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Bounds on Approximating Generalized Waring Distribution In A Generalized Waring Process

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Abstract: *This paper uses Stein's method for Poisson and negative binomial distributions together with the covariance associated with the generalized Waring random variable to determine error bounds for measuring the accuracy of approximations of generalized Waring distribution with parameters* (a,kt,ρ) *in a generalized Waring process, where* $a>0,\ k>0,\ \rho>0$ *and* $t\geq 0$. *The bounds in the present study are pointed out that (i) for* $c=\frac{\rho-1}{k}>0$, *the generalized Waring distribution can be approximated by the negative binomial distribution with parameters* $\left(a,\frac{c}{c+t}\right)$ *when* c *and/or* k *are large and (ii) for* $\lambda=\frac{a}{c}>0$, *the generalized Waring distribution with parameters* (a,kt,c) *can be approximated by the Poisson distribution with mean* λt *when* c *is large or* λt *is small.*

Keywords: *Generalized Waring process, Poisson approximation, Negative binomial approximation, Stein's method.*

I. INTRODUCTION

Let the counting process $\{N(t),\ t\geq 0\}$ be a *generalized Waring process* with parameters (a,k,ρ), where $a>0,\ k>0$ and $\rho>2$. Then the process has to satisfy the following conditions: (i) $N(0)=0$, (ii) $N(t)$ is a Markov process, (iii) $N(t+h)-N(t)$ has a generalized Waring distribution with parameters (a,kh,ρ) for $h>0$ and ([12]). The condition (iii) indicates that $N(t)$ has a generalized Waring distribution $GW(a,kt,\rho)$ with the probability function as follows:

$$P(N(t)=n)=\frac{1}{n!}\frac{\rho_{(kt)}}{(a+\rho)_{(kt)}}\frac{a_{(n)}(kt)_{(n)}}{(\rho+a+kt)_{(n)}},\ n\in\{0,1,...\},\tag{1}$$

where $\alpha_{(\beta)}=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)}$, $\alpha>0$ and $\beta>0$. The mean and variance of $N(t)$ are $\mu_{N(t)}=E[N(t)]=\frac{akt}{\rho-1}$ and $\sigma^2_{N(t)}=Var[N(t)]=\frac{akt(\rho+kt-1)(\rho+a-1)}{(\rho-2)(\rho-1)^2}$, respectively. Applications of the generalized Waring process can be found in modeling informetric data ([3]) and in the context of modeling web access patterns ([12]).

Let us consider the probability function in (1.1), it can be expressed as

$$P(N(t)=n)=\frac{\Gamma(a+n)}{\Gamma(a)n!}\frac{\Gamma(\rho+a)\Gamma(\rho+kt)}{\Gamma(\rho)}\frac{\Gamma(kt+n)}{\Gamma(kt)\Gamma(\rho+kt+a+n)},\ n\in\{0,1,...\}.\tag{2}$$

Let $\xi(a,kt,\rho)=\frac{\Gamma(\rho+a)\Gamma(\rho+kt)}{\Gamma(\rho)\Gamma(\rho+kt+a)}$, we have

$$\begin{aligned}\xi(a,kt,\rho)&=\frac{\Gamma((\rho-1)+a+1)}{\Gamma((\rho-1)+1)}\frac{1}{\frac{\Gamma((\rho-1+kt)+a+1)}{\Gamma((\rho-1+kt)+1)}}\\&=\left(\frac{\rho-1+a}{\rho-1}\right)\frac{\Gamma((\rho-1)+a)}{\Gamma(\rho-1)}\frac{1}{\left(\frac{\rho-1+kt+a}{\rho-1+kt}\right)\frac{\Gamma((\rho-1+kt)+a)}{\Gamma(\rho-1+kt)}}\\&=\left(1+\frac{a}{\rho-1}\right)\frac{\Gamma((\rho-1)+a)}{\Gamma(\rho-1)(\rho-1)^a}\left(\frac{1}{1+\frac{a}{\rho-1+kt}}\right)\frac{1}{\frac{\Gamma((\rho-1+kt)+a)}{\Gamma(\rho-1+kt)(\rho-1+kt)^a}}\left(\frac{\rho-1}{\rho-1+kt}\right)^a.\end{aligned}\tag{3}$$

