International Journal of Mathematics Trends and Technology (IJMTT) - Volume 66 Issue 2 - Feb 2020

# Existence of a weak solution of some quasilinear elliptical system in a weighted Sobolev space

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**Abstract**

We consider, for a bounded open domain $\Omega$ in $I\!R^n$ and a function $u$: $\Omega \to I\!R^m$, the quasilinear elliptic system

$$(\text{QES})\begin{cases} -\text{div}\sigma\left(x, u\left(x\right), Du\left(x\right)\right) = f(x) + g(x,u) \text{ in } \Omega \\ u = 0 \quad \text{ on } \partial\Omega, \end{cases} \qquad (0.1)$$

where $f$ belongs to the dual space $W^{-1,p'}\left(\Omega, \omega^*, I\!R^m\right)$ of $W_0^{1,p}\left(\Omega, \omega, I\!R^m\right)$, $g$ satisfy some standard continuity and growth conditions. We prove existence of a regularity, growth and coercivity conditions for $\sigma$, but with only very mild monotonicity.

## 1 Introduction and statement of results

Throughout all this work, $\Omega$ will denotes a bounded open domain in $I\!R^n$. This work is devoted to establish existence results for the Dirichlet problem in divergence form of type (1.1) below. We study this quasilinear elliptic system in a weighted sobolev space and with

[1]Classification: 58J10, 58J20

[2]Keywords: Quasilinear elliptic systems, Young measure, Galerkin scheme.

mild monotonicity assumptions on $\sigma$ and various hypotheses of monotonicity and $g$ satisfy some standard continuity and growth conditions. We consider the quasilinear elliptic system:

$$\text{(QES)} \begin{cases} -\text{div}\sigma\left(x, u\left(x\right), Du\left(x\right)\right) & = & f(x) + g(x,u) \text{ in } \Omega \\ u & = & 0 \qquad\qquad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $f$ belongs to the dual space $W^{-1,p'}\left(\Omega, \omega^*, I\!R^m\right)$ of $W_0^{1,p}\left(\Omega, \omega, I\!R^m\right)$ $\left(\frac{1}{p} + \frac{1}{p'} = 1,\ p > 1\right)$ and $\omega = \{\omega_{ij} \mid 0 \leq i \leq n,\ \ 1 \leq j \leq m\}$ is a family of weight functions defined on $\Omega$ with $\omega_{ij}(x) > 0$ for almost every $x \in \Omega$ and $\omega^* = \{\omega_{ij}^* = \omega_{ij}^{1-p'},\quad 0 \leq i \leq n,\ 1 \leq j \leq m\}$, $\sigma = (\sigma_{rs})_{1 \leq s \leq n,\ 1 \leq r \leq m}$ and which satisfies some hypotheses (see below).
We denote by $I\!M^{m \times n}$ the real vector space of $m \times n$ matrices equipped with the inner product $M : N = \sum_{ij} M_{ij} N_{ij}$. The Jacobian matrix of a function $u : \Omega \longrightarrow I\!R^m$ is denoted by

$$Du(x) = (D_1 u(x), D_2 u(x), ....., D_n u(x)) \text{ with } D_i = \partial/\partial(x_i)$$

.

The first goal of this paper is that we treat a class of problems for which the classical monotone operator methods developed by Visik [16], Brezis [2], Browder [3], Lions [14], Minty [15] and others do not apply. The reason for this is that $\sigma$ does not need to satisfy the strict monotonicity condition of typical Leray-Lions operator. The tool we use in order to prove the needed compactness of approximating solutions is Young measures. The second goal of this paper is to treat the degenerated or singular case. To do this, we replace the classical Sobolev spaces $W^{1,p}(\Omega, I\!R^m)$ by a general setting of weighted Sobolev spaces $W^{1,p}(\Omega, \omega, I\!R^m)$. So the existence results are proved in this general setting. This paper can be seen as a generalization of Quasilinear elliptic systems with $W^{-1,p'}$-data, by N. Hungerbühler [9], [10] and as a continuation of Y. Akdim [1]. Let $\omega = \{\omega_{ij} \mid\ \ 0 \leq i \leq n,\ 1 \leq j \leq m\}$, and $\overline{\omega_0} = (w_0 j)$ for all $1 \leq j \leq m$ the weight functions system defined in $\Omega$ satisfying the following integrability conditions:

$$\omega_{ij} \in L^1_{loc}(\Omega), \quad \omega_{ij}^{\frac{-1}{p-1}} \in L^1_{loc}(\Omega), \text{ for some } p \in ]1, \infty[ \text{ and } \exists s > 0 \text{ such that } \omega_{ij}^{-s} \in L^1(\Omega). \tag{1.2}$$

The space $W^{1,p}(\Omega, \omega, I\!R^m)$ is the set of functions

$$\left\{ u = u(x) \mid u \in L^p(\Omega, \overline{\omega_0}, I\!R^m),\ D_{ij} u = \frac{\partial u^i}{\partial x_j} \in L^p(\Omega, \omega_{ij}, I\!R^m),\ 1 \leq i \leq n,\ 1 \leq j \leq m\} \right\}$$

with

$$L^p(\Omega, \omega_{ij}, I\!R^m) = \left\{u = u(x) \mid \mid u \mid \omega_{ij}{}^{\frac{1}{p}} \in L^p(\Omega, I\!R^m)\right\}.$$

The weighted space $W^{1,p}(\Omega, \omega, I\!R^m)$ can be equipped by the norm

$$\|u\|_{1,p,\omega} = \left(\sum_{j=1}^{m} \int_\Omega |u_j|^p w_{0j} dx + \sum_{1 \le i \le n, 1 \le j \le m} \int_\Omega |D_{ij} u|^p \omega_{ij} dx\right)^{\frac{1}{p}}.$$

The norm $\|.\|_{1,\omega,p}$ is equivalent to the norm $|||\ .\ |||$ on $W_0^{1,p}(\Omega, \omega, I\!R^m)$, defined by $|||\ u\ |||= (\sum_{1 \le i \le n, 1 \le j \le m} \int_\Omega |D_{ij} u|^p \omega_{ij} dx)^{\frac{1}{p}}$.

**Proposition 1.1** *The weighted Sobolev space $W^{1,p}(\Omega, \omega, I\!R^m)$ is a Banach space, separable and reflexive. The weighted Sobolev space $W_0^{1,p}(\Omega, \omega, I\!R^m)$ is the closure of $C_0^\infty(\Omega, I\!R^m)$ in $W^{1,p}(\Omega, \omega, I\!R^m)$ equipped by the norm $\|.\|_{1,p,\omega}$.*

Proof: see [4] (with a slight modification).

**Definition 1.1** *[7] A Young measure $(\vartheta_x)_{x \in \Omega}$ is called $W^{1,p}$-gradient young measures $(1 \le p < \infty)$ if it is associated to a sequence of gradients $Du_k$ such that $u_k$ is bounded in $W^{1,p}(\Omega)$.*

**Definition 1.2** *The $W^{1,p}$-gradient young measures $(\vartheta_x)_{x \in \Omega}$ is called homogeneous, if it does not depend on $x$, i.e, if $\vartheta_x = \vartheta$ for a.e. $x \in \Omega$.*

**Definition 1.3** *Let $u : \Omega \to I\!R^m$ be a measurable function. We say that $u$ is approximately differentiable at $x \in \Omega$ if there exists a matrix $F_x \in I\!M^{m \times n}$ such that for all $\varepsilon > O$ we have*

$$\lim_{r \to 0} \frac{1}{r^n} |y \in B_r(x) : \mid u(y) - u(x) - F_x(y - x) \mid > r\varepsilon| = 0.$$

*In this case, $F_x$ is unique and write $apDu(x) = F_x$.*

**Hypotheses**

$(H_0)$ (Hardy-Type inequalities):

There exist some constant $c > 0$, some weight function $\gamma$ and some real $q$ $(1 < q < \infty)$ such that

$$\left(\sum_{j=1}^{m} \int_\Omega |u_j(x)|^q \gamma_j(x) dx\right)^{\frac{1}{q}} \le c \left(\sum_{1 \le i \le n;\ 1 \le j \le m} \int_\Omega |D_{ij} u|^p \omega_{ij}\right)^{\frac{1}{p}}$$

for all $u \in W_0^{1,p}(\Omega, \omega, I\!R^m)$, with $\gamma = \{\gamma_j \mid 1 \leq j \leq m\}$.

