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Square sum labeling in context of some graph operations

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Abstract

A graph $G = (V, E)$ with order p and size q is said to be square sum graph, if there exists a bijection mapping $f : V(G) \to \{0, 1, 2, \ldots, p-1\}$ such that the induced function $f^* : E(G) \to \mathbb{N}$ defined by $f^*(uv) = (f(u))^2 + (f(v))^2$, for every $uv \in E(G)$ is injective. In this paper we prove that the graph obtained by joining two copies of a specific graph by a path of arbitrary length admits a square sum labeling. We also discuss here some square sum graphs in the context of arbitrary super subdivision.

AMS Subject classification 2010: 05C78.
Key words: Square sum graph, Arbitrary super subdivision.

1 Introduction

Labeling of graph was discovered by Rosa[1] in 1967. Because of large number of applications the researchers are attracted to this domain. A dynamic survey on graph labeling is regularly updated by Gallian[6] and it is published by *The Electronic Journal of Combinatorics*.

In this paper we consider finite, simple, undirected and connected graph. We refer to Bondy and Murty[5] for the standard terminology and notations related to graph theory and Burton[2] for the terms related to number theory. Square sum graph was defined by Ajitha, Arumugam and Germina[10].

Definition 1.1. *[10] A graph $G = (V, E)$ with order p and size q is said to be a square sum graph, if there exists a bijection $f : V(G) \to \{0, 1, 2, \ldots, p-1\}$ such that the induced function $f^* : E(G) \to \mathbb{N}$ defined by $f^*(uv) = (f(u))^2 + (f(v))^2$, is injective.*

In[10], Ajitha, Arumugam and Germina derived square sum labeling for basic graphs such as trees and cycles. They also proved that complete lattice grids $L_{m,n} = P_m \times P_n$ and cycle-cactus $C^k_{(n)}$ are square sum graphs. R. Sebastian and K. A. Germina[9] explored some planar graphs which are square sum. Godasara and Patel [3, 4] discussed about some square sum graphs in context of duplication of vertex and discovered some bistar related square sum graphs.

2 Main Results

Definition 2.1. *[6] The wheel graph W_n is the graph obtained by joining the graphs C_n and K_1. i.e. $W_n = C_n + K_1$. Here the vertices corresponding to C_n are called rim vertices and C_n is called rim of W_n, while the vertex corresponds to K_1 is called apex vertex.*

Remark 2.1. *The square of any odd number never equals to the sum of square of any even number and 1, because if they are equal then there exists some positive integers k and m such that $(2k+1)^2 = (2m)^2 + 1$. So, $k(k+1) = m^2$, which is not true (Refer [2]).*

Theorem 2.1. *The graph G constructed by joining two copies of wheel W_n at any rim vertex by a path of arbitrary length is a square sum graph.*

Proof. Let $W_n^{(1)}$ be the first copy of wheel with $V(W_n^{(1)}) = \{v_0, v_1, \dots, v_n\}$ and
$E(W_n^{(1)}) = \{v_0v_i \mid 1 \le i \le n\} \bigcup \{v_iv_{i+1} \mid 1 \le i \le n-1\} \bigcup \{v_nv_1\}$, where v_0 is apex.
Let $W_n^{(2)}$ be the second copy of wheel with $V(W_n^{(2)}) = \{u_0, u_1, \dots, u_n\}$ and
$E(W_n^{(2)}) = \{u_0u_i \mid 1 \le i \le n\} \bigcup \{u_iu_{i+1} \mid 1 \le i \le n-1\} \bigcup \{u_nu_1\}$, where u_0 is apex.
Let P_k be a path with $V(P_k) = \{w_1, w_2, \dots, w_k\}$.
Let G denote the resultant graph constructed by joining two copies of W_n at any rim vertex by a path P_k. Without loss of generality take $w_1 = v_1$ and $w_k = u_1$.
Here, $|V(G)| = 2n + k$ and $|E(G)| = 4n + k - 1$.
Let us define a bijection $f : V(G) \to \{0, 1, 2 \dots, 2n + k - 1\}$ which consists of two cases.
Case 1: n is even.

$$f(v_i) = \begin{cases} 0 \ ; i = 0. \\ 2n - 4i + 5 \ ; 1 \le i \le \frac{n}{2}. \\ 3 \ ; i = \frac{n}{2} + 1. \\ 4i - (2n+1) \ ; \frac{n}{2} + 2 \le i \le \ n. \end{cases}$$

$$f(u_i) = \begin{cases} 1 \ ; i = 0. \\ 2n - 4i + 4 \ ; 1 \le i \le \frac{n}{2}. \\ 2 \ ; i = \frac{n}{2} + 1. \\ 4i - 2n - 2 \ ; \frac{n}{2} + 2 \le i \le \ n. \end{cases}$$

$$f(w_i) = \begin{cases} 2n + 2i - 1 \ ; 2 \le i \le \lfloor \frac{k}{2} \rfloor. \\ 2n + 2k - 2i \ ; \lfloor \frac{k}{2} \rfloor < i \le k - 1. \end{cases}$$

For the edge labels in the graph there are five possibilities:

(1) The edge labels $\{f^*(v_0v_{\frac{n+2}{2}}), f^*(v_0v_{\frac{n}{2}}), f^*(v_0v_{\frac{n+4}{2}}), \ldots, f^*(v_0v_1)\}$ are in ascending order of the form $4k+1$ ($k \in \mathbb{N}$), because common end vertex of these edges is labeled by 0 and other end vertices are labeled by consecutive (naturally distinct) odd numbers.

