Relationship of The First Type Stirling Matrix With Tetranacci Matrix

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Abstract —.*The first Stirling Matrix is a matrix whose entries contain the first Stirling number which is denoted by* $\boldsymbol{S_n}(\mathbf{1})$. *The first Stirling number is the number of arrays of* n *objects into non-empty cyclical permutations. Furthermore, the* $\mathcal{M}_n$*tetranacci matrix is the lower triangular matrix of the tetranacci rows with each entry being the main diagonal tetranacci number 1. The tetranacci sequence is a generalization of the Fibonacci sequence which consists of the sum of the four previous terms beginning with the terms*$0, 0, 0, 1$. *In this article discusses the first type Stirling matrix, the tetranacci matrix and the relationship of the first type Stirling matrix with the tetranacci matrix to obtain a new matrix called the* $\mathcal{H}_n$ *matrix. Then the* $\mathcal{H}_n$ *matrix can be expressed in the form* $\mathcal{H}_n = S_n(1)\mathcal{M}_n^{-1}$.

Keywords —.*The first Stirling number, the tetranacci number, the first Stirling matrix and the tetranacci matrix*.

I. Introduction

Stirling Numbers, was first discovered by James Stirling around the 18th century (1692-1770) consisting of the first Stirling numbers and the second Stirling numbers. Bona [1] defines the first Stirling number type is the number of arrays of n objects into non-empty cyclic permutations notated by$s(n,k)$every $\in \mathbb{N}$.Comtet [4] describes the Stirling number of the first type$S_n(1)$in the form of a square matrix, that is, the bottom triangular matrix with each entry being the first Stirling number. Cheon and Kim [2] discuss the relationship of the Pascal matrix with the first Stirlingmatrix $s(n,k)$. From the relationship between the two matrices obtained several combinatoric identities involving the first Stirling number. Maltais and Gulliver [8] discuss the Pascal matrix and the Stirling matrix. Lee et al. [7] discusses the relationship of the Stirling matrix with the Fibonacci matrix, then from the relationship of the two matrices obtained two different matrix factors. Cheon and Kim [3] discuss Stirling matrix with Vandermonde matrix, Stirling matrix with Bernaoulli numbers, Stirling matrix with Eulierian numbers.

The tetranacci sequence was initially studied by Feinberg [5] in 1963, after 760 years the Fibonacci sequence was introduced by Leonardo Pisano, but in 1970 the Fibonacci sequence was known and developed by European mathematicians. Saveri and Patel [12] state the tetranacci sequence begins with the terms 0, 0, 0, and 1, then the next term is obtained by adding up the four previous terms. Then, Sabeth et al. [11] defines the tribonacci matrix and discusses the relationship between the pascal matrix and the tribonacci matrix. Rasmi et al. [10] discusses the relationship of the first Stirling matrix type and the tribonacci matrix. Mirfaturiqa et al. [9].define the tetranacci matrix and discuss the factorization of the Pascal matrix with the tetranacci matrix.

This article, discusses the relationship of the first Stirling matrix and the tetranacci matrix with the same idea from Rasmi et al. [9] and Mirfaturiqa et al. [8] obtained a matrix called the $H_n$matrix.

II. Stirling Matrix and Tetranacci Matrix

This section provides definitions and theorems of the first Stirling matrix. Bona [1, p.133] defines the first Stirling number type

Definition 2.1 The first type of Stirling Number is the number of arrays of n objects into c non-empty cyclic permutations and is denoted by $s(n,k)$ where n,k are positive integers with $n \geq k$. The first Stirling number has the initial conditions $s(n,k) = 0$ if $n < k$, $s(0,0) = 1$ and $s(n,n) = 1$

Example 1

The value of s (4.1) will be determined. Suppose n is a set consisting of four elements namely $1, 2, 3, 4$ and k consisting of one block or set, namely A. the arrangement of four objects with one cyclical permutation is

$$(1234), (1432), (1234), (2134), (1423), \text{ and}(1324)$$

So the number of permutations of four objects with cyclic permutations is 6 or the value of $s\ (4,1) = 6$

Example 2

The value of s (4.2) will be determined. Suppose n is a set consisting of four elements namely 1, 2, 3, 4 and k consisting of two blocks or sets, namely A and B. The arrangement of four objects with two cyclic permutations is

(1)(234), (2)(134), (3)(124), (4)(123), (1)(234), (2)(314), (4)(213), (12)(34), (13)(24), dan (14)(23)

So the number of permutations of four objects with cyclic permutations is 11 or the value of s (4, 2) = 11

To determine the Stirling number of the first type $s\ (n,k)$ in addition to using the definition (2.1) can also be determined using the following theorem

Theorem 2.2 For each positive integer n and k where $n \geq k$ satisfies the recursive requirements

$$s(n,k) = s(n-1,k-1) + (n-1)s(n-1,k) \qquad (1)$$

Proof. Suppose there are n positive integers $1,2,\cdots,n$. There are two ways to arrange these n objects into k circles. The first way, n is alone in a circle, so there are $s\ (n-1,k-1)$ how to arrange $n-1$ objects into $k-1$ cyclic permutations. The second way,n is in a circle with other objects, so there are n-1 objects that must be arranged in the k circle. Because n objects can be placed in $1, 2, \cdots, n-1$ in $n-1$ ways, the number of arrangements of this second method is $(n-1)s(n-1,k)$.■

