0}d(\eta^T(s,0)\psi(-s)), & s=0.\end{cases} \tag{24}$$

For $\varphi(\theta)\in C[-\tau,0)$ and $\psi\in C[0,\tau]$, we define a Bilinear form

$$<\psi,\varphi>=\overline{\psi}^T(0)\varphi(0)-\int_{\theta=-\tau}^{0}\int_{\xi=0}^{\theta}\overline{\psi}(\xi-\theta)[d\eta(\theta)]\varphi(\xi)d\xi, \tag{25}$$

where $\eta(\theta)=\eta(\theta,0)$.

Lemma 5. The eigenvectors $q(\theta)=Ve^{i\omega_0\theta}$ and $q^*(s)=DV^*e^{-i\omega_0\theta}$ are respectively the eigenvectors corresponding to the eigenvalues $i\omega_0$ and $-i\omega_0$ of $A(0)$ and $A^*(0)$, and

$$<q^*,q>=1,\ <q^*,\bar{q}>=0,$$

where

$$V=(1,\frac{b_1}{i\omega_0})^T,V^*=(-\frac{b_1}{a_1+i\omega_0},1)^T,\overline{D}=[\overline{V}^{*T}V-\tau_0e^{i\omega_0\tau}\overline{V}^{*T}B_2V].$$

Proof. Since $\pm i\omega_0$ is the eigen value of $A(0)$, they are also eigen values of $A^*(0)$. In order to determine the standard form of the operator τ_0, let's assume that $q(\theta)$ and $q^*(s)$ are eigen vectors corresponding to $A(0)$ and $A^*(0)$'s eigen values $i\omega_0$ and $-i\omega_0$, respectively.

$$\begin{cases}A(0)q(\theta)=i\omega_0q(\theta),\\ A^*(0)q^*(s)=i\omega_0q^*(s).\end{cases} \tag{26}$$

From (19) and (21), (26) can be written as

$$\frac{dq(\theta)}{d\theta} = i\omega_0 q(\theta), \qquad \theta \in [-\tau, 0).$$
$$L(0)q(0) = i\omega_0 q(0), \qquad \theta = 0. \tag{27}$$

Therefore,

$$q(\theta) = q(0)e^{i\omega_0 \theta}, \quad \theta \in [-\tau, 0].$$

$q(0) = (q_1(0), q_2(0))^T \in C^2$ is a constant vector, which can be obtained from (17) and (26)

$$B_1 q(0) + B_2 e^{-i\omega_0 \tau_0} q(0) = i\omega_0 I q(0),$$

we get

$$q(0) = \begin{pmatrix} 1 \\ \rho_1 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{b_1}{i\omega_0} \end{pmatrix},$$

we make

$$V = q(0)^T = (1, \frac{b_1}{i\omega_0}),$$

then

$$q(\theta) = Ve^{i\omega_0 \theta}.$$

For non-zero vectors $q^*(s), s \in [0, \tau]$, we have

$$B_1^T q^*(0) + B_2^T e^{-i\omega_0 \tau_0} q^*(0) + i\omega_0 I q^*(0) = 0 .$$

Similarly

$$q^*(0) = \begin{pmatrix} \rho_2 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{b_1}{i\omega_0 + a_1} \\ 1 \end{pmatrix}.$$

We make

$$V^* = q_1^*(0)^T = (-\frac{b_1}{i\omega_0 + a_1}, 1).$$

Then $q^*(s) = V^* e^{-i\omega_0 s}$, we make $q^*(s) = DV^* e^{-i\omega_0 s}$.

Now let's prove that $< q^*, q >= 1$ and $< q^*, q >= 1$, from equation (25), we get

$$\begin{aligned} \langle q^*, q \rangle &= \overline{q^*}^T q(0) - \int_{\theta=-\tau_0}^{0} \int_{\xi=0}^{\theta} \overline{q^*}^T (\xi - \theta) d\eta(\theta) q(\xi) d\xi \\ &= \overline{D}[\overline{V^*}^T V - \int_{\theta=-\tau_0}^{0} \int_{\xi=0}^{\theta} \overline{V^*}^T e^{-i\omega_0(\xi-\theta)} d\eta(\theta) V e^{i\omega_0 \xi} d\xi] \\ &= \overline{D}[\overline{V^*}^T V - \int_{\theta=-\tau_0}^{0} \overline{V^*}^T [d\eta(\theta)] \theta e^{i\omega_0 \theta} V] \\ &= \overline{D}[\overline{V^*}^T V - \tau_0 e^{-i\omega_0 \tau_0} \overline{V^*}^T B_2 V]. \end{aligned} \tag{28}$$

So, let $\overline{D}=[\overline{V^*}^T V-\tau_0 e^{-i\omega_0\tau_0}\overline{V^*}^T B_2 V]^{-1}$, we can obtain $<q^*,q>=1$.

Since $\langle\psi,A\varphi\rangle=\langle A^*\psi,\varphi\rangle$, we have

$$-i\omega_0\langle q^*,\overline{q}\rangle=\langle q^*,A\overline{q}\rangle=\langle A^*q^*,\overline{q}\rangle=\langle -i\omega_0 q^*,\overline{q}\rangle=i\omega_0\langle q^*,\overline{q}\rangle. \tag{29}$$

Therefore $<q^*,\overline{q}>=0$, this completes the proof.