The injection $W_0^{1,p}(\Omega, \omega, I\!R^m) \hookrightarrow\hookrightarrow L^q(\Omega, \gamma, I\!R^m)$ is compact, and

$W_0^{1,p}(\Omega, \omega, I\!R^m) \hookrightarrow\hookrightarrow L^r(\Omega, I\!R^m)$ is compact, with

$$\begin{cases} 1 \leq r < \frac{nps}{n(s+1)-ps} & \text{if} \quad ps < n(s+1) \\ \qquad\qquad r \geq 1 & \text{if} \quad n(s+1) \leq ps \end{cases}$$

$(H_1)$ Continuity :

$\sigma : \Omega \times I\!R^m \times I\!M^{m\times n} \longrightarrow I\!M^{m\times n}$ is a Carathéodory function

(i.e: $x \longmapsto \sigma(x,u,F)$ is measurable for every $(u,F) \in I\!R^m \times I\!M^{m\times n}$ and $(u,F) \longmapsto \sigma(x,u,F)$ is continuous for almost every $x \in \Omega$).

$(H_2)$ Growth and coercivity conditions:

There exist $c_1 \geq 0$, $c_2 > 0$, $\lambda_1 \in L^{p'}(\Omega)$, $\lambda_2 \in L^1(\Omega)$, $\lambda_3 \in L^{(p/\alpha)'}(\Omega)$, $0 < \alpha < p$, $1 < q < \infty$ and $\beta > 0$ such that for all $1 \leq r \leq n$, $1 \leq s \leq m$, we have:

$$|\sigma_{rs}(x,u,F)| \leq \beta w_{rs}^{1/p}[\lambda_1(x) + c_1 \sum_{j=1}^{m} |\gamma_j|^{1/p'} \cdot |u_j|^{q/p'} + c_1 \sum_{1\leq i\leq n;\ 1\leq j\leq m} w_{ij}^{1/p'} |D_{ij}u|^{p-1}] \quad (1.3)$$

and

$$\sigma(x,u,F) : F \geq -\lambda_2(x) - \sum_{j=1}^{m} w_{0j}(x)^{\alpha/p} \lambda_3(x) |u_j|^\alpha + c_2 \sum_{1\leq i\leq n;\ 1\leq j\leq m} w_{ij}(x) \cdot |F_{ij}|^p \qquad (1.4)$$

$(H_3)$ Monotonicity conditions: $\sigma$ satisfies one of the following conditions:

a) For all $x \in \Omega$, and all $u \in I\!R^m$, the map $F \longmapsto \sigma(x,u,F)$ is a $C^1$-function and is monotone i.e, $(\sigma(x,u,F) - \sigma(x,u,G)) : (F - G) \geq 0$, for all $x \in \Omega$, all $u \in I\!R^m$ and all $F, G \in I\!M^{m\times n}$).

b) There exists a function $W : \Omega \times I\!R^m \times I\!M^{m\times n} \longrightarrow I\!M^{m\times n}$ such that $\sigma(x,u,F) = \dfrac{\partial W}{\partial F}(x,u,F)$ and $F \longmapsto W(x,u,F)$ is convex and $C^1$ function.

c) For all $x \in \Omega$, and for all $u \in I\!R^m$ the map $F \longmapsto \sigma(x,u,F)$ is strictly monotone i.e, $\sigma(x,u,.)$ is monotone and, $(\sigma(x,u,F) - \sigma(x,u,G)) : (F - G) = 0$ implies that $F = G$).

d) $\sigma(x,u,F)$ is strictly $p$-quasi-monotone in $F$, i.e:

$$\int_{I\!M^{m\times n}} (\sigma(x,u,\lambda) - \sigma(x,u,\overline{\lambda})) : (\lambda - \overline{\lambda})) d\vartheta(\lambda) > 0,$$

for all homogeneous $W^{1,p}$-gradient young measures $\vartheta$ with center of mass $\overline{\lambda} = \langle \vartheta, id \rangle$ which are not a single Dirac mass.

$(G_0)$ : (continuity) the map $g : \Omega \times \mathbb{R}^m \to \mathbb{M}^{m\times n}$ is a Carathéodory function.

$(G_1)$: (growth condition) there exist : $b_2 \in L^{p'}(\Omega)$

$$|g_{rs}| \leq \omega_{rs}^{\frac{1}{p}}[b_2 + \sum_j \gamma_j^{\frac{1}{p'}} |u_j|^{\frac{q}{p'}}]$$

For all $1 \leq r \leq n$ and $1 \leq s \leq m$.

$(F_0)$:( continuity) $f : \Omega \times \mathbb{R}^m \to \mathbb{R}^m$ is a Carathéodory function i-e: $x \mapsto f(x,u)$ is measurable for every $u \in \mathbb{R}^m$, and , $u \mapsto f(x,u)$ is continuous for almost every $x \in \Omega$ .

$(F_1)$: (growth condition ): The exist : $b_1 \in L^{p'}(\Omega)$ such that :

$$|f_j(x,u)| \leq [b_1(x) + \gamma_j^{\frac{1}{p'}} |u_j|^{\frac{q}{p'}}]\omega_{0j}^{\frac{1}{P}}; \;\; \forall 1 \leq j \leq m$$

Our aim of this paper is to prove the existence of the problem (1.1) in the space $W_0^{1,P}(\Omega,\omega,\mathbb{R}^m)$. The main point is that we do not require strict monotonicity or monotonicity in the variables $(u,F)$ in $(H_3)$ as it is usually assumed in previous work see ([12] or [13]).

Our aim of this paper is to prove the existence of the problem (QES) in the space $W_0^{1,p}(\Omega,\omega,\mathbb{R}^m)$, when the second member $f$ is lying in $W_0^{-1,p'}(\Omega,\omega^*,\mathbb{R}^m)$.