(2) The edge labels $\{f^*(v_{\frac{n+2}{2}}v_{\frac{n}{2}}), f^*(v_{\frac{n+2}{2}}v_{\frac{n+4}{2}}), \ldots, f^*(v_nv_1)\}$ are in ascending order of the form $4k+2$ ($k \in \mathbb{N}$), because end vertices of edges are labeled by distinct odd numbers.

(3) The edge labels $\{f^*(u_0u_{\frac{n+2}{2}}), f^*(u_0u_{\frac{n}{2}}), f^*(u_0u_{\frac{n+4}{2}}), \ldots, f^*(u_0u_1)\}$ are in ascending order of the form $4k+1$ ($k \in \mathbb{N}$), because common end vertex of these edges is labeled by 1 and other end vertices are labeled by consecutive (naturally distinct) even numbers.

(4) The edge labels $\{f^*(u_{\frac{n+2}{2}}u_{\frac{n}{2}}), f^*(u_{\frac{n+2}{2}}u_{\frac{n+4}{2}}), \ldots, f^*(u_nu_1)\}$ are in ascending order of the form $4k$ ($k \in \mathbb{N}$), because end vertices of edges are labeled by distinct even numbers.

(5) The edge labels $\{f^*(w_1w_2), f^*(w_kw_{k-1}), f^*(w_2w_3) \ldots, f^*(w_{\lfloor\frac{k}{2}\rfloor}w_{\lfloor\frac{k+1}{2}\rfloor})\}$ are in ascending order.

It is clear that the labels of possibilities (2) and (4) are distinct from possibilities (1) and (3). The labels of the possibilities (1) and (3) are distinct from each other because of Remark 2.1.
The labels of possibility (5) are greater than labels of possibilities (1) to (4).
So, the labels of above all possibilities are internally as well as externally distinct.
Case 2: n is odd.

$$f(v_i) = \begin{cases} 0 \ ; i = 0. \\ 2n+1 \ ; i = 1. \\ (2n-1) - 4(i-2) \ ; 1 \leq i \leq \frac{n+1}{2}. \\ 3 \ ; i = \frac{n+3}{2}. \\ 4(i-1) - (2n-1); \ ; \frac{n+3}{2} < i \leq \ n. \end{cases}$$

$$f(u_i) = \begin{cases} 1 \ ; i = 0. \\ 2n - 4(i-1) \ ; 1 \leq i \leq \frac{n}{2}. \\ 2 \ ; i = \frac{n}{2}+1. \\ 4i - (2n+2) \ ; \frac{n}{2}+1 < i \leq \ n. \end{cases}$$

$$f(w_i) = \begin{cases} 2(n+1) + (2i-3) \ ; 2 \leq i \leq \lfloor\frac{k}{2}\rfloor. \\ 2(n+1) + 2k - 2(i+1) \ ; \lfloor\frac{k}{2}\rfloor < i \leq k-1. \end{cases}$$

For the edge labels in the graph there are five possibilities:

(1) The edge labels $\{f^*(v_0v_{\frac{n+3}{2}}), f^*(v_0v_{\frac{n+1}{2}}), f^*(v_0v_{\frac{n+5}{2}}), \ldots, f^*(v_0v_1)\}$ are in ascending order of the form $4k+1$ ($k \in \mathbb{N}$), because common end vertex of these edges is labeled by 0 and other end vertices are labeled by consecutive (naturally distinct) odd numbers.

(2) The edge labels $\{f^*(v_{\frac{n+3}{2}}v_{\frac{n+1}{2}}), f^*(v_{\frac{n+3}{2}}v_{\frac{n+5}{2}}), \ldots, f^*(v_1v_2)\}$ are in ascending order of the form $4k+2$ ($k \in \mathbb{N}$), because end vertices of edges are labeled by distinct odd numbers.

(3) The edge labels $\{f^*(u_0u_{\frac{n+3}{2}}), f^*(u_0u_{\frac{n+1}{2}}), f^*(u_0u_{\frac{n+5}{2}}), \dots, f^*(u_0u_1)\}$ are in ascending order of the form $4k+1$ $(k \in \mathbb{N})$, because common end vertex of these edges is labeled by 1 and other end vertices are labeled by consecutive (naturally distinct) even numbers.