Next, we will use the method proposed by Hassard *et al.* to construct coordinates on the central epidemic C_0 at $W_{20}(\theta)$. Define

$$z(t)=<q^*,u_t>, \tag{30}$$

and

$$W(t,\theta)=u_t(\theta)-2\operatorname{Re}\{z(t)q(\theta)\}. \tag{31}$$

On the center manifold C_0, we have

$$W(t,\theta)=W(z(t),\overline{z}(t),\theta). \tag{32}$$

Where $W(z,\overline{z},\theta)=W_{20}(\theta)\frac{z^2}{2}+W_{11}(\theta)z\overline{z}+W_{02}(\theta)\frac{\overline{z}^2}{2}+\cdots$.

For central manifold C_0, z and $\overline{z}$ represent the local coordinates of the central epidemic in the directions of q and $\overline{q^*}$ respectively. If u_t is real, then W is real, and we're only looking at the real solution here, Since $\mu=0$, it is easy to see that

$$\begin{aligned}\dot{z}(t)&=\langle q^*,\dot{\mu}_t\rangle=\langle q^*,(A(0)+R(0))\mu_t\rangle\\&=<q^*,A\mu_t>+<q^*,R\mu_t>\\&=i\omega_0 z+\overline{q}^{*T}f_0(z,\overline{z}).\end{aligned} \tag{33}$$

Abbreviate (33) as follows

$$\dot{z}(t)=i\omega_0 z+g(z,\overline{z}), \tag{34}$$

Where

$$g(z,\overline{z})=g_{20}\frac{z^2}{2}+g_{11}z\overline{z}+g_{02}\frac{\overline{z}^2}{2}+\cdots, \tag{35}$$

from (23) and (35), we have

$$\dot{W}=\dot{u}_t-\dot{z}q-\dot{\overline{z}}\dot{q}=\begin{cases}AW-2\operatorname{Re}\overline{q^*}^T(0)f_0(z,\overline{z})q(\theta), & \theta\in[-\tau_0,0)\\ AW-2\operatorname{Re}\{\overline{q^*}^T(0)f_0(z,\overline{z})q(\theta)\}+f_0(z,\overline{z}), & \theta=0\end{cases} \tag{36}$$

The above equation can be rewritten as

$$\dot{W}=AW+H(z,\overline{z},\theta). \tag{37}$$

Where

$$H(z,\overline{z},\theta)=H_{20}(\theta)\frac{z^2}{2}+H_{11}(\theta)z\overline{z}+H_{02}(\theta)\frac{\overline{z}^2}{2}+\cdots. \tag{38}$$

On the other hand, on the central manifold C_0, there is

$$\dot{W} = W_z \dot{z} + W_{\bar{z}} \dot{\bar{z}}. \tag{39}$$

Substitute Equations (33) and (35) for W_z and $\dot{z}$ into (39), respectively, we can get another expression of $\dot{W}$

$$\dot{W} = i\omega_0 W_{20}(\theta) z^2 - i\omega_0 W_{02}(\theta) \bar{z}^2 + \cdots. \tag{40}$$

Comparing the coefficients of the above equation with those of (38) and (41), we get

$$\begin{cases} (A - 2i\omega_0) W_{20}(\theta) = -H_{20}(\theta), \\ A W_{11}(\theta) = -H_{11}(\theta), \\ (A + 2i\omega_0) W_{02}(\theta) = -H_{02}(\theta). \end{cases} \tag{41}$$

Notice that $u_t(\theta) = W(z(t), \bar{z}(t), \theta) + zq + \overline{zq}$ and $q(\theta) = (1, \rho_1)^T e^{i\omega_0\theta}$, we have

$$u_t = \begin{pmatrix} x_1(t+\theta) \\ x_2(t+\theta) \end{pmatrix} = \begin{pmatrix} W^{(1)}(z, \bar{z}, \theta) \\ W^{(2)}(z, \bar{z}, \theta) \end{pmatrix} + z \begin{pmatrix} 1 \\ \rho_1 \end{pmatrix} e^{i\omega_0\theta} + \bar{z} \begin{pmatrix} 1 \\ \overline{\rho_1} \end{pmatrix} e^{-i\omega_0\theta}. \tag{42}$$

Therefore, we can obtain

$$Y_1(t+\theta) = ze^{i\omega_0\theta} + \bar{z}e^{-i\omega_0\theta} + W_{20}{}^{(1)}(\theta)\frac{z^2}{2} + W_{11}{}^{(1)}(\theta)z\bar{z} + W_{02}{}^{(1)}(\theta)\frac{\bar{z}^2}{2} + \cdots,$$
$$Y_2(t+\theta) = z\rho_1 e^{i\omega_0\theta} + \bar{z}\overline{\rho_1}e^{-i\omega_0\theta} + W_{20}{}^{(2)}(\theta)\frac{z^2}{2} + W_{11}{}^{(2)}(\theta)z\bar{z} + W_{02}{}^{(2)}(\theta)\frac{\bar{z}^2}{2} + \cdots.$$

Therefore, we have

$$\varphi_1(0) = z + \bar{z} + W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0)z\bar{z} + W_{02}^{(1)}(0)\frac{\bar{z}^2}{2} + \cdots,$$

$$\varphi_1{}^2(0) = z^2 + 2z\bar{z} + \bar{z}^2 + [2W_{11}{}^{(1)}(0) + W_{20}{}^{(1)}(0)]z^2\bar{z} + \cdots,$$