Replacing $\xi(a,kt,\rho)=\frac{\Gamma(\rho+a)\Gamma(\rho+kt)}{\Gamma(\rho)\Gamma(\rho+kt+a)}$ in to (2), we obtain

$$P(N(t)=n) = \begin{cases} \xi(a,kt,\rho) & , n=0, \\ \dfrac{a\xi(a,kt,\rho)kt}{\rho+kt+a} & , n=1, \\ \dfrac{\Gamma(a+n)}{\Gamma(a)n!}\dfrac{\xi(a,kt,\rho)(kt+n-1)\cdots kt}{(\rho+kt+a+n-1)\cdots(\rho+kt+a)} & , n=2,\ldots \end{cases}$$

$$= \begin{cases} \xi(a,kt,\rho) & , n=0, \\ \dfrac{a\xi(a,kt,\rho)\frac{kt}{\rho-1+kt}}{1+\frac{a+1}{\rho-1+kt}} & , n=1, \\ \dfrac{\Gamma(a+n)}{\Gamma(a)n!}\dfrac{\xi(a,kt,\rho)\left(\frac{kt}{\rho-1+kt}+\frac{n-1}{\rho-1+kt}\right)\cdots\frac{kt}{\rho-1+kt}}{\left(1+\frac{a+n}{\rho-1+kt}\right)\cdots\left(1+\frac{a+1}{\rho-1+kt}\right)} & , n=2,\ldots \end{cases} \quad (4)$$

From (4), it can be seen that if $\rho,k\to\infty$ and $\frac{kt}{\rho-1+kt}$ tends to a constant, then by the property (15) in [7] and using [11], it follows that $\xi(a,kt,\rho)\to\left(\frac{\rho-1}{\rho-1+kt}\right)^a$ and $P(N(t)=n)\to\frac{\Gamma(a+n)}{\Gamma(a)n!}\left(\frac{kt}{\rho-1+kt}\right)^n\left(\frac{\rho-1}{\rho-1+kt}\right)^a$ for every $n\in\{0,1,\ldots\}$. That is, if $\rho,k\to\infty$ and $c=\frac{\rho-1}{k}>0$ is a constant, then $P(N(t)=n)\to\frac{\Gamma(a+n)}{\Gamma(a)n!}\left(\frac{t}{c+t}\right)^n\left(\frac{c}{c+t}\right)^a$ for every $n\in\{0,1,\ldots\}$. Thus the generalized Waring distribution with parameters (a,kt,c) can be approximated by the negative binomial distribution with parameters $\left(a,\frac{c}{c+t}\right)$ when c and/or k are large. Let $M(t)$ be the negative binomial random variable with the probability function

$$P(M(t)=n)=\frac{\Gamma(a+n)}{\Gamma(a)n!}\left(\frac{t}{c+t}\right)^n\left(\frac{c}{c+t}\right)^a,\ n\in\{0,1,\ldots\}. \quad (5)$$

Let $Z(t)$ be the Poisson random variable with the probability function

$$P(Z(t)=n)=\frac{e^{-\lambda t}(\lambda t)^n}{n!},\ n\in\{0,1,\ldots\}. \quad (6)$$

Additionally, in the case of $a=\frac{\lambda k}{\rho-1}=\lambda c$ and $\lambda>0$, the probability function in (1.5) can be written as

$$P(M(t)=n)=\begin{cases} \left(\dfrac{1}{\frac{\lambda c+\lambda t}{\lambda c}}\right)^{\lambda c} & , n=0, \\ \dfrac{c\lambda t}{c+t}\left(\dfrac{1}{\frac{\lambda c+\lambda t}{\lambda c}}\right)^{\lambda c} & , n=1, \\ \dfrac{1}{n!}\left[\dfrac{(c\lambda+n-1)t}{c+t}\cdots\dfrac{c\lambda t}{c+t}\right]\left(\dfrac{1}{\frac{\lambda c+\lambda t}{\lambda c}}\right)^{\lambda c} & , n=2,\ldots \end{cases} \quad (7)$$

From which, if $c\to\infty$ and λt remains a constant, then $P(M(t)=n)\to P(Z(t)=n)$ for every $n\in\{0,1,\ldots\}$. Therefore, the negative binomial distribution with parameters $\left(a,\frac{c}{c+t}\right)$ converges to the Poisson distribution with mean λt when c is large.