(4) The edge labels $\{f^*(u_{\frac{n+3}{2}}u_{\frac{n+1}{2}}), f^*(u_{\frac{n+3}{2}}u_{\frac{n+5}{2}}), \dots, f^*(u_1u_2)\}$ are in ascending order of the form $4k$ $(k \in \mathbb{N})$, because end vertices of edges are labeled by distinct even numbers.

(5) The edge labels $\{f^*(w_1w_2), f^*(w_kw_{k-1}), f^*(w_2w_3)\dots, f^*(w_{\lfloor\frac{k}{2}\rfloor}w_{\lfloor\frac{k+1}{2}\rfloor})\}$ are in ascending order.

Using the arguments similar to the case 1, one can observe that in this case the function f is bijective and for every $uv \in E(G)$ the induced edge labels $f^*(uv) = (f(u))^2 + (f(v))^2$ are all distinct. Hence, two copies of wheel W_n joined by a path of arbitrary length at rim vertices admit a square sum labeling. □

Example 2.1. *Square sum labeling in the graph constructed by joining two copies of W_8 at rim vertices v_1 and u_1 by path P_8 is shown in the following Figure 1.*

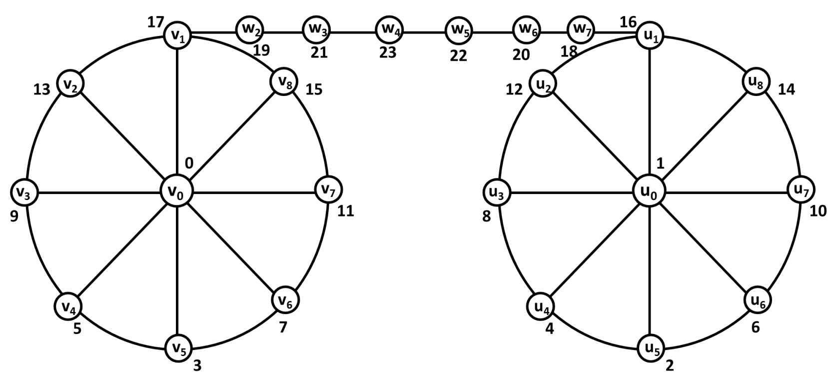

Figure 1

Definition 2.2. *[6] A gear graph G_n is obtained from the wheel graph W_n by adding a vertex between every pair of adjacent vertices of the cycle C_n.*

Corollary 2.1. *The graph G constructed by joining two copies of gear G_n at any rim vertex by a path of arbitrary length is a square sum graph.*

Proof. Consider G be the graph constructed by joining two copies of gear G_n at any rim vertex by a path of arbitrary length.
Looking in other way, G is obtained by deleting alternative edges between apex to rim vertices in wheel W_{2n}. Consider the same labeling which is defined in the above Theorem 2.1.
So, for every $uv \in E(G)$ the induced edge labels $f^*(uv) = (f(u))^2 + (f(v))^2$ are all distinct. Hence, two copies of gear G_n joined by a path of arbitrary length at rim vertices admit a square sum labeling. □

Definition 2.3. *[6] The shell graph S_n is the graph obtained by taking $n-3$ concurrent chords in a cycle C_n. The vertex at which all the chords are concurrent is called the apex. The shell graph S_n is also called fan graph F_{n-1}. That is, $S_n = F_{n-1} = P_{n-1} + K_1$.*

Theorem 2.2. *The graph G constructed by joining two copies of shell S_n at apex by a path of arbitrary length is a square sum graph.*

Proof. Let $S_n^{(1)}$ be the first copy of shell graph with $V(S_n^{(1)}) = \{v_1, v_2, \dots, v_n\}$ and $E(S_n^{(1)}) = \{v_1 v_i \mid 3 \le i \le n-1\} \bigcup \{v_i v_{i+1} \mid 1 \le i \le n-1\} \bigcup \{v_n v_1\}$, where v_1 is apex.
Let $S_n^{(2)}$ be the second copy of shell graph with $V(S_n^{(2)}) = \{u_1, u_2, \dots, u_n\}$ and $E(S_n^{(2)}) = \{u_1 u_i \mid 3 \le i \le n-1\} \bigcup \{u_i u_{i+1} \mid 1 \le i \le n-1\} \bigcup \{u_n u_1\}$, where u_1 is apex.
Let P_k be a path with $V(P_k) = \{w_1, w_2, \dots, w_k\}$.
Let G denote the resultant graph constructed by joining two copies of S_n at apex by a path P_k. Without loss of generality take $w_1 = v_1$ and $w_k = u_1$.
Here, $|V(G)| = 2n + k - 2$ and $|E(G)| = 4n + k - 7$.
Let us define a bijection $f : V(G) \to \{0, 1, 2 \dots, 2n + k - 3\}$ as follows.

$$f(u_i) = \begin{cases} 1 \text{ ; } i = 1. \\ 2(i-1) \text{ ; } 2 \le i \le n. \end{cases}$$

$$f(v_i) = \begin{cases} 0 \text{ ; } i = 0. \\ 2i - 1 \text{ ; } 2 \le i \le n. \end{cases}$$

$$f(w_i) = \begin{cases} 2n + (2i - 3) \text{ ; } 2 \le i \le \lfloor \frac{k}{2} \rfloor. \\ 2n + 2k - 2(i+1) \text{ ; } \lfloor \frac{k}{2} \rfloor < i \le k - 1. \end{cases}$$