**Theorem 1.1** *If $p \in (1,\infty)$ and $\sigma$ satisfies the conditions $(H_1)-(H_3)$, then the Dirichlet problem (QES) has a weak solution $u \in W_0^{1,p}(\Omega,\omega,\mathbb{R}^m)$, for every $f \in W^{-1,p'}(\Omega,\omega^*,\mathbb{R}^m)$ and $g$ satisfies $(F_0)$ and $(F_1)$.*

**Lemma 1.1** *For arbitrary $u \in W_0^{1,p}(\Omega,\omega,\mathbb{R}^m)$ and $f \in W^{-1,p'}(\Omega,\omega^*,\mathbb{R}^m)$, the functional*

$$\begin{array}{rcl} F(u) : W_0^{1,p}(\Omega,\omega,\mathbb{R}^m) & \longrightarrow & \mathbb{R} \\ v & \longmapsto & \int_\Omega \sigma(x,u(x),Du(x)) : Dv(x)dx - \langle f,v\rangle + \int_\Omega g(x,u) : vdx, \end{array}$$

*is well defined, linear and bounded.*

**Proof** For all $u \in W_0^{1,p}(\Omega,\omega,\mathbb{R}^m)$, we denote

$$F(u)(v) = I_1 - I_2 + I_3,$$

with

$$I_1 = \int_\Omega \sigma(x,u(x),Du(x)) : Dv(x)dx,$$

and

$$I_2 = \langle f, v \rangle .$$

$$I_3 = \int_\Omega g(x,u) : v dx$$

We define

$$I_{rs} = \int_\Omega \sigma_{rs}(x, u(x), Du(x)) : D_{rs}v(x)dx.$$

Firstly, by virtue of the growth conditions $(H_2)$ and the Hölder inequality, one has

$$\begin{aligned}
|I_{rs}| &\leq \int_\Omega |\sigma_{rs}(x, u(x), Du(x))| : |D_{rs}v(x)|dx \\
&\leq \int_\Omega \beta w_{rs}^{1/p}(x)[\lambda_1(x) + c_1 \sum_{j=1}^{m} |\gamma_j(x)|^{1/p'} |u(x)|^{q/p'} \\
&+ c_1 \sum_{1\leq i\leq n;\ 1\leq j\leq m} \omega_{ij}^{1/p'} |D_{ij}|^{p-1}]|D_{rs}v|dx \\
&\leq \beta\Big[(\int_\Omega |\lambda_1(x)|^{p'}dx)^{1/p'}(\int_\Omega |D_{rs}v(x)|^p w_{rs}dx)^{1/p} \\
&+ c_1(\int_\Omega |D_{rs}v(x)|^p w_{rs})^{1/p}(\sum_{j=1}^{m}\int_\Omega |u_j|^q \gamma_j dx)^{1/p'} \\
&+ c_1(\sum_{1\leq i\leq n;\ 1\leq j\leq m}\int_\Omega |D_{ij}u|^p w_{ij}dx)^{1/p'}(\int_\Omega |D_{rs}v|^p w_{rs}dx)^{1/p}\Big]
\end{aligned}$$

with $p = p'(p-1)$, and thanks to the Hardy-Type inequalities $(H_0)$ we have:

$$\begin{aligned}
|I_{rs}| &\leq c\beta\Big[\|\lambda_1\|_{p'}\|v\|_{1,p,w_{rs}} + c_1\|Dv\|_{p,w_{rs}}(\int_\Omega |u|^q\gamma dx)^{1/p'} \\
&\quad +c_1\sum_{ij}\|Dv\|_{p,w_{ij}}\|Du\|_{p,w_{rs}}\Big] \\
&\leq c'\beta[\|\lambda_1\|_{p'}\|v\|_{1,p,w_{rs}} + \|v\|_{1,p,w_{rs}}\|u\|_{q,\gamma} + \|u\|_{1,p,w}\|v\|_{1,p,w_{rs}}
\end{aligned}$$

with $c' = \max(c, 1)$, we have:

$$|I_1| \leq c'\beta[\|\lambda_1\|_{p'} + \|u\|_{1,p,w}^{q/p'} + \|u\|_{1,p,w}]\|v\|_{1,p,w} < \infty$$

and

$$I_3 = \sum_j \int_\Omega f_j(x,u)\varphi_j(x)dx$$

We denote $I_{3,j} = |\int_\Omega f_j(x,u)\varphi_j(x)dx|$.

$$|I_2| \leq \int_\Omega |f||v|dx \leq \|f\|_{-1,p',w*}\|v\|_{1,p,w} < \infty.$$

$$I_{3,j} \leq \int_\Omega |f_j(x,u)||v_j(x)|dx|$$

$$\leq \int_\Omega b_1(x)|v_j(x)|\omega_{0j}^{\frac{1}{P}}dx + \int_\Omega \gamma_j^{\frac{1}{p'}}|u_j|^{\frac{q}{p'}}|v_j(x)|\omega_{0j}^{\frac{1}{P}}dx$$

$$\leq (\int_\Omega |b_1(x)|^{p'})^{\frac{1}{p'}}(\int_\Omega |v_j(x)|^p\omega_{0j}dx)^{\frac{1}{p}} + (\int_\Omega \gamma_j(x)|u_j|^q dx)^{\frac{1}{p'}}(\int_\Omega |v_j(x)|^p\omega_{0j}dx)^{\frac{1}{p}}$$

$$\leq \| b_1 \|_{p'} \| v \|_{1,p,\omega} + (\sum_j \int_\Omega \gamma_j(x)|u_j|^q dx)^{\frac{1}{p'}} \| v \|_{1,p,\omega}$$

$$\leq \| b_1 \|_{p'} \| v \|_{1,p,\omega} + c \| Du \|_{1,p,\omega} \| v \|_{1,p,\omega}$$

$$\leq (\| b_1 + c \| Du \|_{1,p,\omega}) \| v \|_{1,p,\omega}$$

.

hence $I \leq c_3 \| v \|_{1,p,\omega}$. With $c_3 < \infty$.

Finally the functional $F(\cdot)$ is bounded. □

**Lemma 1.2** *The restriction of $F$ to a finite dimensional linear subspace $V$ of $W_0^{1,p}(\Omega,\omega,\mathbb{R}^m)$ is continuous.*

**Proof** Let $d$ be the dimension of $V$ and $(e_1, e_2, ..., e_d)$ a basis of $V$. Let $u_j = \sum_{1\leq i\leq d} a_j^i e_i$ be a sequence in $V$ which converges to $u = \sum_{1\leq i\leq d} a^i e_i$ in $V$. The sequence $(a_j)$ converge to $a \in \mathbb{R}^d$, so $u_j \longrightarrow u$ and $Du_j \longrightarrow Du$ a.e. On the other hand $\|u_j\|_{1,p,\omega}$ and $\|Du_j\|_{p,\omega}$ are bounded by a constant $c$. Thus, it follows by the continuity conditions $(H_1)$, that

$$\sigma(x, u_j, Du_j) : Dv \longrightarrow \sigma(x, u, Du) : Dv$$

for all $v \in W_0^{1,p}(\Omega,\omega,\mathbb{R}^m)$ and a.e. in $\Omega$. Let $\Omega'$ be a measurable subset of $\Omega$ and let $v \in W_0^{1,p}(\Omega,\omega,\mathbb{R}^m)$.

Thanks to the condition $(H_2)$, we get

$$\int_{\Omega'} |\sigma(x, u_j, Du_j) : Dv| dx < \infty,$$

By the continuity conditions $(F_0)$ we have:

$$g(x, u_j).v \to g(x, u).v$$

almost everywhere. Moreover we infer from the growth conditions $(F_1)$ that the sequences: $(\sigma(x, u_j, Du_j) : Dv)$ and $(g(x, u_j).v)$

Are equiintegrable. Indeed, if $\Omega' \subset \Omega$ is a measurable subset and $v \in W_0^{1,p}(\Omega, \omega, I\!R^m)$ then:

$\int_{\Omega'} |g(x, u_j).v| dx < \infty$ (by $(F_1)$ and Hölder inequality )

$\int_{\Omega'} |\sigma(x, u_j, Du_j) : Dv| dx < \infty$ (by Hölder inequality)

which implies that $\sigma(x, u_j, Du_j) : Dv$ is equiintegrable. And by applying the Vitali's theorem, it follows that

$$\int_{\Omega} \sigma(x, u_j, Du_j) : Dv dx \longrightarrow \int_{\Omega} \sigma(x, u, Du) : Dv dx,$$

for all $v \in W_0^{1,p}(\Omega, \omega, I\!R^m$.