$$\varphi_2(-\tau_0) = z\rho_1 e^{-i\omega_0\tau_0} + \bar{z}\bar{\rho}_1 e^{i\omega_0\tau_0} + W_{11}{}^{(2)}(-\tau_0)z\bar{z} + W_{20}{}^{(2)}(-\tau_0)\frac{z^2}{2} + W_{02}{}^{(2)}(-\tau_0)\frac{\bar{z}^2}{2} + \cdots,$$
$$\begin{aligned}\varphi_1(0)\varphi_2(-\tau_0) &= \rho_1 e^{-i\omega_0\tau_0} z^2 + (\bar{\rho}_1 e^{i\omega_0\tau_0} + \rho_1 e^{-i\omega_0\tau_0})z\bar{z} + \bar{\rho}_1 e^{i\omega_0\tau_0}\bar{z}^2 \\ &\quad + [W_{11}{}^{(2)}(-\tau_0) + \frac{1}{2}W_{20}{}^{(2)}(-\tau_0) + \frac{1}{2}\bar{\rho}_1 e^{i\omega_0\tau_0}W_{20}{}^{(1)}(0) + \rho_1 e^{-i\omega_0\tau_0}W_{11}{}^{(1)}(0)]z^2\bar{z} + \cdots,\end{aligned}$$

$$\varphi_1{}^2(0)\varphi_2(-\tau_0) = (\overline{\rho_1}e^{i\omega_0\tau_0} + 2\rho_1 e^{-i\omega_0\tau_0})z^2\bar{z} + \cdots.$$

From (34) and (35), we obtain

$$f_0(z, \bar{z}) = \begin{pmatrix} K_1 z^2 + K_2 z\bar{z} + K_3\bar{z}^2 + K_4 z^2\bar{z} \\ 0 \end{pmatrix}.$$

Where

$$K_1 = a_3 + a_4\rho_1 e^{-i\omega_0\tau_0},$$

$$K_2 = 2a_3 + a_4(\bar{\rho}_1 e^{i\omega_0\tau_0} + \rho_1 e^{-i\omega_0\tau_0}),$$

$K_3 = a_3 + a_4 \bar{\rho}_1 e^{i\omega_0\tau_0}$,

$$K_4 = a_3[2W_{11}^{(1)}(0) + W_{20}^{(1)}(0)] + a_4[W_{11}^{(2)}(-\tau_0) + \frac{1}{2}W_{20}^{(2)}(-\tau_0) + \frac{1}{2}\bar{\rho}_1 e^{i\omega_0\tau_0} W_{20}^{(1)}(0) + \bar{\rho}_1 e^{-i\omega_0\tau_0} W_{11}^{(1)}(0)]$$
$$+ a_5(\overline{\rho_1} e^{i\omega_0\tau_0} + 2\rho_1 e^{-i\omega_0\tau_0}).$$

Since $\bar{q}^{*T}(0) = \bar{D}(\bar{\rho}_2, 1)$, we obtain

$$\begin{aligned} g(z,\bar{z}) &= \bar{q}^{*T}(0) f_0(z,\bar{z}) \\ &= \bar{D}(\bar{\rho}_2, 1)\begin{pmatrix} K_1 z^2 + K_2 z\bar{z} + K_3 \bar{z}^2 + K_4 z^2 \bar{z} \\ 0 \end{pmatrix} \\ &= \bar{D}\bar{\rho}_2 (K_1 z^2 + K_2 z\bar{z} + K_3 \bar{z}^2 + K_4 z^2 \bar{z}). \end{aligned}$$

Comparing the coefficients of the above equation with those in (35), we have

$$\begin{aligned} g_{20} &= 2\bar{D}\bar{\rho}_2 K_1, \quad g_{11} = \bar{D}\bar{\rho}_2 K_2, \\ g_{02} &= 2\bar{D}\bar{\rho}_2 K_3, \quad g_{21} = 2\bar{D}\bar{\rho}_2 K_4. \end{aligned} \tag{43}$$

In order to get the value of g_{21}, we also need to compute $W_{20}(\theta)$ and $W_{11}(\theta)$. For $\theta \in [-\tau_0, 0)$, we have

$$\begin{aligned} H(z,\bar{z},\theta) &= -2\,\mathrm{Re}[\bar{q}^{*T}(0) f_0(z,\bar{z}) q(\theta)] \\ &= -(g_{20}\frac{z^2}{2} + g_{11} z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + \cdots) q(\theta) \\ &\quad -(\bar{g}_{20}\frac{\bar{z}^2}{2} + \bar{g}_{11} z\bar{z} + \bar{g}_{02}\frac{z^2}{2} + \cdots)\bar{q}(\theta). \end{aligned} \tag{44}$$

Comparing the coefficients of the above equation with those in (38), we have

$$\begin{aligned} H_{20}(\theta) &= -g_{20} q(\theta) - \bar{g}_{02}\bar{q}(\theta), \\ H_{11}(\theta) &= -g_{11} q(\theta) - \bar{g}_{11}\bar{q}(\theta). \end{aligned} \tag{45}$$

When $\theta = 0$, we have

$$\begin{aligned} H(z,\bar{z},\theta) &= -2\,\mathrm{Re}[\bar{q}^{*T}(0) f_0(z,\bar{z}) q(0)] + f_0(z,\bar{z}) \\ &= -(g_{20}\frac{z^2}{2} + g_{11} z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + \cdots) q(0) \\ &\quad -(\bar{g}_{20}\frac{\bar{z}^2}{2} + \bar{g}_{11} z\bar{z} + \bar{g}_{02}\frac{z^2}{2} + \cdots)\bar{q}(0) + \begin{pmatrix} K_1 z^2 + K_2 z\bar{z} + K_3 \bar{z}^2 + K_4 z^2 \bar{z} \\ 0 \end{pmatrix}. \end{aligned}$$

Comparing the coefficients with (44), we have

$$\begin{aligned} H_{20}(0) &= -g_{20} q(0) - \bar{g}_{20}\bar{q}(0) + 2\begin{pmatrix} K_1 \\ 0 \end{pmatrix}, \\ H_{11}(0) &= -g_{11} q(0) - \bar{g}_{11}\bar{q}(0) + \begin{pmatrix} K_2 \\ 0 \end{pmatrix}. \end{aligned} \tag{46}$$