Repeating the arguments similar to the Theorem 2.1, one can observe that the function f is bijective and for every $uv \in E(G)$ the induced edge labels $f^*(uv) = (f(u))^2 + (f(v))^2$ are all distinct. Hence, two copies of shell S_n joined by a path of arbitrary length at apex vertices admit a square sum labeling. $\square$

Example 2.2. *Square sum labeling in the graph constructed by joining two copies of S_8 at apex vertices by path P_8 is shown in the following Figure 2.*

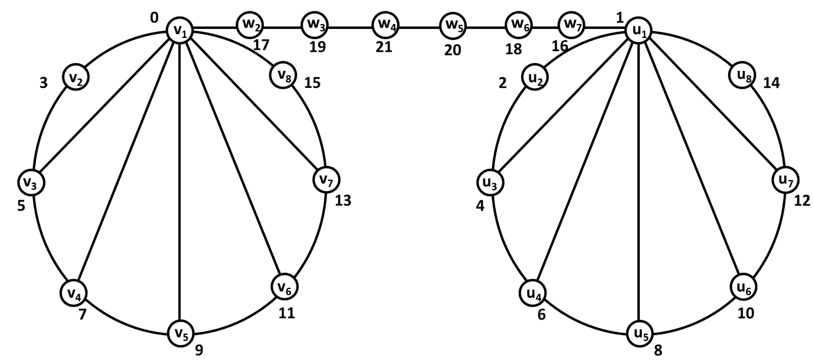

Figure 2

Definition 2.4. *[5] Generalized Petersen graph, denoted by $P(n,k)(n \geq 5,\ 1 \leq k \leq n)$ is a graph with vertex set $\{a_0, a_1, \dots a_{n-1}, b_0, b_1, \dots b_{n-1}\}$ and edge set $\{a_i a_{i+1} \setminus i = 0, 1, \dots n-1\} \cup \{a_i b_i \setminus i = 0, 1, \dots n-1\} \cup \{b_i b_{i+k} \ \setminus i = 0, 1, \dots n-1\}$, where all subscripts are taken over modulo n. The standard Petersen graph is $P(5,2)$.*

Theorem 2.3. *The graph G constructed by joining two copies of Petersen graph $P(5,2)$ by a path of arbitrary length is a square sum graph.*

Proof. Let $P^{(1)}(5,2)$ be the first copy of Petersen graph with
$V(P^{(1)}(5,2)) = \{v_1, v_2, \dots v_{10}\}$, where $v_1, v_2, \dots v_5$ are external vertices, $v_6, v_7, \dots\ v_{10}$ are internal vertices and
$E(P^{(1)}(5,2)) = \{v_i v_{i+1} \mid 1 \leq i \leq 4\} \bigcup \{v_5 v_1\} \bigcup \{v_i v_{i+5} \mid 1 \leq i \leq 5\} \bigcup \{v_i v_{i+2} \mid 6 \leq i \leq 8\} \bigcup \{v_9 v_6, v_{10} v_7\}$.
Let $P^{(2)}(5,2)$ be the second copy of Petersen graph with
$V(P^{(2)}(5,2)) = \{u_1, u_2, \dots u_{10}\}$, where $u_1, u_2, \dots u_5$ are external vertices, $u_6, u_7, \dots, u_{10}$ are internal vertices and
$E(P^{(2)}(5,2)) = \{u_i u_{i+1} \mid 1 \leq i \leq 4\} \bigcup \{u_5 u_1\} \bigcup \{u_i u_{i+5} \mid 1 \leq i \leq 5\} \bigcup \{u_i u_{i+2} \mid 6 \leq i \leq 8\} \bigcup \{u_9 u_6, u_{10} u_7\}$.
Let P_k be a path with $V(P_k) = \{w_1, w_2, \dots, w_k\}$.
Let G denote the resultant graph constructed by joining two copies of $P(5,2)$ by path P_k.
Without loss of generality take $w_1 = v_1$ and $w_k = u_1$.
Here, $|V(G)| = 18 + k$, and $|E(G)| = 19 + k$.