The first Stirling number $s\ (n,k)$ can be constructed into the first Stirling matrix $S_n(1)$ with $n \times n$ where each entry is the first Stirling number. Comtet [9, p. 144] provides the following definition:

$$s_{i,j} = \begin{cases} s(i,j), & if\ \ i \geq j, \\ 0, & \text{otherwise.} \end{cases} \qquad (2)$$

The first Stirling number $s\ (n,k)$ can be constructed in the form of the first Stirling matrix $S_n(1)$. In general the Stirling matrix of the first type of order $n \times n$ can be expressed in the form of the following matrix

$$S_n(1) = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & 0 & \cdots & 0 \\ 2 & 3 & 1 & 0 & \cdots & 0 \\ 6 & 11 & -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ s(n,1) & s(n,2) & s(n,3) & s(n,4) & \cdots & s(n,n) \end{bmatrix} \quad (3)$$

The tetranacci sequence is a generalization of the Fibonacci sequence which consists of the sum of the four previous terms beginning with the terms 0, 0, 0,and 1. So that the tetranacci sequence can be formed as follows:

$$0, 0, 0, 1, 1, 2, 4, 8, 15, 29, 56, 108, 208, 401, \cdots .$$

Definition 2.4Tetranacci numbers M_n for all n members of natural numbers, meeting the recursive conditions as follows:

$$M_n = \begin{cases} 0 & \text{if n } = 0, 1, 2; \\ 1 & \text{if n} = 3 \\ M_{n-1} + M_{n-2} + M_{n-3} + M_{n-3} & \text{otherwise.} \end{cases} \quad (4)$$

Similar to Fibonacci numbers, the term tetranacci is also expressed as a square matrix. Mirfaturriqa et al. [9] determined $n \times n$tetranacci matrix as follows :

Definition 2.5 For all natural numbers n, the tetranacci matrix has the method $n \times n$ with each entry from the tetranacci matrix $\mathcal{M}_n = [m_{ij}]$, $\forall i, j = 1,2,3, \cdots, n$

$$m_{ij} = \begin{cases} M_{i-j+3}, & \text{if } i \geq j, \\ 0, & \text{if } i < j. \end{cases} \quad (5)$$

For definition (2.5) the tetranacci matrix M_n is the bottom triangular matrix with the main diagonal of 1 and the determinant of the tetranacci matrix $M_n$is the result of multiplying its diagonal entries, so that$\det(M_n) = 1$is obtained. Because $\det(M_n) \neq 0$then the tetranacci matrix M_n has an inverse. Then, Mirfaturiqa et al. [9] gives the formula for the inverse of the tetranacci matrix, for example ${M_n}^{-1}$is the inverse of the tetranacci matrix, $n \epsilon N$ with each element of the inverse of the tetranacci matrix $\mathcal{M}_n^{-1} = [m'_{ij}]$, $\forall i, j = 1, 2, 3, \cdots, n$is defined as

$$m'_{ij} = \begin{cases} 1, & \text{if } i = j, \\ -1, & \text{if } i - 4 \leq j \leq i - 1, \\ 0, & \text{otherwise}. \end{cases} \quad (6)$$

In general the tetranacci matrix can be stated in the following matrix form

$$\mathcal{M}_n = \begin{bmatrix} M_3 & 0 & 0 & 0 & \cdots & 0 \\ M_4 & M_3 & 0 & 0 & \cdots & 0 \\ M_5 & M_4 & M_3 & 0 & \cdots & 0 \\ M_6 & M_5 & M_4 & M_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ M_n & M_{n-1} & M_{n-2} & M_{n-3} & \cdots & 1 \end{bmatrix} \quad (7)$$

From the calculation result obtain invers matrix tetranacci$\mathcal{M}_6$as following

$$\mathcal{M}_6^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 & 0 \\ -1 & -1 & -1 & 1 & 0 & 0 \\ -1 & -1 & -1 & -1 & 1 & 0 \\ 0 & -1 & -1 & -1 & -1 & 1 \end{bmatrix} \quad (8)$$

Based on equation (8) it can be seen that the inverse of the tetranacci matrix $\mathcal{M}_6$and so on, applies to each pattern in the entry column $[m_{i,j}]$are $1, -1, -1, -1$ and -1. It can be stated for the tetranacci $n \times n$ matrix where the pattern in the column entry $[m_{i,j}]$cannot be exchanged.

Because the tetranacci matrix M_n has an inverse, the matrix $\mathcal{M}_n$ $\mathcal{M}_n^{-1} = I_n = \mathcal{M}_n^{-1}\mathcal{M}_n$.. So the matrix $\mathcal{M}_n$is the matrix $\mathcal{M}_n$ invertible.