Finally

$$\lim_{j\to\infty} \langle F(u_j), v\rangle = \langle F(u), v\rangle,$$

which means that

$$F(u_j) \longrightarrow F(u) \text{ in } W^{-1,p'}(\Omega, \omega^*, I\!R^m).$$

□

## 2 Galerkin approximation

Let $V_1 \subset V_2 \subset ..... \subset W_0^{1,p}(\Omega, \omega, I\!R^m)$ be a sequence of finite dimensional subspaces with $\bigcup_{k\in I\!N} V_k$ dense in $W_0^{1,p}(\Omega, \omega, I\!R^m)$. Let us fix some $k$, We assume that $V_k$ has dimension $d$ and that $(e_1, e_2, ...., e_d)$ is a basis of $V_k$. Then, we define the map

$$\begin{array}{llll} G: & I\!R^d & \longrightarrow & I\!R^d \\ & (a_1, ....., a_d)^t & \longmapsto & \left( \left\langle F(\sum_{i=1}^d a_i e_i), e_1 \right\rangle, ..., \left\langle F(\sum_{i=1}^d a_i e_i), e_d \right\rangle \right)^t. \end{array}$$

**Proposition 2.1** *The map $G$ is continuous and $G(a) \cdot a$ tends to infinity when $\|a\|_{I\!R^k}$ tends to infinity.*

**Proof** Since $F$ restricted to $V_k$ is continuous by Lemma 1.2, so $G$ is continuous.
Let $a = (a_1, ....., a_d)^t$ and $u = \sum_{1\le i\le d} a^i e_i$, Then $G(a)\cdot a = \langle F(u), u\rangle$ and the fact that $\|a\|_{\mathbb{R}^d}$ tends to infinity is equivalent to the fact that $\|u\|_{1,p,\omega}$ tends to infinity. In fact, we have:

$$G(a)\cdot a = \sum_{1\le i\le d} \langle F(u), a^i e_i\rangle = \langle F(u), u\rangle$$

and

$$\begin{aligned}\|u\|^p_{1,p,\omega} = \left\|\sum_{1\le i\le d} a^i e_i\right\|^p_{1,p,\omega} &\le \left(\sum_{1\le i\le d} |a^i|\cdot\|e_i\|_{1,p,\omega}\right)^p\\ &\le \max_{1\le i\le d}\left(\|e_i\|^p_{1,p,\omega}\right)\cdot\left(\sum_{1\le i\le d}|a^i|\right)^p\\ &\le c\cdot\|a\|_{\mathbb{R}^d}\,,\end{aligned}$$

which implies that $\|a\|_{\mathbb{R}^d}$ tends to infinity if $\|u\|_{1,p,\omega}$ tends to infinity. Now, it suffices to prove that

$$\langle F(u), u\rangle \to \infty \quad \text{when} \quad \|u\|_{1,p,\omega} \to \infty.$$

Indeed, thanks to the first coercivity condition and the Hölder inequality, we obtain

$$I = \int_\Omega \sigma(x,u,Du) : Dudx \ge -\|\lambda_2\|_1 - \int_\Omega \lambda_3\omega_{0j}^{\alpha/p}|u_j|^\alpha dx + c_2 \sum_{1\le i<n,\ 1<j\le m}\int_\Omega |D_{ij}u|^p\omega_{ij}dx.$$

By the Hölder inequality, we have

$$\begin{aligned}\int_\Omega \lambda_3|u_j|^\alpha\omega_{0j}^{\alpha/p}dx &\le \|\lambda_3\|_{(p/\alpha)'}\left(\int_\Omega \omega_{0j}|u_j|^{(p/\alpha)\cdot\alpha}\,dx\right)^{\alpha/p}\\ &\le c'\,\|\lambda_3\|_{(p/\alpha)'}\,\|u_j\|_{1,p,\omega_{0j}}\,,\end{aligned}$$

where $c'$ is a constant positive. For $\|u\|_{1,p,\omega}$ large enough, we can write

$$\begin{aligned} I &\ge -\|\lambda_2\|_1 - c'\,\|\lambda_3\|_{(p/\alpha)'}\cdot\|u_j\|^\alpha_{1,p,\omega_{0j}} + c_2\cdot\textstyle\sum_{1\le i,j\le n,m}\|Du_j\|^p_{1,p,\omega_{ij}}\\ &\ge -\|\lambda_2\|_1 - c'\,\|\lambda_3\|_{(p/\alpha)'}\cdot\|u\|^\alpha_{1,p,\omega} + c_2c'\cdot\|u\|^p_{1,p,\omega}\,.\end{aligned}$$

And since

$$I' = |\,\langle f,u\rangle\,| \le \|f\|_{-1,p',\omega^*}\cdot\|u\|_{1,p,\omega}\,,$$

Finally , it follows from the growth condition $F_1$ that :

$$|I''| = |\int_\Omega f(x,u).udx| \le (\|b_1\|_{p'} + c.\|Du\|_{1,p,\omega}).\|u\|_{1,p,\omega}$$

.

$$\leq c_3.\|u\|_{1,p,\omega}$$

With; $0 < \alpha < p$ and $p > 1$, we get :

$$I - I' - I'' \geq c_2.c'. \cdot \|u\|^p_{1,p,\omega} - \|v\|_{-1,p',\omega^*} \cdot \|u\|_{1,p,\omega} - c' \|\lambda_3\|_{(p/\alpha)'} \cdot \|u\|^\alpha_{1,p,\omega} - \|\lambda_2\|_1 - c_3. \|u\|_{1,p,\omega} \tag{2.1}$$

Consequently, by using (2.1), we deduce

$$I - I' - I'' \longrightarrow \infty \text{ as } \|u\|_{1,p,\omega} \longrightarrow \infty$$

.

$$\langle F(u), u\rangle \longrightarrow \infty \text{ as } \|u\|_{1,p,\omega} \longrightarrow \infty$$

.

The properties of $G$ allows us to construct our Galerkin approximations.

**Corollary 2.1** *For all $k \in \mathbb{N}$, there exists $(u_k) \subset V_k$ such that $\langle F(u_k), v\rangle = 0$, for all $v \in V_k$.*

**Proof** By the proposition 2.1, there exists $R > 0$, such that for all $a \in \partial B_R(0) \subset \mathbb{R}^d$, we have $G(a) \cdot a > 0$. And the usual topological argument see e.g [14] or [15], implies that $G(x) = 0$ has a solution $x \in B_R(0)$. Hence, for all $k \in \mathbb{N}$, there exists $(u_k) \subset V_k$, such that

$$\langle (Fu_k), v\rangle = 0, \text{ for all } v \in V_k.$$

Taking $(u_k = \Sigma^d_{i=1} x^i_k e_i)$ □

**Proposition 2.2** *The Galerkin approximations sequence constructed in Corollary 2.1 is uniformly bounded in $W^{1,p}_0(\Omega, \omega, \mathbb{R}^m)$; i.e:*

*there exists a constant $R > 0$, such that $\|u_k\|_{1,p,\omega} \leq R$, for all $k \in \mathbb{N}$.*

**Lemma 2.1** *Let $p > 1$ and $u_k$ be a sequence which is uniformly bounded in $W^{1,p}_0(\Omega, \omega, \mathbb{R}^m)$. There exists a subsequence of $u_k$ ( for convenience not relabeled ) and a function $u \in W^{1,p}_0(\Omega, \omega, \mathbb{R}^m)$ such that $u_k \rightharpoonup u$ in $W^{1,p}_0(\Omega, \omega, \mathbb{R}^m)$*
*And such that $u_k \to u$ in measure on $\Omega$ and in $L^r(\Omega, \mathbb{R}^m)$, with :*

$$\begin{cases} 1 \leq r \prec \frac{nps}{n(s+1)-ps} & if \quad ps \prec n(s+1) \\ r \geq 1 & if \quad n(s+1) \prec ps \end{cases}$$

**Lemma 2.2** *Let $p > 1$ and $u_k$ be a sequence which is uniformly bounded in $W_0^{1,p}(\Omega, \omega, I\!R^m)$. There exists a subsequence of $u_k$ ( for convenience not relabeled ) and a function $u \in W_0^{1,p}(\Omega, \omega, I\!R^m)$ such that $u_k \rightharpoonup u$ in $W_0^{1,p}(\Omega, \omega, I\!R^m)$*
*And such that $u_k \to u$ in measure on $\Omega$ and in $L^r(\Omega, I\!R^m)$, with :*

$$\begin{cases} 1 \leq r \prec \frac{nps}{n(s+1)-ps} & \text{if} \quad ps \prec n(s+1) \\ \qquad\qquad r \geq 1 & \text{if} \quad n(s+1) \prec ps \end{cases}$$

For the Proof, see [9], [10], with a slight modification.