Let us define a bijection $f : V(G) \to \{0, 1, \dots, k+17\}$ as follows.

$$f(v_i) = \begin{cases} i-1 \; ; 1 \le i \le 5. \\ 9 \; ; i = 6. \\ i-2 \; ; 7 \le i \le 10. \end{cases}$$

$$f(u_i) = \begin{cases} 9+i \; ; 1 \le i \le 5. \\ 19 \; ; i = 6. \\ 8+i \; ; 7 \le i \le 10. \end{cases}$$

$$f(w_i) = \begin{cases} 17+2i \; ; 2 \le i \le [\frac{k}{2}]. \\ 18+2k-2i \; ; [\frac{k}{2}] < i \le k-1. \end{cases}$$

Repeating the arguments similar to the Theorem 2.1, one can observe that the function f is bijective and for every $uv \in E(G)$ the induced edge labels $f^*(uv) = (f(u))^2 + (f(v))^2$ are all distinct. Hence, two copies of $P(5,2)$ joined by a path of arbitrary length admit a square sum labeling. □

Example 2.3. *Square sum labeling in the graph constructed by joining two copies of $P(5,2)$ by path P_8 is shown in the following Figure 3.*

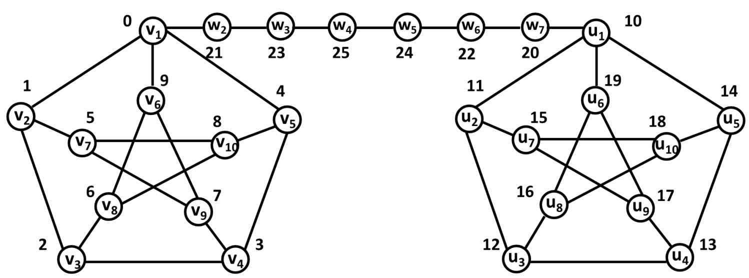

Figure 3

Definition 2.5. *[9] $P_n(+)N_m$ is the graph with vertex set*

$$V(P_n(+)N_m) = \{v_1, v_2, \dots, v_n, y_1, y_2, \dots, y_m\}$$

and edge set

$$E(P_n(+)N_m) = E(P_n) \cup \{(v_1, y_i), (v_n, y_i) / 1 \le i \le m\}.$$

Here $V(P_n) = \{v_1, v_2, \dots, v_n\}$ and $V(N_m) = \{y_1, y_2, \dots, y_m\}$.

Theorem 2.4. *The graph G constructed by joining two copies of $P_n(+)N_m$ at any vertex of N_m by a path of arbitrary length is a square sum graph.*

Proof. Let $(P_n(+)N_m)^{(1)}$ be the first copy of graph $P_n(+)N_m$ with

$$V((P_n(+)N_m)^{(1)}) = \{v_1, v_2, \ldots, v_n, y_1, y_2, \ldots, y_m\}.$$

Let $(P_n(+)N_m)^{(2)}$ be the second copy of graph $P_n(+)N_m$ with

$$V((P_n(+)N_m)^{(2)}) = \{u_1, u_2, \ldots, u_n, x_1, x_2, \ldots, x_m\}.$$

Let P_k be a path with $V(P_k) = \{w_1, w_2, \ldots, w_k\}$.
Let G denote the resultant graph constructed by joining two copies of graph $P_n(+)N_m$ by path P_k. Without loss of generality let us take $w_1 = y_m$ and $w_k = x_m$.
Here, $|V(G)| = 2(n+m-1)+k$, and $|E(G)| = 2(n+2m)+k-3$.
Let us define a bijection $f : V(G) \to \{0, 1, \ldots, 2n+2m+k-3\}$ which consists of two cases.
Case 1: n is odd.

$$\begin{aligned}
f(v_i) &= \begin{cases} 2n-4i \; ; 1 \le i \le \frac{n-1}{2}. \\ 4i-2(n+1) \; ; \frac{n-1}{2} < i \le n. \end{cases} \\
f(y_i) &= 2n+2(i-1) \; ; 1 \le i \le m. \\
f(u_i) &= \begin{cases} 2n-4i+1 \; ; 1 \le i \le \frac{n-1}{2}. \\ 4i-2(n+1)+1 \; ; \frac{n-1}{2} < i \le n. \end{cases} \\
f(x_i) &= 2n+2(i-1)+1 \; ; 1 \le i \le m. \\
f(w_i) &= \begin{cases} 2(n+m)+2(i-2) \; ; 2 \le i \le [\frac{k}{2}]. \\ 2(n+m+k)-2(i)-1 \; ; [\frac{k}{2}] < i \le k-1. \end{cases}
\end{aligned}$$

Case 2: n is even.

$$\begin{aligned}
f(v_i) &= \begin{cases} 2(n+1)-4i \; ; 1 \le i \le \frac{n}{2}. \\ 4i-2(n+2) \; ; \frac{n}{2} < i \le n. \end{cases} \\
f(y_i) &= 2n+2(i-1) \; ; 1 \le i \le m. \\
f(u_i) &= \begin{cases} 2(n+1)-4i+1 \; ; 1 \le i \le \frac{n}{2}. \\ 4i-2(n+2)+1 \; ; \frac{n}{2} < i \le n. \end{cases} \\
f(x_i) &= 2n+2(i-1)+1 \; ; 1 \le i \le m. \\
f(w_i) &= \begin{cases} 2(n+m)+2(i-2) \; ; 2 \le i \le [\frac{k}{2}]. \\ 2(n+m+k)-2(i)-1 \; ; [\frac{k}{2}] < i \le k-1. \end{cases}
\end{aligned}$$

Repeating the arguments similar to the Theorem 2.1, one can observe that the function f is bijective and for every $uv \in E(G)$ the induced edge labels $f^*(uv) = (f(u))^2 + (f(v))^2$ are all distinct. Hence, the graph constructed by joined by two copies of $P_n(+)N_m$ at any vertex of N_m by a path of arbitrary length is a square sum graph. □

Example 2.4. *Square sum labeling in the graph constructed by joining two copies of* $P_7(+)N_4$ *at* 4^{th} *vertex of* N_4 *by path* P_8 *is shown in the following Figure* 4.