Using (46) and (50), we obtain

$$
\begin{aligned}
W_{20}(\theta) &= \frac{ig_{20}}{\omega_0} q(0)e^{i\omega_0\theta} + \frac{i\overline{g}_{02}}{3\omega_0}\overline{q}(0)e^{-i\omega_0\theta} + E_1 e^{2i\omega_0\theta}, \\
W_{11}(\theta) &= -\frac{ig_{11}}{\omega_0} q(0)e^{i\omega_0\theta} + \frac{i\overline{g}_{11}}{\omega_0}\overline{q}(0)e^{-i\omega_0\theta} + E_2.
\end{aligned} \tag{47}
$$

Where $E_1 = (E_1^{(1)}, E_1^2) \in R^2, E_2 = (E_2^{(1)}, E_2^2) \in R^2$ are two two-dimensional vectors.

According to the definition of $A(0)$ and formula (41), we have

$$
\begin{aligned}
&\int_{-\tau_0}^{0} d\eta(\theta) W_{20}(\theta) = 2i\omega_0 W_{20}(0) - H_{20}(0), \\
&\int_{-\tau_0}^{0} d\eta(\theta) W_{11}(\theta) = -H_{11}(0).
\end{aligned}
$$

And

$$
\begin{aligned}
&(i\omega_0 I - \int_{-\tau_0}^{0} e^{i\omega_0\theta} d\eta(\theta))q(0) = 0, \\
&(-i\omega_0 I - \int_{-\tau_0}^{0} e^{-i\omega_0\theta} d\eta(\theta))\overline{q}(0) = 0.
\end{aligned}
$$

Hence, we can get

$$
\begin{aligned}
&(2i\omega_0 I - \int_{-\tau_0}^{0} e^{2i\omega_0\theta} d\eta(\theta))E_1 = 2\begin{pmatrix} K_1 \\ 0 \end{pmatrix}, \\
&(\int_{-\tau_0}^{0} d\eta(\theta))E_2 = -\begin{pmatrix} K_2 \\ 0 \end{pmatrix}.
\end{aligned}
$$

Therefore, we have

$$
\begin{cases}
\begin{pmatrix} i2\omega_0 - a_1 & -a_2 e^{-2i\omega_0\tau} \\ -b_1 & 2i\omega_0 \end{pmatrix}\begin{pmatrix} E_1^{(1)} \\ E_1^{(2)} \end{pmatrix} = 2\begin{pmatrix} K_1 \\ 0 \end{pmatrix}, \\
\begin{pmatrix} a_1 & a_2 \\ b_1 & 0 \end{pmatrix}\begin{pmatrix} E_2^{(1)} \\ E_2^{(2)} \end{pmatrix} = -\begin{pmatrix} K_2 \\ 0 \end{pmatrix}.
\end{cases} \tag{48}
$$

By calculation, we have

$$
\begin{cases}
E_1^{(1)} = \dfrac{i4\omega_0(a_3 + a_4\rho_1 e^{-i\omega_0\tau_0})}{-4\omega_0^{\ 2} - 2ia_1\omega_0 - a_2 b_1 e^{-2i\omega_0\tau_0}}, \\
E_1^{(2)} = \dfrac{2b_1(a_3 + a_4\rho_1 e^{-i\omega_0\tau_0})}{-4\omega_0^{\ 2} - 2ia_1\omega_0 - a_2 b_1 e^{-2i\omega_0\tau_0}},
\end{cases} \tag{49}
$$

and

$$
\begin{cases}
E_2^{(1)} = 0, \\
E_2^{(2)} = \dfrac{2a_3 + a_4(\overline{\rho}_1 e^{i\omega_0\tau_0} + \rho_1 e^{-i\omega_0\tau_0})}{a_2}.
\end{cases} \tag{50}
$$

Based on the above analysis, we next determine several important values of the properties of Hopf periodic solutions at the critical value τ_0 :

$$
\begin{aligned}
&C_1(0)=\frac{i}{2\omega_0}(g_{20}g_{11}-2|g_{11}|^2-\frac{1}{3}|g_{02}|^2)+\frac{g_{21}}{2},\\
&\mu_2=-\frac{\operatorname{Re}\{C_1(0)\}}{\operatorname{Re}\{\lambda'(0)\}},\\
&\beta_2=2\operatorname{Re}\{C_1(0)\},\\
&T_2=-\frac{\operatorname{Im}\{C_1(0)\}+\mu_2(\operatorname{Im}\{\lambda'(0)\})}{\omega_0}.
\end{aligned} \tag{51}
$$

Theorem 2. In the case of system (5), the conclusion holds:

(1) The direction of the Hopf bifurcation is determined by the parameter μ_2. If $\mu_2 > 0$ ($\mu_2 < 0$), the Hopf bifurcation is supercritical (subcritical) .

(2) The value of β_2 determines the stability of Hopf bifurcation periodic solution. If $\beta_2 < 0$ ($\beta_2 > 0$), then the branching periodic solution is asymptotically stable (unstable).

(3) The value of T_2 determines the period of the Hopf bifurcation periodic solution. If $T_2 > 0$ $(T_2 < 0)$, then the period of the periodic solution increases (decreases).