**Proof** As in the proof of Proposition 2.1, we can see that $\langle F(u), u\rangle \longrightarrow \infty$ as $\|u\|_{1,p,\omega} \longrightarrow \infty$. Then, there exists $R$ satisfying $\langle F(u), u\rangle > 1$ when $\|u\|_{1,p,\omega} > R$. Now, for the sequence of Galerkin approximations $(u_k) \subset V_k$ of Corollary 2.1, which satisfy $\langle F(u_k), u_k\rangle = 0$, we have the uniform bound $\|u_k\|_{1,p,\omega} \leq R$ for all $k \in I\!N$. And there exists a subsequence $(u_k)$ of the sequence $(u_k) \subset V_k$, such that:

$$u_k \rightharpoonup u \text{ in } W_0^{1,p}(\Omega, \omega, I\!R^m) \text{ and } u_k \longrightarrow u \text{ in measure in } L^r(\Omega, I\!R^m).$$

The gradient sequence $(Du_k)$ generates the Young measure $\vartheta_x$. Since $u_k \longrightarrow u$ in measure,We have that $(u_k, Du_k)$ generates the Young measure $(\delta_{u(x)} \otimes \vartheta_x)$, see e.g [5]. Moreover, for almost all $x$ in $\Omega$, we have,

**(i)** $\vartheta_x$ is a probability measure, i.e, $\|\vartheta_x\|_{mes} = 1$.

**(ii)** $\vartheta_x$ is a homogeneous $W^{1,p}$ - gradient Young measure.

**(iii)** $\langle \vartheta_x, id\rangle = Du(x)$, see e.g [3].

**Proof.** See [5] and [6].

# 3 Passage to the limit in (QES)

Now, we are in a position to prove our main result under convenient hypotheses.
Let

$$I_k = (\sigma(x, u_k, Du_k) - \sigma(x, u, Du)) : (Du_k - Du). \tag{3.1}$$

**Lemma 3.1** *(Type-Fatou lemma)(See [5]) Let $F : \Omega \times I\!R^m \times I\!M^{m\times n} \longrightarrow I\!R$ be a Carathéodory function, and $u_k : \Omega \longrightarrow I\!R^m$ a measurable sequence, such that $(Du_k)$ generates the Young* *measure* $\vartheta_x$*, with* $\|\vartheta_x\|_{mes} = 1$*, for a.e.* $x \in \Omega$*. Then*

$$\liminf_{k\to\infty} \int_\Omega F(x, u_k, Du_k)dx \geq \int_\Omega \int_{\mathbb{M}^{m\times n}} F(x, u, \zeta)d\vartheta_x(\zeta)dx, \tag{3.2}$$

*provided that the negative part of* $F(x, u_k, Du_k)$ *is equiintegrable.*

**Lemma 3.2** *The sequence* $(I_k)_k$ *is equiintegrable.*

**Proof** We have

$$\begin{aligned} I_k &= (\sigma(x, u_k, Du_k) - \sigma(x, u, Du)) : (Du_k - Du) \\ &= [\sigma(x, u_k, Du_k) : Du_k] - [\sigma(x, u_k, Du_k) : Du] - [\sigma(x, u, Du) : Du_k] + [\sigma(x, u, Du) : Du] \\ &= I_k^1 + I_k^2 + I_k^3 + I_k^4. \end{aligned} \tag{3.3}$$

We denote $(I_k^1)^- = -[\sigma(x, u_k, Du_k : Du_k]^-$. Thanks to the coercivity condition $(H_2)$, we have

$$\begin{aligned} \int_{\Omega'} |(I_k^1)^-|dx &\leq \int_\Omega |\lambda_2| + c_2 \sum_{1\leq j\leq m} \omega_{0j}^{\frac{\alpha}{p}} |\lambda_3| \cdot |u_{kj}|^\alpha + c \sum_{1\leq i<n,\ 1<j\leq m} \omega_{ij}|D_{ij}u_k|^p dx \\ &\leq \|\lambda_2\|_1 + \int_{\Omega'} (\sum_{1\leq j\leq m} \omega_{0j}^{\alpha/p} |u_{kj}|^\alpha)^{p/\alpha} \|\lambda_3\|_{(p/\alpha)'} + c_2 \|u_k\|_{1,\omega,p}^p \end{aligned} \tag{3.4}$$

which $p/\alpha \geq 1$. Therefore,

$$\begin{aligned} \int_{\Omega'} |(I_k^1)^-|dx &\leq \|\lambda_2\|_1 + \Big(\sum_{1\leq j\leq m} \omega_{0j}|u_{kj}|^p\Big)^{\alpha/p} \|\lambda_3\|_{(p/\alpha)'} + c_2 \|u_k\|_{1,\omega,p}^p \\ &\leq \|\lambda_2\|_1 + \|u_k\|_{p,\overline{\omega}_{00}}^\alpha \|\lambda_3\|_{(p/\alpha)'} + c_2 \|u_k\|_{1,\omega,p}^p \\ &< \infty \end{aligned}$$

for all $\Omega' \subset \Omega$.

Similarly for $(|(I_k^4)^-|$.

Now, by using the growth condition $(H_2)$ and the Hardy-Type inequalities $(H_0)$, we have

$$\begin{aligned} \int_{\Omega'} |(I_k^2)^-|dx &= \int_{\Omega'} |\sigma(x, u_k, Du_k) : Du_k|dx \\ &\leq \beta \int_{\Omega'} \omega_{rs}^{1/p} \lambda_1(x) D_{rs}u_k dx \\ &+ c_1\beta \sum_{1\leq j\leq m} \int_{\Omega'} \gamma_j^{1/p'} |u_{kj}|^{q/p'} D_{rs}u_k dx \\ &+ c_1\beta \sum_{1\leq i<n,\ 1<j\leq m} \int_{\Omega'} \omega_{ij}^{1/p'} |D_{ij}u_k|^{p-1} D_{rs}u_k dx. \end{aligned} \tag{3.5}$$

Thus, by the Hölder inequality, we obtain

$$\begin{aligned}
\int_{\Omega'} |(I_k^2)^-|dx &\leq \beta\Big[ \|\lambda_1\|_{p'} \left(\int_{\Omega'} |D_{rs}u_k|^p \omega_{rs} dx\right)^{1/p} \\
&+ c_1 \left(\int_{\Omega'} |D_{rs}u_k|^p \omega_{rs} dx\right)^{1/p} \left(\int_{\Omega'} \left(\sum_{1\leq j\leq m} \gamma_j^{1/p'} |u_{kj}|^{q/p'}\right)^{p'} dx\right)^{1/p'} \\
&+ c_1 \left(\sum_{1\leq i<n} \sum_{1<j\leq m} \int_{\Omega'} \left(|D_{ij}u_k(x)|^{p'(p-1)} \omega_{ij} dx\right)^{1/p'}\right) \\
&\times \left(\int_{\Omega'} |D_{rs}u_k|^p \omega_{rs} dx\right)^{1/p} \Big].
\end{aligned} \tag{3.6}$$

So, by combining (3.5) and (3.6), we deduce that

$$\int_{\Omega'} |\sigma(x, u_k, Du_k) : Du_k| dx \leq c'\beta \left(\|\lambda_1\|_{p'} \| \|u_k\|_{1,p,\omega} + \|u_k\|_{1,p,\omega}\right) < \infty. \tag{3.7}$$

Similarly to $(|(I_k^2)^-|$, we obtain $(|(I_k^3)^-|$. Finally: $I_k$ is equiintegrable.