The results mentioned above indicate that both negative binomial and Poisson distribution are limiting distributions of the generalized Waring distribution in the process. However, these limiting forms do not give any criteria for measuring the accuracy of each approximation. In this paper, we give two error bounds to be the mentioned criteria, that is, one bound for the total variation distance between the distributions of $N(t)$ and $M(t)$, denoted by $d(N(t),M(t))$, and the other bound for the total variation distance between the distributions of $N(t)$ and $Z(t)$, denoted by $d(N(t),Z(t))$. The distances can be defined as follows:

$$d(N(t),M(t))=\sup_{A\subseteq\mathbb{N}\cup\{0\}}\left|P(N(t)\in A)-P(M(t)\in A)\right| \quad (8)$$

and

$$d(N(t),Z(t)) = \sup_{A\subseteq \mathrm{N}\cup\{0\}} \left|P(N(t)\in A) - P(Z(t)\in A)\right|. \tag{9}$$

The following theorems present two error bounds for $d(N(t),M(t))$ and $d(N(t),Z(t))$ which are our main results.

Theorem 1.1. *For* $A\subseteq \mathrm{N}\cup\{0\}$, *let* $p=\frac{c}{c+t}$, *then the following inequality holds:*

$$d(N(t),M(t)) \le \frac{a}{ck}\min\left\{1, \frac{k(c+t)(1-p^{a+1})}{ck-1}\right\}. \tag{10}$$

Theorem 1.2. *For* $A\subseteq \mathrm{N}\cup\{0\}$, *let* $\lambda=\frac{a}{c}$, *then we have the following:*

$$d(N(t),Z(t)) \le \frac{1-e^{-\lambda t}}{c}\left[t + \frac{(a+1)(c+t)}{ck-1}\right]. \tag{11}$$

Remark. 1. Theorem 1.1 tell us that the generalized Waring distribution with parameters (a,kt,c) can be approximated by the negative binomial distribution with parameters $\left(a,\frac{c}{c+t}\right)$ when c and/or k are large. Similarly, the result in Theorem 1.2 indicates that the generalized Waring distribution with parameters (a,kt,c) can be approximated by the Poisson distribution with mean λt when c is large or λt is small. In these situations, simpler forms of both negative binomial and Poisson distributions are appropriate choices for approximating the generalized Waring distribution.

2. Consider the bounds in Theorems 1.1 and 1.2, because $\frac{1-p^{a+1}}{(a+1)q} < \frac{1-p^{a}}{aq}$ ([6]) and by Taylor's expansion, $e^{\frac{1-p}{p}} = 1 + \frac{1-p}{p} + \frac{\left[\frac{1-p}{p}\right]^2}{2!} + \mathrm{L} > \frac{1}{p} \Rightarrow p > e^{-\frac{1-p}{p}} \Rightarrow p^a > e^{-\frac{a(1-p)}{p}} \Rightarrow 1-p^a < 1-e^{-\lambda t}$, these give $\frac{a}{ck}\min\left\{1,\frac{k(c+t)(1-p^{a+1})}{ck-1}\right\} \le \frac{(1-p^{a+1})(c+t)(a+1)q}{c(ck-1)(a+1)q} < \frac{(1-p^{a})(c+t)(a+1)}{c(ck-1)} < \frac{(1-e^{-\lambda t})(a+1)(c+t)}{c(ck-1)} \le \frac{1-e^{-\lambda t}}{c}\left[t+\frac{(a+1)(c+t)}{ck-1}\right]$. Therefore, it can be concluded that the bound in Theorem 1.1 is sharper than the bound in Theorem 1.2. In addition, it is seen that $\frac{a}{ck}\min\left\{1,\frac{k(c+t)(1-p^{a+1})}{ck-1}\right\} < \frac{(1-e^{-\lambda t})(a+1)(c+t)}{c(ck-1)}$ and $d(N(t),Z(t)) \le d(N(t),M(t)) + d(M(t),Z(t)) \le \frac{a}{ck}\min\left\{1,\frac{k(c+t)(1-p^{a+1})}{ck-1}\right\} + \frac{(1-e^{-\lambda t})t}{c}$. Because the second term is obtained from Theorem 1.C (ii) in [1]. Thus, the shaper bound for $d(N(t),Z(t))$ in Theorem 1.2 is of the form

$$d(N(t),Z(t)) \le \frac{1}{c}\left\{(1-e^{-\lambda t})t + \frac{a}{k}\min\left\{1,\frac{k(c+t)(1-p^{a+1})}{ck-1}\right\}\right\}, \tag{12}$$

which yields a good approximation when c is large or k and c are large.

II. METHOD

The tools for giving the desired results are Stein's method for negative binomial and Poisson distributions and the covariance associated with the generalized Waring random variable $N(t)$.