III. Relation Between Stirling's Matrix and Tetranacci Matrix

In this section, we obtain the relationship between the first Stirling matrix and the tetranacci matrix. Rasmi et.al. [10] which obtains a new matrix $D_n$from the relationship of the second Stirling matrix type with the tribonacci matrix.

Definition 3.1 For all natural numbers , $n \times n$matrix with$D_n = = [d_{i,j}]$, $\forall i, j = 1,2,3, \cdots, n$ as,

$$d_{i,j} = s(i,j) - s(i-1,j) - s(i-2,j) - s(i-3,\mathrm{j}). \quad (9)$$

From tthe relationship of the second type Stirling matrix with the tribonacci matrix in equation (9) a new matrix is obtained,namely the $\mathcal{H}_n$matrix of the relationship of the first Stirling matrix type with the tetranacci matrix with the following

Definition 3.2 For all natural numbers $n, n \times n$matrixwith$\mathcal{H}_n = [h_{i,j}]$, $\forall i, j = 1,2,3, \cdots, n$ as,

$$h_{i,j} = s(i,j) - s(i-1,j) - s(i-2,j) - s(i-3,j) - s(i-4,j). \quad (10)$$

From definition (3.2) be obtained, $h_{1,1} = 1, h_{1,2} = 0, \forall j \geq 2; h_{2,1} = 0, h_{2,2} = 1, h_{2,j} = 0, \forall j \geq 3; h_{3,1} = -2, h_{3,2} = 2, h_{3,3} = 1, h_{3,j} = 0, \forall j \geq 4; h_{4,1} = -12, h_{4,2} = 4, h_{4,3} = 5, h_{4,4} = 1, h_{4,j} = 0, h_{5,1} = -72, h_{5,2} = 4, h_{4,3} = 24, h_{5,4} = 9, h_{5,5} = 1, h_{5,6} = 0$, dan untuk $i, j \geq 2, h_{i,j} = h_{i-1,j-1} + j.h_{i-1,j}$

Based on the definition of the H_n matrix in equation (10) theorem 3.2 can be derived as follows

Theorem 3.2 There is a matrix H_n, for each original number n with the first Stirling matrix $S_n(1)$defined in equation (2) and inverse of the tetranacci matrix $\mathcal{M}_n$defined in equation (6) obtained $S_n(1) = \mathcal{H}_n\mathcal{M}_n^{-1}$.

Proof. Will be proven tetranacci matrix $\mathcal{M}_n$ is an invertible matrix.

$$\mathcal{M}_n^{-1}S_n(1) = \mathcal{H}_n. \quad (11)$$

Pay attention to the left hand side of equation (11) if $\forall i,j = 1$ dan $\forall j \geq 2$, kemudian $m'_{i,j} = m'_{1,j} = 0$. Kemudian $\forall i,j = 1$obtained the following equation

$$\sum_{k=1}^{n} m'_{i,k}S_{k,j} = \sum_{k=1}^{n} m'_{1,k}S_{k,1} = 1 = h_{1,1}.$$

if $\forall i = 1$ dan $\forall j \geq 2$ then$m'_{1,j} = 0$ dan $S_{1,j} = 0$. Then$\forall i = 1$ dan $\forall j \geq 2$, the following equation is

$$\sum_{k=1}^{n} m'_{i,k}S_{k,j} = \sum_{k=1}^{n} m'_{1,k}S_{k,1} = 0 = h_{1,j}.$$

From the equation(6) dan (2)$\forall i,j \geq 2$ was obtaine

$$\sum_{k=1}^{n} m'_{i,k}S_{k,j} = S(i,j) - S(i-1,j) - S(i-2,j) - S(i-3,j) - S(i-4,j),$$

$$= (1)S(i,j) + (-1)S(i-1,j) + (-1)S(i-2,j) + (-1)S(i-3,j) + (-1)S(i-4,j) + (0)S(i-5,j) + \cdots + (0)S$$

$$= h_{i,j}$$

Thus proven $\mathcal{M}_n^{-1}S_n(1) = \mathcal{H}_n$.

Given an example suppose $n = 5$, we will get the H_5 matrix as follows:

$$S_5(1).\ M_5^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 \\ -1 & -1 & -1 & 1 & 0 \\ -1 & -1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -2 & 2 & 1 & 0 & 0 \\ -12 & 4 & 5 & 1 & 0 \\ -72 & 4 & 24 & 9 & 1 \end{bmatrix} \quad (12)$$

So based on the matrix multiplication in equation (12) an entry from matrix$\mathcal{H}_5$is obtained

$$\mathcal{H}_5 = [h_{ij}] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -2 & 2 & 1 & 0 & 0 \\ -12 & 4 & 5 & 1 & 0 \\ -72 & 4 & 24 & 9 & 1 \end{bmatrix} \quad (13)$$

IV. CONCLUSIONS

This article, we discuss the relationship between the first type of Striling matrix and the tetranacci matrix. Then from the relationship between the two matrices obtained a formula for a new matrix, namely the $\boldsymbol{H_n}$ matrix.For further research, a research about the relationship between the Stirling matrix and Pentanacci matrix, and the Stirling matrix with other matrices.

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