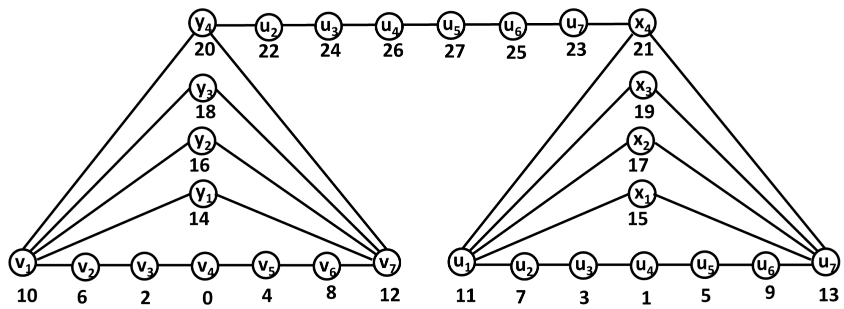

Figure 4

Theorem 2.5. *The graph* G *constructed by joining two copies of complete graph* K_n *by a path of arbitrary length is a square sum graph for* $n \leq 5$.

Proof. Let $K_n^{(1)}$ be the first copy of complete graph with $V(K_n^{(1)}) = \{v_1, v_2, \ldots, v_n\}$.
Let $K_n^{(2)}$ be the second copy of complete graph with $V(K_n^{(2)}) = \{u_1, u_2, \ldots, u_n\}$.
Let P_k be a path with $V(P_k) = \{w_1, w_2, \ldots, w_k\}$.
Let G denote the resultant graph constructed by joining two copies of K_n by path P_k.
Without loss of generality let us take $w_1 = v_1$ and $w_k = u_1$.
Here, $|V(G)| = 2(n-1) + k$, and $|E(G)| = n^2 - n + k - 1$.
Let us define a bijection $f : V(G) \to \{0, 1, 2 \ldots, 2n + k - 3\}$ as follows.

$$f(v_i) = \begin{cases} 2n - 1\ ; i = 1. \\ 2i - 3\ ; 2 \leq i \leq n. \end{cases}$$

$$f(u_i) = \begin{cases} 2n - 2\ ; i = 1. \\ 2i - 4\ ; 2 \leq i \leq n. \end{cases}$$

$$f(w_i) = \begin{cases} 2n + (2i - 3)\ ; 2 \leq i \leq \lfloor \frac{k}{2} \rfloor. \\ 2n + 2k - 2(i + 1)\ ; \lfloor \frac{k}{2} \rfloor < i \leq k - 1. \end{cases}$$

Repeating the arguments similar to the Theorem 2.1, one can observe that the function f is bijective and for every $uv \in E(G)$ the induced edge labels $f^*(uv) = (f(u))^2 + (f(v))^2$ are all distinct. Hence, the graph constructed by joining two copies of K_n by a path of arbitrary length is a square sum graph for $n \leq 5$. □

Remark 2.2. *We strongly believe that the above result is also true for* $n > 5$.

Example 2.5. *Square sum labeling in the graph constructed by joining two copies of K_5 by path P_8 is shown in the following Figure 5.*

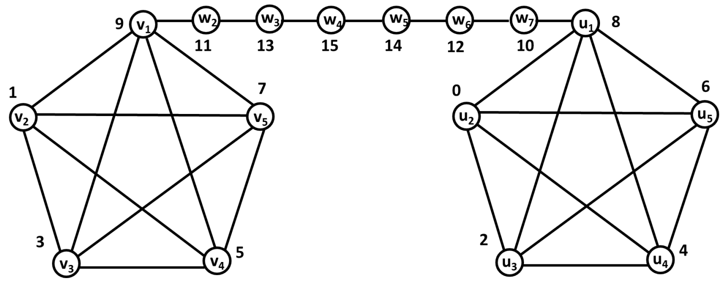

Figure 5

Definition 2.6. *[6] The graph obtained from G by replacing every edge e_i of G by a complete bipartite graph K_{2,m_i} for some positive integer m_i $(1 \leq i \leq q)$ is called arbitrary supersubdivision of G.*

Theorem 2.6. *The graph G constructed by arbitrary supersubdivision of path P_n is a square sum graph.*