For Stein's method, Stein [8] introduced a power full method to give a bound on the normal approximation to the distribution of a sum of dependent random variables. Later, Chen [5] applied this method to give a bound on the Poisson approximation to the distribution of a sum of dependent Bernoulli random variables. Brown and Phillips [2] also applied this method to give a bound on the negative binomial approximation to the distribution of a sum of dependent Bernoulli random variables. Additionally, this method was applied to other discrete distributions, including binomial, hypergeometric and geometric. The important tool of Stein's method for each approximation is called Stein's equation. Stein's equation of each distribution is a simple tool for giving a bound on the distance between the two interesting distributions.

In this study, Stein's equations for Poisson and negative binomial distributions are used to give bounds for approximating the generalized Waring distribution. Stein's equation for the Poisson distribution with parameter $\lambda t > 0$, for given *h*, is of the form

$$h(x) - \boldsymbol{P}_{\lambda t}(h) = \lambda t g(x+1) - x g(x), \tag{13}$$

where $\boldsymbol{P}_{\lambda t}(h) = \sum_{i=0}^{\infty} h(k)\frac{e^{-\lambda t}(\lambda t)^i}{i!}$ and *g* and *h* are bounded real valued functions defined on $\mathrm{N}\cup\{0\}$. Similarly, Stein's equation for the negative binomial distribution with parameters $a>0$ and $0<p=1-q<1$, for given *h*, is of the form

$$h(x) - \boldsymbol{NB}_{a,p}(h) = q(a+x)f(x+1) - xf(x), \tag{14}$$

where $\boldsymbol{NB}_{a,p}(h)=\sum_{k=0}^{\infty}h(k)\frac{\Gamma(a+k)}{k!\Gamma(a)}p^a q^k$ and f and h are bounded real valued functions defined on $\mathrm{N}\cup\{0\}$.

For $A\subseteq \mathrm{N}\cup\{0\}$, let $h_A:\mathrm{N}\cup\{0\}\to \mathrm{R}$ be defined by

$$h_A(x)=\begin{cases}1 \text{ , if } x\in A,\\ 0 \text{ , if } x\notin A.\end{cases} \tag{15}$$

Following [1], the solution g_A of (13) can be expressed as

$$g_A(x)=\begin{cases}(x-1)!(\lambda t)^{-x}e^{\lambda t}[\boldsymbol{P}_{\lambda t}(h_{A\cap C_{x-1}})-\boldsymbol{P}_{\lambda t}(h_A)\boldsymbol{P}_{\lambda t}(h_{C_{x-1}})] \text{ , if } x\geq 1,\\ 0 \text{ , if } x=0,\end{cases} \tag{16}$$

where $x\in \mathrm{N}$ and $C_{x-1}=\{0,...,x-1\}$. For negative binomial approximation, by [2] and [9], the solution f_A of (14) is as follows:

$$f_A(x)=\begin{cases}\dfrac{\boldsymbol{NB}_{a,p}(h_{A\cap C_{x-1}})-\boldsymbol{NB}_{a,p}(h_A)\boldsymbol{NB}_{a,p}(h_{C_{x-1}})}{\frac{x\Gamma(a+x)p^a q^x}{x!\Gamma(a)}}, \text{ if } x\geq 1,\\ 0 \text{ , if } x=0.\end{cases} \tag{17}$$

Let $\Delta g_A(x)=g_A(x+1)-g_A(x)$ and $\Delta f_A(x)=f_A(x+1)-f_A(x)$. The following lemma gives bounds for Δg_A and Δf_A.

Lemma 2.1. *For* $A\subseteq \mathrm{N}\cup\{0\}$ *and* $x\in \mathrm{N}$, *then we have the following*:

$$\sup_A|\Delta g_A(x)|\leq \frac{1-e^{-\lambda t}}{\lambda t} \text{ ([1])}, \tag{18}$$

$$\sup_A|\Delta f_A(x)|\leq \frac{1-p^{a+1}}{(a+1)q} \text{ ([10])} \tag{19}$$

and

$$\sup_A|\Delta f_A(x)|\leq \frac{1}{(a+x)q} \text{ ([10])}. \tag{20}$$

For the covariance associated with the random variable $N(t)$, by following [4], the covariance of $N(t)$ and $g_A(N(t))$ and the covariance of $N(t)$ and $f_A(N(t))$ can be expressed as

$$Cov[N(t),g_A(N(t))]=\sum_{n=0}^{\infty}\left[\Delta g_A(n)\sum_{j=0}^{n}(\mu_{N(t)}-j)P(N(t)=j)\right] \tag{21}$$

and

$$Cov[N(t),f_A(N(t))]=\sum_{n=0}^{\infty}\left[\Delta f_A(n)\sum_{j=0}^{n}(\mu_{N(t)}-j)P(N(t)=j)\right], \tag{22}$$

respectively.