We choose a sequence $\varphi_k$ such that $\varphi_k$ belongs to the same space $V_k$ and $\varphi_k \to \varphi$ in $W_0^{1,p}(\Omega, \omega, I\!R^m)$ this allows us in particular, to use $u_k - \varphi_k$ as a test function in (2.1). We have :

$$\begin{aligned}
\int_{\Omega} |\sigma(x, u_k, Du_k) : (Du_k - D\varphi_k)| dx &= \langle v, u_k - \varphi_k\rangle \\
&+ \int_{\Omega} g(x, u_k) : (u_k - \varphi_k) dx.
\end{aligned} \tag{3.8}$$

The first term on the right in (3.8) converge to zero since $(u_k - \varphi_k) \rightharpoonup 0$ in $W_0^{1,p}(\Omega, \omega, I\!R^m)$. By the choice of $\varphi_k$, the sequence $\varphi_k$ uniformly bounded in $W_0^{1,p}(\Omega, \omega, I\!R^m)$.

By the equivalence of the norm in $W_0^{1,p}(\Omega, \omega, I\!R^m)$ and the sequence $(u_k$ is uniformly bounded in $W_0^{1,p}(\Omega, \omega, I\!R^m)$, $\|u_k\|_{1,p,\omega}$ is bounded. Moreover, by the construction of $\varphi_k$, and lemma (2.2) we have :

$$\|u_k - \varphi_k\|_{1,p,\omega} \leq \|u_k - u\|_{1,p,\omega} + \|u - \varphi_k\|_{1,p,\omega}$$

$$(\|u_k - u\|_{1,p,\omega} + \|u - \varphi_k\|_{1,p,\omega}) \to 0$$

We infer that the second term in (3.8) vanishes as $k \to \infty$. Finally, for the last term And lemma (2.1) Next, for the second term: $II_k = \int_{\Omega} f(x, u_k)(u_k - \varphi_k)dx$ in (3.8) it follows from the growth condition $F_1$ and the Hölder inequality that :

$$|II_k| \leq (\|b_1\|_{p'} + c.\|D(u_k - \varphi_k)\|_{1,p,\omega})\|u_k - \varphi_k\|_{1,p,\omega}$$

$$\leq (\|b_1\|_{p'} + c.\|D(u_k - \varphi_k)\|_{1,p,\omega}).\|u_k - \varphi_k\|_{1,p,\omega}.$$

By the equivalence of the norm in $W_0^{1,p}(\Omega, \omega, I\!R^m)$ and the sequence ($u_k$ is uniformly bounded in $W_0^{1,p}(\Omega, \omega, I\!R^m)$, $\|u_k\|_{1,p,\omega}$ is bounded. We infer that the second term in (3.8) vanishes as $k \to \infty$. $\|\varphi_k - u\|_{1,p,\omega} \to 0$ , $\|u_k - u\|_{1,p,\omega} \to 0$ and $\|u_k - \varphi_k\|_{q,\gamma}^{\frac{q}{p'}} \to 0$
. Now, we consider $(I_k)' = (\sigma(x, u_k, Du_k) : (Du_k - Du)$. We have that $I_k'$ is equiintegrable because $I_k$ is. So, we define

$$X = \liminf_k \int_\Omega I_k dx = \liminf_k \int_\Omega (I_k)' dx \geq \int_\Omega \int_{I\!M^{m\times n}} (\sigma(x, u, \lambda) : (\lambda - Du) d\vartheta_x(\lambda).$$

So to prove (3.2), it suffices to prove that

$$X \leq 0. \tag{3.9}$$

Let $\varepsilon > 0$, so there exists $k_0 \in I\!N$ such that, for all $k > k_0$, we have $\operatorname{dist}(u, V_k) < \varepsilon$ since $\liminf_{v_k \in V_k} \|u - v_k\|_{1,p,\omega} < \varepsilon$, $(u_k \rightharpoonup u)$.
Or in an equivalent manner $\operatorname{dist}(u_k - u, V_k) < \varepsilon$, $\forall k > k_0$ Then for all $v_k \in V_k$ we have

$$\begin{aligned} X &= \liminf_{k\to\infty} \int_\Omega \sigma(x, u_k, Du_k) : (Du_k - Du) dx \\ &= \liminf_{k\to\infty} \left[ \int_\Omega \sigma(x, u_k, Du_k) : D(u_k - u - v_k) dx + \int_\Omega \sigma(x, u_k, Du_k) : D(v_k) dx \right] \end{aligned}$$

Combining $(H_2)$ and (1.1), we get

$$\begin{aligned} X &\leq \lim_{k\to\infty}\inf (\int_\Omega \beta \omega_{rs}^{1/p} [\lambda_1 + c_1 \sum_{1\leq j\leq m} \gamma_j^{1/p'} |u_{k_j}|^{q/p'} + c_1 \sum_{i=1,j=1}^{n,m} \omega_{ij}^{1/p'} |D_{ij} u_k|^{p-1}] \\ &. \quad |D_{rs}(u_k - u - v_k)| dx) \\ &+ \quad \langle f, v_k \rangle. \end{aligned}$$

For all $\varepsilon > 0$, we choice $v_k \in V_k$ such that

$$\|u_k - u - v_k\|_{1,p,\omega} \leq 2\varepsilon, \tag{3.10}$$

for all $k \geq k_0$. Which implies that

$$|\langle f, v_k \rangle| \leq |\langle f, v_k + (u - u_k) \rangle| + |\langle f, u_k - u \rangle| \leq 2\varepsilon \|f\|_{-1,p',\omega^*} + o(k).$$

Hence $\lim_{k\to\infty}\langle f, u_k - v\rangle = 0$. According to the Hölder and Hardy-Type inequalities, and by Lemma 3.1 we deduce that

$$\begin{aligned} X &\leq \liminf_{k\to\infty} c\beta\Bigg( \|\lambda_1\|_{p'}(\int_\Omega |D_{rs}(u_k - u - v_k)|^p \omega_{rs}dx\Big)^{1/p} \\ &\quad + c_1 \left(\int_\Omega |u_k|^q\gamma\right)^{1/p'} \left(\int_\Omega |D_{rs}(u_k - u - v_k)|^p \omega_{rs}dx\right)^{1/p} \\ &\quad + c_1 \left(\sum \int_\Omega \omega_{ij}|D_{ij}u|^{p'(p-1)}\right)^{1/p'} \left(\int_\Omega \omega_{rs}|D_{rs}(u_k - u - v_k)|^p\right)^{1/p} \Bigg) + |\langle f, v_k\rangle| \\ &\leq \liminf_{k\to\infty} c\left(\|\lambda_1\|_{p'}\|u_k - u - v_k\|_{1,p,\omega}\right) + \|u_k\|^q_{1,p,\omega}\|u_k - u - v_k\|_{1,p,\omega} + 2\varepsilon\|f\|_{-1,p',\omega^*} + o(k). \end{aligned}$$