V. NUMERICAL SIMULATION

In this section, we verified the validity of the above theoretical analysis results by using mathematica, a mathematical software, for numerical simulation. In order to facilitate the comparison, we chose the same parameters as the literature:

$$N=50, K=0.001, C=1000, \alpha=\beta=0.1.$$

When $k_d = k_q = 0$, the system (5) is in the state of no control system, which can be obtained by calculation:

$$W_0 = 4.4444,\ p_0 = 0.0960,\ \tau_0 = 0.2103,\ \omega_0 = 3.0620,$$

When $\tau = 0.2 < \tau_0$ is taken and $\tau \in (0, \tau_0)$ is known, the system (1) is asymptotically stable at the equilibrium point, as shown in Fig. 1. When $\tau = 0.24 > \tau_0$, the system (1) loses stability at the equilibrium point, Hopf bifurcation occurs and the system is in the limit cycle state, as shown in Fig. 2. Next, the control effect is verified. The above parameters are still selected. By selecting an appropriate PD control coefficient of $k_d = -0.5$ and $k_q = -0.5$, when $\tau = 0.24$, the system finally stabilizes at the equilibrium point, as shown in Fig. 3. However, as τ continues to increase, as in $\tau = 0.29$, the wireless network congestion model with PD controller added still generates Hopf branch, and the system loses stability, generating limit cycle, as shown in Fig 4. Then, by selecting PD control coefficient $k_d = -5$ and $k_d = -5$, when $\tau = 0.29$, the system finally stabilizes at the equilibrium point, as shown in Fig. 5. Therefore, choosing an appropriate PD control coefficient can delay the Hopf branch.

VI. CONCLUSIONS

Based on the simplified mathematical model of wireless congestion control algorithm, this paper studies a wireless congestion control fluid flow model with PD controller. On the basis of theoretical analysis, we simply introduce the Hopf bifurcation behavior of the controlless system model. In order to delay this behavior, PD controller is added. By selecting appropriate control parameters, we can get the critical value of communication delay to keep the controlled system stable, thus effectively delaying the generation of Hopf branch.However, when the delay is large, the system will still block, or even crash.The numerical simulation results verify the correctness of the theoretical analysis. Therefore, we come to the conclusion that although the bifurcation behavior is not eliminated by PD controller, we can effectively delay the generation of Hopf branch, expand the stable interval of wireless network, and achieve better service performance of wireless network.

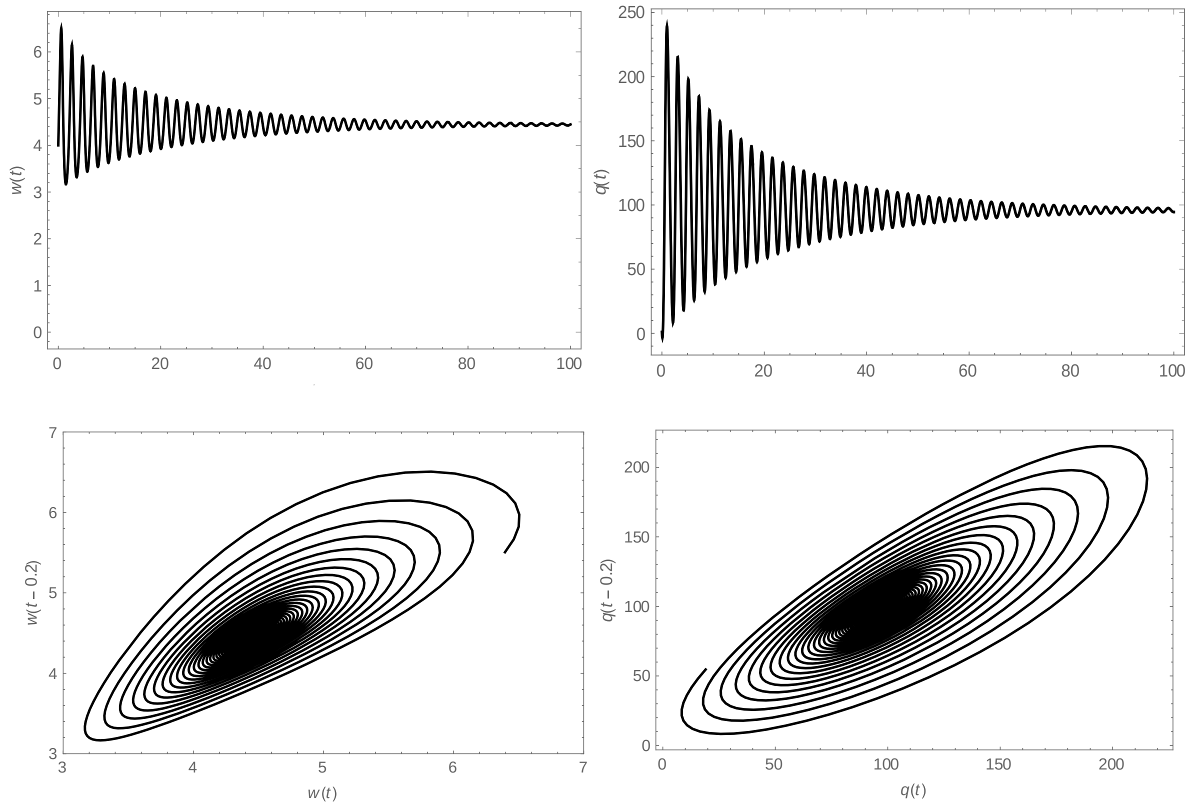

Fig. 1 State and Phase plot of $W(t)$ and $p(t)$ with $\tau = 0.2$.