Lemma 2.2. *With the above definitions, we have the following.*

$$Cov[N(t),g_A(N(t))]=\frac{1}{ck}E[\Delta g_A(N(t))(kt+N(t))(a+N(t))] \tag{23}$$

and

$$Cov[N(t),f_A(N(t))]=\frac{1}{ck}E[f_A(N(t))(kt+N(t))(a+N(t))]. \tag{24}$$

Proof. Because the proof of (24) is similar to that of (23), it suffices to show that (23) holds. The probability function of $N(t)$, $c=\frac{\rho-1}{k}$, can be expressed as

$$P(N(t)=n)=\frac{\Gamma(a+n)}{\Gamma(a)n!}\frac{\Gamma(ck+a+1)\Gamma(k(c+t)+1)\Gamma(kt+n)}{\Gamma(ck+1)\Gamma(kt)\Gamma(k(c+t)+a+n+1)},\ n\in\{0,1,...\}.$$

Let $\gamma(n)=\frac{\sum_{j=0}^{n}(\mu_{N(t)}-j)P(N(t)=j)}{P(N(t)=n)}$, $n=0,1,...,$ then we have to show that

$$\gamma(n)=\frac{(kt+n)(a+n)}{ck},\ n=0,1,... \tag{25}$$

It is seen that $\gamma(0)=\mu_{N(t)}=\frac{at}{c}$ and $\gamma(1)=\frac{(kt+1)(a+1)}{ck}$. For $m\in \mathrm{N}$, let $\gamma(m)=\frac{(kt+m)(a+m)}{ck}$, we shall show that $\gamma(m+1)=\frac{(kt+m+1)(a+m+1)}{ck}$. Because

$$\begin{aligned}
\gamma(m+1) &= \frac{\sum_{j=0}^{m+1}(\mu_{N(t)}-j)P(N(t)=j)}{P(N(t)=m+1)} \\
&= \frac{P(N(t)=m)}{P(N(t)=m+1)}\frac{\sum_{j=0}^{m}(\mu_{N(t)}-j)P(N(t)=j)}{P(N(t)=m)} + \mu_{N(t)} - (m+1) \\
&= \frac{(m+1)[k(c+t)+a+m+1]}{(a+m)(kt+m)}\frac{(kt+m)(a+m)}{ck} + \frac{at}{c} - (m+1) \\
&= \frac{(m+1)[k(c+t)+a+m+1]}{ck} + \frac{at}{c} - (m+1) \\
&= \frac{(kt+m+1)(a+m+1)}{ck},
\end{aligned}$$

by mathematical induction, (25) is obtained.

Substituting (25) into (21), it becomes

$$\begin{aligned}
Cov\big[N(t), g_A(N(t))\big] &= \sum_{n=0}^{\infty}\left[\frac{\Delta g_A(n)(kt+n)(a+n)}{ck}P(N(t)=n)\right] \\
&= \frac{1}{ck}E\big[\Delta g_A(N(t))(kt+N(t))(a+N(t))\big],
\end{aligned}$$

this implies that (23) holds. □

III. PROOF OF MAIN RESULTS

This section uses Stein's method for negative binomial and Poisson distributions and the covariance associated with the random variable $N(t)$ to prove our main results, Theorems 1.1 and 1.2.

Proof of Theorem 1.1. From (14), substituting h by h_A and x by $N(t)$ and taking expectation in this equation, we have

$$\begin{aligned}
P(N(t)\in A) - P(N(t)\in A) &= E\{q(a+N(t))f(N(t)+1) - N(t)f(N(t))\} \\
&= E\{aqf(N(t)+1) + qN(t)\Delta f(N(t)) - pN(t)f(N(t))\} \\
&= aqE\{f(N(t)+1)\} + qE\{N(t)\Delta f(N(t))\} - p\{Cov[N(t), f(N(t))] + \mu_{N(t)}E[f(N(t))]\} \\
&= aqE\{\Delta f(N_k(t))\} + qE\{N_k(t)\Delta f(N_k(t))\} - p\{Cov[N(t), f(N(t))]\},
\end{aligned}$$