Therefore,

$$X \leq 2\varepsilon c\beta\left(\|\lambda_1\|_{p'} + \|u\|^q_{1,p,\omega} + \|f\|_{-1,p',\omega*}\right).$$

This proves that $X \leq 0$, and finally

$$\int_\Omega \int_{I\!M^{mn}} \sigma(x, u, \lambda) : \lambda d\vartheta_x dx \leq \int_\Omega \int_{I\!M^{mn}} \sigma(x, u, \lambda) : Dud\vartheta_x(\lambda)dx.$$

**Proof of Theorem 1.1** For arbitrary $v$ in $W_0^{1,p}(\Omega, \omega, I\!R^m)$. It follows from the continuity condition $(F_0)$ that

$$g(x, u_k)v(x) \to g(x, u)v(x)$$

almost everywhere. Since, by the growth conditions $(F_1)$ and the uniform bound of $u_k$, $g(x, u_k)v(x)$ is equiintegrable, it follows that the Vitali's theorem . This implies that:

$$\lim_{k\to\infty} \int_\Omega f(x, u_k)v(x)dx = \int_\Omega f(x, u)v(x)dx$$

for all$v \in \cup_{k=1}^\infty V_k$

We will start with the easiest case

$$(d) \qquad F \longmapsto \sigma(x, u, F) \text{ is stictly } p\text{-quasi-monotone.} \tag{3.11}$$

Indeed, we assume that $\vartheta_x$ is not a Dirac mass on the set $M$ with $x \in M$ of positive Lebesgue measure $|M| > 0$. Moreover, by the strict $p$-quasi-monotonicity of $\sigma(x, u, \cdot)$ and $\vartheta_x$ is a homogeneous $W^{1,p}$-gradient Young measure for a.e. $x \in M$. So, for a.e. $x \in M$, with $\bar\lambda = \langle\vartheta_x, Id\rangle = apDu(x))$, Where $apDu(x))$ is the differentiable approximation in $x$. We get

$$\begin{aligned} \int_{I\!M^{m\times n}} \sigma(x, u, \lambda) : (\lambda - Du)d\vartheta_x(\lambda) &> \textstyle\int_{I\!M^{m\times n}} \sigma(x, u, Du) : (\lambda - Du)d\vartheta_x(\lambda) \\ &> \sigma(x, u, Du) : \int_{I\!M^{m\times n}} \lambda d\vartheta_x(\lambda) \\ &- \sigma(x, u, Du) : Du \textstyle\int_{I\!M^{m\times n}} d\vartheta_x(\lambda) \\ &> (\sigma(x, u, Du) : Du - \sigma(x, u, Du) : Du)) = 0 \\ &> 0. \end{aligned}$$

On the other hand by (3.9), integrating over $\Omega$, and using the div-cul inequality(See [9]) we have:

$$\begin{aligned}\int_\Omega \int_{\mathbb{M}^{m\times n}} \sigma(x,u,\lambda):\lambda d\vartheta_x(\lambda)dx &> \int_\Omega \int_{\mathbb{M}^{m\times n}} \sigma(x,u,\lambda):Dud\vartheta_x(\lambda)dx \\ &\geq \int_\Omega \int_{\mathbb{M}^{m\times n}} \sigma(x,u,\lambda):\lambda d\vartheta_x(\lambda)dx.\end{aligned}$$

This contradicts (3.8). Thus $\vartheta_x = \delta_{\overline{\lambda}} = \delta_{Du(x)}$ for a.e. $x \in \Omega$. Therefore, $Du_k \longrightarrow Du$ in measure when $k$ tends to infinity. And thus, $\sigma(x,u_k,Du_k) \longrightarrow \sigma(x,u,Du)$ almost everywhere. Since, by the growth condition in $(H_2)$, $\sigma(x,u_k,Du_k)$ is equiintegrable, it follows that $\sigma(x,u_k,Du_k) \longrightarrow \sigma(x,u,Du)$ in $L^1(\Omega)$ by the Vitali's theorem. The other hand, for all $v \in \bigcup_{k\in\mathbb{N}} \vartheta_k$ we have that $\sigma(x,u_k,Du_k):Dv \longrightarrow \sigma(x,u,Du):Dv$ a.e. $x \in \Omega$. Moveover, for all $\Omega' \subset \Omega$ measurable, it is easy to see that:

$$\int_{\Omega'} \sigma(x,u_k,Du_k):Dvdx \leq c\beta\left(\|\lambda_1\|_{p'} + \|u_k\|_{1,p,\omega}^{q/p'} + \|u_k\|_{1,p,\omega}^{p/p'}\right)\|u\|_{1,p,\omega} < \infty,$$

because $\|u_k\|_{1,p,\omega} \leq R$. And thanks to Vitali's theorem, we obtain:

$$\langle F(u),v\rangle = 0, \text{ for all } v \in \bigcup_{k\in\mathbb{N}} \vartheta_k.$$

Which proves the theorem in this case.

**Remark 3.1** *Before treating the cases* $(a)$, $(b)$ *and* $(c)$ *of* $(H_3)$, *we note that*

$$\int_\Omega \int_{\mathbb{M}^{m\times n}} (\sigma(x,u,\lambda)-\sigma(x,u,Du)):(\lambda - Du)d\vartheta_x(\lambda)dx \leq 0. \tag{3.12}$$

We conclude that

$$\int_\Omega \int_{\mathbb{M}^{m\times n}} \sigma(x,u,\lambda):(\lambda - Du)d\vartheta_x(\lambda)dx = 0,$$

thanks to the (3.9). On the other hand, the integrand in (3.12) is non negative, by the monotonicity of $\sigma$. Consequently, the integrating should be null, a.e., with respect to the product measure $d\vartheta_x \otimes dx$, which means

$$(\sigma(x,u,\lambda)-\sigma(x,u,Du)):(\lambda - Du) = 0 \text{ in spt}\vartheta_x. \tag{3.13}$$

Thus,

$$\text{spt}\vartheta_x \subset \left\{\lambda \in \mathbb{M}^{m\times n} \mid (\sigma(x,u,\lambda)-\sigma(x,u,Du)):(\lambda - Du) = 0\right\}. \tag{3.14}$$

**Case c:** We prove that the map $F \longmapsto \sigma(x,u,F)$ is strictly monotone for all $x \in \Omega$ and for all $u \in \mathbb{R}^m$.

Since $\sigma$ is strictly monotone, and according to (3.14),

$$\mathrm{spt}\vartheta_x = \{Du\}\,,\ \text{i.e, } \vartheta_x = \delta_{Du}, \quad \text{a.e. in } \Omega,$$

which implies that, $Du_k \longrightarrow Du$ in measure. The rest of our prove is similarly to case d.