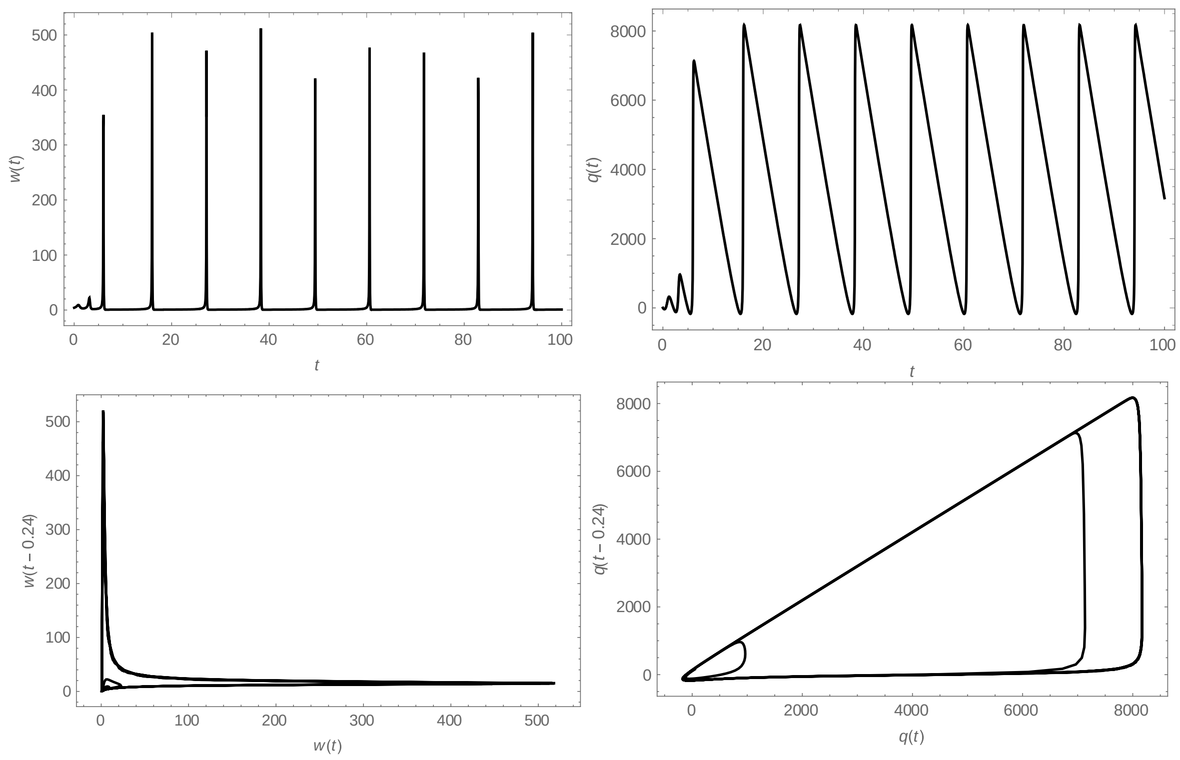

Fig. 2 State and Phase plot of $W(t)$ and $p(t)$ with $\tau = 0.24$.