where $f = f_A$ is defined in (17). By applying (6) and (24), we obtain

$$\begin{aligned}
d(N(t), M(t)) &= \sup_A\left|aqE\{\Delta f(N(t))\} + qE\{N(t)\Delta f(N(t))\} - \tfrac{p}{ck}\{E[\Delta f(N(t))(kt+N(t))(a+N(t))]\}\right| \\
&= p\sup_A\left|E\left\{\left[\frac{aq}{p} + \frac{q}{p}N(t) - \frac{(kt+N(t))(a+N(t))}{ck}\right]\Delta f(N(t))\right\}\right| \\
&= p\sup_A\left|E\left\{\left[\frac{t}{c}(a+N(t)) - \frac{(kt+N(t))(a+N(t))}{ck}\right]\Delta f(N(t))\right\}\right| \\
&\le pE\left\{\left[\frac{N(t)(a+N(t))}{ck}\right]\sup_A|\Delta f(N(t))|\right\} &(26) \\
&\le \frac{p(1-p^{a+1})}{(a+1)q}E\left[\frac{N(t)(a+N(t))}{ck}\right] \qquad \text{(by (19))} \\
&= \frac{1-p^{a+1}}{(a+1)t}E\left[\frac{N(t)(a+N(t))}{k}\right] \\
&= \frac{a(c+t)(1-p^{a+1})}{c(ck-1)}. &(27)
\end{aligned}$$

From (26), we also obtain

$$\begin{aligned}
d(N(t), M(t)) &\le pE\left[\frac{N(t)(a+N(t))}{ck}\frac{1}{(a+N(t))q}\right] \qquad \text{(by (20))} \\
&= \frac{a}{ck}. &(28)
\end{aligned}$$

Hence, following (27) and (28), the result in Theorem 1.1 is obtained. □

Proof of Theorem 1.2. From (13), substituting h by h_A and x by $N(t)$ and taking expectation in this equation, we have

$$P(N(t)\in A)-P(Z(t)\in A) = E\{\lambda t g(N(t)+1)-N(t)g(N(t))\},$$

where $g=g_A$ is defined in (16). Applying (7), we have

$$\begin{aligned} d(N(t),Z(t)) &= \sup_A\left|\lambda t E[g(N(t)+1)]-E[N(t)g(N(t))]\right| \\ &= \sup_A\left|\lambda t E[g(N(t)+1)]-Cov[N(t),g(N(t))]-\mu_{N(t)}E[g(N(t))]\right| \\ &= \sup_A\left|\lambda t E[\Delta g(N(t))]-Cov[N(t),g(N(t))]\right|. \end{aligned}$$

By (23), we obtain

$$\begin{aligned} d(N(t),Z(t)) &= \sup_A\left|\frac{at}{c}E[\Delta g(N(t))]-\frac{1}{ck}E[\Delta g_A(N(t))(kt+N(t))(a+N(t))]\right| \\ &= \frac{1}{c}\sup_A\left|E\left\{\left[at-\frac{(kt+N(t))(a+N(t))}{k}\right]\Delta g(N(t))\right\}\right| \\ &\le \frac{1}{c}E\left\{\left|-\frac{ktN(t)+N(t)(a+N(t))}{k}\right|\sup_A\left|\Delta g(N(t))\right|\right\} \\ &\le \frac{1-e^{-\lambda t}}{\lambda t c}E\left[\frac{ktN(t)+N(t)(a+N(t))}{k}\right] \text{ (by (18))} \\ &\le \frac{1-e^{-\lambda t}}{c}\left[t+\frac{(a+1)(c+t)}{ck-1}\right], \end{aligned}$$

which gives the Theorem 1.2. □

IV. CONCLUSION

In a generalized Waring process with parameters (a,k,ρ), the negative binomial and Poisson distribution are limiting distribution forms of the generalized Waring distribution. However, these limiting forms do not give any criteria for measuring the accuracy of each approximation. This study gave these criteria in the terms of two error bounds for the total variations distances, the total variations distance between the generalized Waring distribution with parameters (a,kt,c) and the negative binomial distribution with parameters $\left(a,\frac{c}{c+t}\right)$ and the total variations distances between the generalized Waring distribution with parameters (a,kt,c) and the Poisson distribution with mean λt, where $c=\frac{\rho-1}{k}>0$. With these bounds, it is pointed out that (i) the generalized Waring distribution can be approximated by the negative binomial distribution when c and/or k are large and (ii) for $\lambda=\frac{a}{c}>0$, the generalized Waring distribution can be approximated by the Poisson distribution when c is large or λt is small. Following these conditions, each distribution, negative binomial and Poisson, can be used as an appropriate approximation of the generalized Waring distribution.

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