Proof. Let $V(P_n) = \{v_i \mid 1 \leq i \leq n\}$ and $E(P_n) = \{v_i v_{i+1} \mid 1 \leq i \leq n-1\}$.
Let G denote the arbitrary supersubdivision of path P_n then each edge $v_i v_{i+1}$ of P_n is replaced by a K_{2,m_i}, for some natural number m_i.
Let $u_{i,j}$ be the vertices of m_i vertex section, where $1 \leq i \leq n-1$ and $1 \leq j \leq m_i$.
Let $M = \sum_{i=1}^{n} m_i$. Then $|V(G)| = n + M$ and $|E(G)| = 2M$.
Let us define a bijection $f : V(G) \rightarrow \{0, 1, 2, \ldots, n + M - 1\}$ as follows.

$$
\begin{aligned}
f(v_1) &= 0. \\
f(u_{i,j}) &= f(v_i) + j \; ; 1 \leq j \leq m_i. \\
f(v_{i+1}) &= f(u_{i,m_i}) + 1; 1 \leq i \leq n-1.
\end{aligned}
$$

From above defined function f, it is easy to see that the induced edge labels in the graph G are in ascending order of the form
$f^*(v_1 u_{1,1}) < f^*(v_1 u_{1,2}) < \ldots < f^*(v_1 u_{1,m_1}) < f^*(v_2 u_{1,1}) < f^*(v_2 u_{1,2}) < \ldots < f^*(v_2 u_{1,m_1}) < f^*(v_2 u_{2,1}) < f^*(v_2 u_{2,2}) < \ldots < f^*(v_2 u_{2,m_2}) < f^*(v_3 u_{2,1}) < \ldots < f^*(v_n u_{n-1,m_{n-1}})$.
So, the function f is bijective and for every $uv \in E(G)$ the induced edge labels $f^*(uv) = (f(u))^2 + (f(v))^2$ are all distinct. This concludes that f is a square sum labeling of graph G and hence G is square sum graph. □

Example 2.6. *Square sum labeling in arbitrary supersubdivision of P_4 is shown in the following Figure* 6.

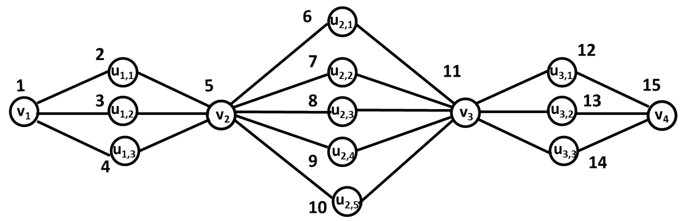

Figure 6

Theorem 2.7. *The graph G constructed by arbitrary supersubdivision of star $K_{1,n}$ is a square sum graph.*

Proof. Let $V(K_{1,n}) = \{v_i \mid 0 \leq i \leq n\}$ and $E(K_{1,n}) = \{v_0v_i \mid 1 \leq i \leq n\}$, where v_0 is apex.
Let G denote the arbitrary supersubdivision of star $K_{1,n}$.
Then each edge v_0v_i of $K_{1,n}$ is replaced by K_{2,m_i}, for some natural number m_i.
Let $u_{i,j}$ be the vertices of m_i vertex section, where $1 \leq i \leq n$ and $1 \leq j \leq m_i$.
Let $M = \sum\limits_{i=1}^{n} m_i$ then $|V(G)| = n + M + 1$ and $|E(G)| = 2M$.
Let us define a bijection $f : V(G) \rightarrow \{0, 1, 2, \ldots, n + M\}$ as follows.

$$f(v_i) = \begin{cases} 0 \; ; i = 0. \\ cm_n + i \; ; 1 \leq i \leq n. \end{cases}$$
$$f(u_{i,j}) = cm_{i-1} + j \; ; 1 \leq j \leq m_i \text{ and } 1 \leq i \leq n.$$

Here $m_0 = 0$ and cm_i = cumulative values of m_i.
It is easy to see that the edge labels in the graph G are in ascending order of the form
$f^*(v_0u_{1,1}) < f^*(v_0u_{1,2}) < \ldots < f^*(v_0u_{1,m_1}) < f^*(v_0u_{2,1}) < f^*(v_0u_{2,2}) < \ldots < f^*(v_0u_{n,m_n}) < f^*(v_1u_{1,1}) < f^*(v_1u_{1,2}) < \ldots < f^*(v_1u_{1,m_1}) < f^*(v_2u_{2,1}) < \ldots < f^*(v_nu_{n,m_n})$.
So, the function f is bijective and for every $uv \in E(G)$ the induced edge labels $f^*(uv) = (f(u))^2 + (f(v))^2$ are all distinct.
This concludes that f is a square sum labeling of graph G and hence G is a square sum graph. □

Example 2.7. *Square sum labeling in arbitrary supersubdivision of $K_{1,4}$ is shown in the following Figure 7.*