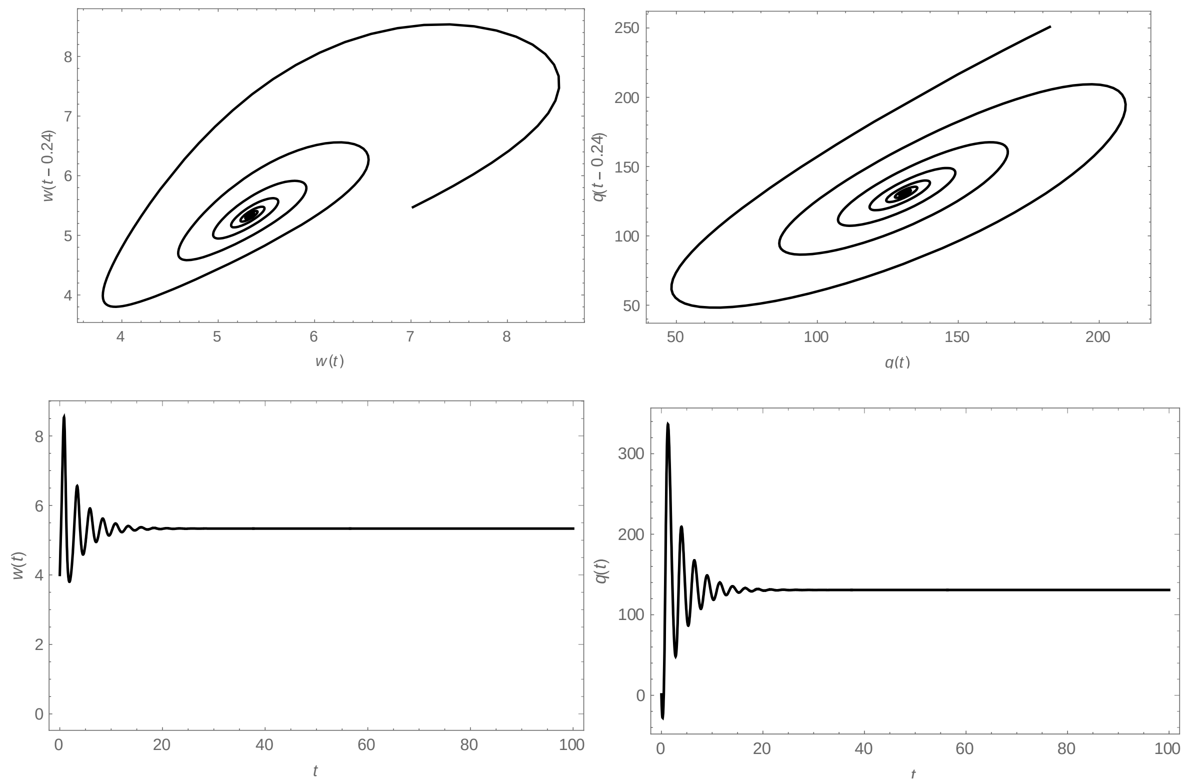

Fig. 3 State and Phase plot of $W(t)$ **and** $p(t)$ **with** $\tau = 0.24$

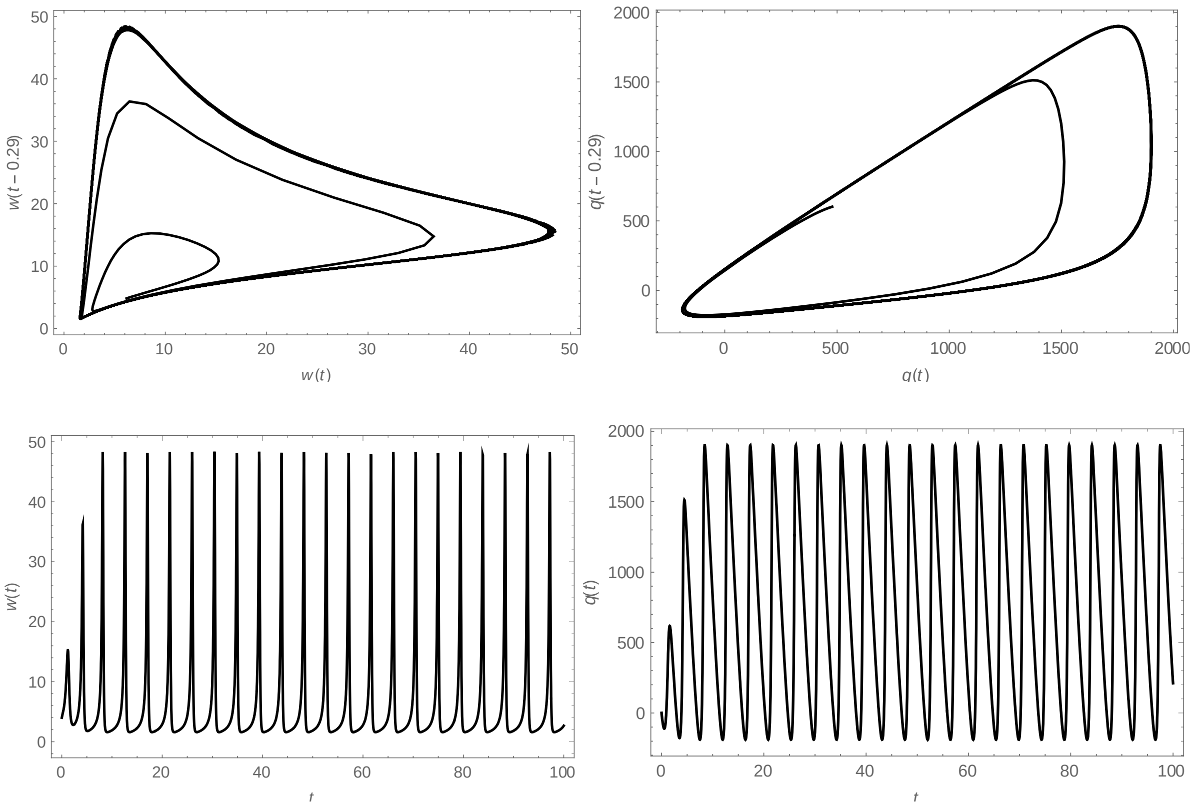

Fig. 4 State and Phase plot of $W(t)$ **and** $p(t)$ **with** $\tau = 0.29$

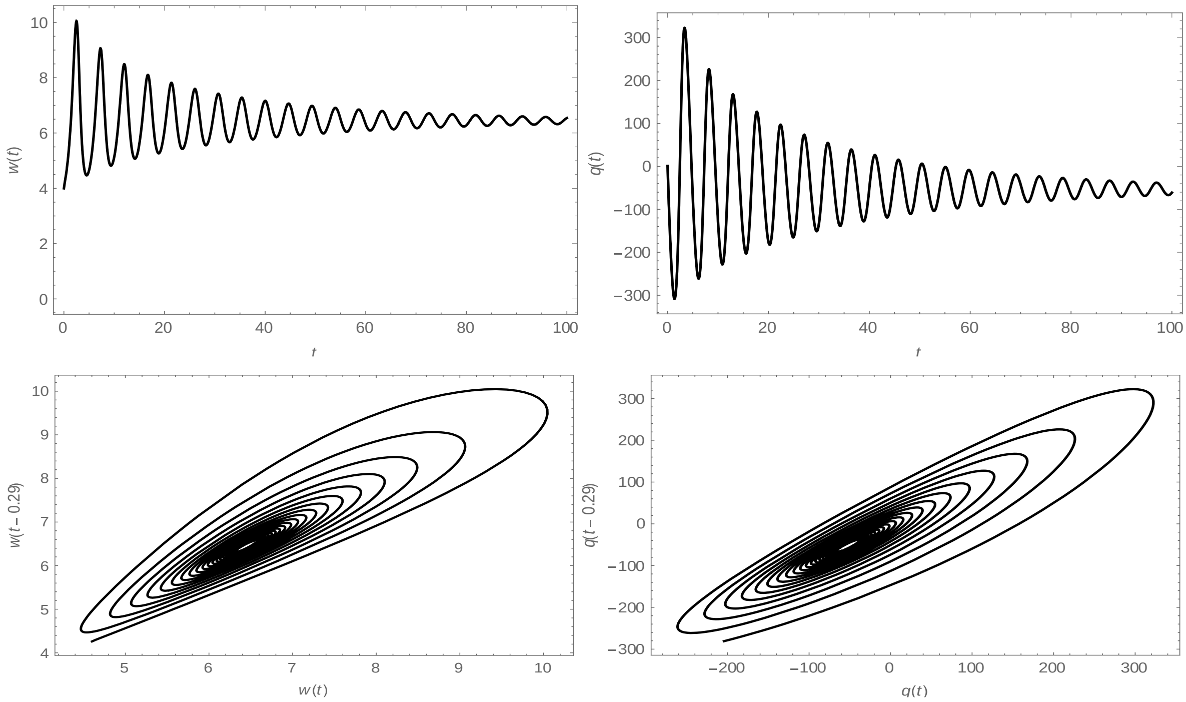

Fig. 5 State and Phase plot of $W(t)$ and $p(t)$ with $\tau = 0.29$

REFERENCES

[1] Y. L. Liu, Nonlinear Stability of Internet Congestion Control System, Shanghai: Shanghai Jiao Tong University. 2010.

[2] D. H. Zhang, Q. H. Wang, L. F. Wei, Congestion control research of Ad hoc wireless networks, Chinese Journal of Scientific Instrument. (Suppl. 1) (2008) 486-487.

[3] Zh. W. Wu, D. Ch. Yuan, K. Han, Congestion control model of TCP network based on hybrid automaton, Chinese Journal of Scientific Instrument. (Suppl. 3) (2008) 2001-2003.

[4] W. Sun, J. P. Wang, Q. Y. Li, et al, Research on MAC layer communication performance model of wireless sensor network, Journal of Electronic Measurement and Instrument. 26(2) (2012) 144-149.

[5] S. Guo, X. Liao, C. Li, Stability and Hopf bifurcation analysis in a novel congestion control model with communication delay, Nonlinear analysis:real world applications. 9(4) (2008) 1292-1309.

[6] F. Liu, Z. H. Guan, H. O. Wang, Stability and Hopf bifurcation analysis in a TCP fluid model, Nonlinear Analysis: Real World Applications. 12(1) (2011) 353-363.

[7] B. Rezaie, M.R. Jahed Motlagh, S. Khorsandi, et al, Hopf bifurcation analysis on an Internet congestion control system of arbitrary dimension with communication delay, Nonlinear Analysis: Real Word Applications. 11(5) (2010) 3842-3857.

[8] Y. L. Liu, J. Zhu, X. S. Luo, Bifurcation and stability of an improved time-delayed fluid flow model in internet congestion control, Chinese Physics B. 18 (2009) 3772-3776.

[9] T. Dong, X. F. Liao, T. W. Huang, Dynamics of a congestion control model in a wireless access network, Nonlinear Analysis: Real World Applications. 14(1) (2013) 671-683.

[10] P. Hou, Z. Q. Wang, Stability and Hopf bifurcation of a fluid-flow model for control in wireless networks, Journal of Electronics and Information. 32(4) (2010) 983-987.