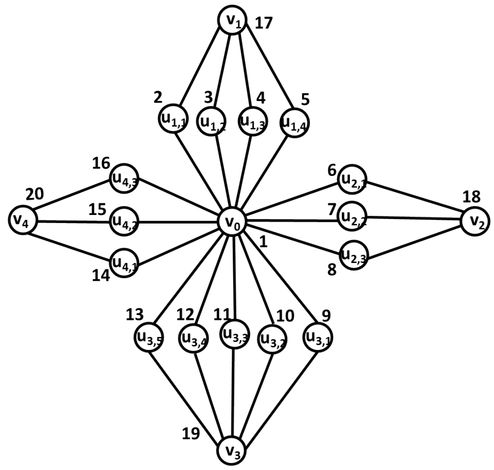

Figure 7

Theorem 2.8. *The graph constructed by arbitrary supersubdivision of cycle $C_n(n \geq 3)$ is a square sum graph.*

Proof. Let $V(C_n) = \{v_i \mid 1 \leq i \leq n\}$ and $E(C_n) = \{v_i v_{i+1} \mid 1 \leq i \leq n-1\} \bigcup \{v_n v_1\}$.
Let G denote the arbitrary supersubdivision of cycle C_n.
Then each edge $v_i v_{i+1}$ (consider v_{n+1} as v_1) is replaced by K_{2,m_i}, for some natural number m_i.
Let $u_{i,j}$ be the vertices of m_i vertex section, where $1 \leq i \leq n$ and $1 \leq j \leq m_i$.
Let $M = \sum\limits_{i=1}^{n} m_i$.
Then $|V(G)| = n + M$ and $|E(G)| = 2M$.
Let us define a bijection $f : V(G) \to \{0, 1, 2 \ldots, n + M - 1\}$ which consists of two cases.
Case 1: n is even.

$$f(v_i) = \begin{cases} 0 \;; i = 1. \\ f(u_{n+2-i,m_{n+2-i}}) + 1 \;; 2 \leq i \leq \frac{n}{2} + 1. \\ f(v_{n+2-i}) + 1 \;; \frac{n}{2} + 1 < i \leq n. \end{cases}$$

$$f(u_{i,j}) = \begin{cases} f(v_{n+2-i}) + j \;; 1 \leq j \leq m_i,\ 2 \leq i \leq \frac{n}{2} \\ f(u_{n+2-i,m_{n+2-i}}) + 1 \;; 1 \leq j \leq m_i,\ \frac{n}{2} < i \leq n. \end{cases}$$

Case 2: n is odd.

$$f(v_i) = \begin{cases} 0 \;; i = 1. \\ f(u_{n+2-i,m_{n+2-i}}) + 1 \;; 2 \leq i \leq \frac{n+1}{2}. \\ f(v_{n+2-i}) + 1 \;; \frac{n+1}{2} < i \leq n. \end{cases}$$

$$f(u_{i,j}) = \begin{cases} f(v_{n+2-i}) + j \;; 1 \leq j \leq m_i,\ 2 \leq i \leq \frac{n+1}{2}. \\ f(u_{n+2-i,m_{n+2-i}}) + 1 \;; 1 \leq j \leq m_i,\ \frac{n+1}{2} < i \leq n. \end{cases}$$

For the edge labels in graph G, we have
$f^*(v_1u_{1,1}) < f^*(v_1u_{1,2}) < \ldots < f^*(v_1u_{1,m_1}) < f^*(v_1u_{n,1}) < f^*(v_1u_{n,2}) < \ldots < f^*(v_1u_{n,m_n}) < f^*(v_2u_{1,1}) < f^*(v_2u_{1,2}) < \ldots < f^*(v_2u_{1,m_1}) < f^*(v_nu_{n,1}) < \ldots$.
So, the function f is bijective and for every $uv \in E(G)$ the induced edge labels $f^*(uv) = (f(u))^2 + (f(v))^2$ are all distinct.
Hence, f is a square sum labeling and hence G is a square sum graph. □

Example 2.8. *Square sum labeling in arbitrary supersubdivision of C_5 is shown in the following Figure* 8.

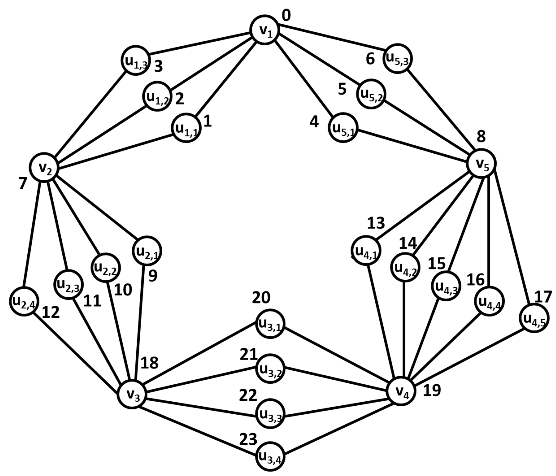

Figure 8

3 Concluding Remarks

Labeling of discrete structure is a potential area of research. Some square sum graphs in context of joining two copies of a particular graph by a path of arbitrary length have been discovered. Arbitrary supersubdivision of path, cycle and star are proved to be square sum graphs. To investigate more results for various graphs as well as in the context of different graph operations is an open area of research.

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