[11] F. Liu, Z. H. Guan, H. O. Wang, Stability analysis and control of bifurcation in a TCP fluid flow model of wireless networks, Proceedings of the 10th World Congress on Intelligent Control and Automation, Beijing: IEEE. (2012) 1026-1030.

[12] D. W. Ding, J. Zhu, X. S. Luo, et al, Controlling Hopf bifurcation in fluid flow model of internet congestion control system, International Journal of Bifurcation and Chaos. 19(4) (2010) 1415-1424.

[13] N. Li, H. Q. Yuan, H. Y. Sun, et al, Adaptive control of bifurcation and chaos in a time-delayed system, Chinese Phys. 22(3) (2013) 030508.

[14] S. T. Guo, G. Feng, X.F. Liao, et al, Hopf bifurcation control in a congestion control model via dynamic delayed feedback, AIP Chaos. 18(4) (2008) 3104-3117.

[15] M. Xiao, W. X. Zheng, Bifurcation control of a congestion control model via state feedback, International Journal of Bifurcation and Chaos. 23(6) (2013) 1330018.

[16] Z. G. Liu, K. W. Chong, Hybird control of bifurcation in continuous nonlinear dynamical systems, International Journal of Bifurcation and Chaos. 15(12) (2005) 3895-3903.

[17] X. S. Luo, G. R. Chen, B. H. Wang, et al, Hybird control of period-doubling bifurcation and chaos in discrete nonlinear dynamical systems, Chaos, Solitons and Fractals. 18(4) (2003) 775-783.

[18] Y. L. Zhou, Ch. Xu, Hopf Bifurcation of Wireless Network Congestion Control Fluid Flow Model based on State feedback Control, Computational Physics. 32(03) (2015) 352-360.

[19] J. Y. Zhang, B. Y. Feng, Geometric theory and Bifurcating problems of ordinary differential equations, Beijing: Peking University Press. 2000.

[20] B. D. Hassard, N. D. Kazarinoff, Y. H. Wan, Theory and Applications of Hopf Bifurcation, Cambridge University Press, Cambridge. 1981.

[21] Ch. Xu, Y. L. Zhou, Y. Wang, Control of Hopf Bifurcation of wireless Network fluid flow model, Journal of Shanghai Jiaotong University. 48(10) (2014) 1479-1484.

[22] D. W. Ding, Ch. Wang, N. N. Kong, PD Control of Hopf Bifurcation in wireless Network model, Journal of Electronic Measurement and Instrumentation. 29(11) (2015) 1683-1690.

[23] L. Deng, X. Wang, M. Peng, Hopf bifurcation analysis for a ratio-dependent predator–prey system with two delays and stage structure for the predator, Applied Mathematics & Computation. 231(231) (2014)214-230.