**Case b:** We start by showing that for almost all $x \in \Omega$, the support of $\vartheta_x$ is contained in the set where $W$ agrees with the supporting hyper-plane.

$$L = \left\{(\lambda, W(x,u,\overline{\lambda}) + \sigma(x,u,\overline{\lambda}) : (\lambda - \overline{\lambda}))\right\} \text{ with } \overline{\lambda} = Du(x).$$

So, it suffices to prove that

$$\mathrm{spt}\vartheta_x \subset K_x = \left\{\lambda \in I\!M^{m\times n} \mid W(x,u,\lambda) = W(x,u,\overline{\lambda}) + \sigma(x,u,\overline{\lambda}) : (\lambda - \overline{\lambda})\right\}. \tag{3.15}$$

If $\lambda \in \mathrm{spt}\vartheta_x$, thanks to (3.14), we have

$$(1-t)\left(\sigma(x,u,Du) - \sigma(x,u,\lambda)\right) : (Du - \lambda) = 0\ ,\ \text{for all } t \in [0,1]. \tag{3.16}$$

On the other hand, since $\sigma$ is monotone, for all $t \in [0,1]$ we have

$$(1-t)\left(\sigma(x,u,Du + t.(\lambda - Du)) - \sigma(x,u,\lambda)\right) : (Du - \lambda) \geq 0. \tag{3.17}$$

By subtracting (3.16) from (3.17), we get

$$(1-t)\left[\sigma(x,u,\overline{\lambda} + t(\lambda - \overline{\lambda})) - \sigma(x,u,\overline{\lambda})\right] : (\overline{\lambda} - \lambda) \geq 0, \tag{3.18}$$

for all $t \in [0,1]$. Doing the same by the monotonicity in (3.18), we obtain

$$(1-t)\left[\sigma(x,u,\overline{\lambda} + t(\lambda - \overline{\lambda})) - \sigma(x,u,\overline{\lambda})\right] : (\overline{\lambda} - \lambda) \leq 0. \tag{3.19}$$

Combining (3.18) and (3.19), we conclude that

$$(1-t)\left[\sigma(x,u,\overline{\lambda} + t(\lambda - \overline{\lambda})) - \sigma(x,u,\overline{\lambda})\right] : (\overline{\lambda} - \lambda) = 0, \tag{3.20}$$

for all $t \in [0,1]$ and for all $\lambda \in \mathrm{spt}\vartheta_x$.

Now, it follows from (3.19) that

$$\begin{aligned} W(x,u,\lambda) &= W(x,u,\overline{\lambda}) + (W(x,u,\lambda) - W(x,u,\overline{\lambda})) \\ &= W(x,u,\overline{\lambda}) + \int_0^1 [\sigma(x,u,\overline{\lambda} + t(\lambda - \overline{\lambda}))] : (\lambda - \overline{\lambda})dt \\ &= W(x,u,\overline{\lambda}) + \sigma(x,u,\overline{\lambda}) : (\lambda - \overline{\lambda}). \end{aligned}$$

This proves (3.15).

Now, by the coercivity of $W$, we get

$$W(x,u,\lambda) \geq W(x,u,\overline{\lambda}) + \sigma(x,u,\overline{\lambda}) : (\lambda - \overline{\lambda}),$$

for all $\lambda \in I\!M^{m\times n}$. Therefore,

$$L \text{ is a supporting hyper-plane, for all } \lambda \in K_x. \tag{3.21}$$

Moveover, the mapping $\lambda \longmapsto W(x,u,\lambda)$ is continuously differentiable, so we obtain

$$\sigma(x,u,\lambda) = \sigma(x,u,\overline{\lambda}), \text{ for all } \lambda \in K_x. \tag{3.22}$$

Thus,

$$\overline{\sigma}(x) = \int_{I\!M^{m\times n}} \sigma(x,u,\lambda) d\vartheta_x(\lambda) = \sigma(x,u,\overline{\lambda}). \tag{3.23}$$

Now, we consider the Carathéodory function

$$h(x,u,\rho) = |(\sigma(x,u,\rho) - \overline{\sigma}(x))|,$$

and observe that $h_k(x) = h(x,u_k,Du_k)$ is equiintegrable. Thus, thanks to Ball's theorem, see [8]and [11] $h_k \rightharpoonup \overline{h}$ weakly in $L^1(\Omega)$, and the weak limit of $\overline{h}$ is given by

$$\begin{aligned}
\overline{h}(x) &= \int\int_{I\!R^m \times I\!M^{m\times n}} |\sigma(x,\eta,\lambda) - \overline{\sigma}(x)| d\delta_{u(x)}(\eta) \otimes d\vartheta_x(\lambda) \\
&= \int_{\mathrm{spt}\vartheta_x} |\sigma(x,u(x),\lambda) - \overline{\sigma}(x)| d\vartheta_x(\lambda) \\
&= 0.
\end{aligned}$$

According to (3.22) and (3.23), and since $h_k \geq 0$, it follow that $h_k \longrightarrow 0$ strongly in $L^1(\Omega)$ by Fatou's lemma, which gives

$$\lim_{k\to\infty} \int_\Omega \sigma(x,u_k,Du_k) : Dv.dx = \int_\Omega \sigma(x,u,Du) : Dvdx.$$

Thus

$$\langle F(u), v\rangle = 0, \quad \forall v \in \bigcup_{k\in I\!N} V_k.$$

This completes the proof of the case (b).

**Case (a):** In this case, on $\mathrm{spt}\vartheta_x$, we affirm that,

$$\sigma(x,u,\lambda) : M = \sigma(x,u,Du) : M + (\nabla_F \sigma(x,u,Du) : M) : (Du - \lambda), \tag{3.24}$$

for all $M \in I\!M^{m\times n}$, where $\nabla_F$ is the derivative with respect to the third variable of $\sigma$ and $\overline{\lambda} = Du(x)$.

Thanks to the monotonicity of $\sigma$, we have

$$(\sigma(x,u,\lambda) - \sigma(x,u,Du+tM)) : (\lambda - Du - tM) \geq 0, \text{ for all } t \in I\!R.$$

By invoking (3.19), we obtain

$$-\sigma(x,u,\lambda) : (tM) \geq -\sigma(x,u,Du) : (\lambda - Du) + \sigma(x,u,Du+tM) : (\lambda - Du - tM).$$

On the other hand, $F \longmapsto \sigma(x,u,F)$ is a $C^1$ function, so

$$\sigma(x,u,Du+tM) = \sigma(x,u,Du) + \nabla_F\sigma(x,u,Du).(tM) + o(t).$$

Thus

$$-\sigma(x,u,\lambda) : (tM) \geq -\sigma(x,u,Du) : (tM) + \nabla_F\sigma(x,u,Du)(tM) : (\lambda - Du) + o(t),$$

which gives

$$-\sigma(x,u,\lambda) : (tM) \geq t[(\nabla_F\sigma(x,u,Du) : (M) : (\lambda - Du) - \sigma(x,u,Du) : (M)] + o(t).$$

The claim follows from this inequality since the sign of $t$ is arbitrary in (3.24). Finally for all $v \in \bigcup_{k\in I\!N} V_k$ the sequence $\sigma(x,u_k,Du_k) : Dv$ is equiintegrable. Then, by the Ball's theorem, see [8]and [11] the weak limit is $\int_{\mathrm{spt}\vartheta_x} \sigma(x,u,\lambda) : Dvd\vartheta_x(\lambda)$ By choosing $M = Du$ in (3.24), we obtain $\displaystyle\int_{\mathrm{spt}\vartheta_x} (Du-\lambda)(\sigma(x,u,\lambda) : Dv) : Dvd\vartheta_x(\lambda)$

$$= \int_{\mathrm{spt}\vartheta_x} \sigma(x,u,Du) : Dvd\vartheta_x(\lambda) + (\nabla_F\sigma(x,u,Du) : Dv)^t \int_{\mathrm{spt}\vartheta_x} (Du-\lambda)d\vartheta_x(\lambda)$$

$$= (\sigma(x,u,Du) : Dv) \int_{\mathrm{spt}\vartheta_x} d\vartheta_x(\lambda) = \sigma(x,u,Du) : Dv.$$

Hence we have

$$\sigma(x,u_k,Du_k) : Dv \longrightarrow \sigma(x,u,Du) : Dv \text{ strongly }.$$

This proves that

$$\langle F(u), v\rangle = 0 \text{ for all } v \in \bigcup V_k.$$

And since $\bigcup V_k$ is dense in $W_0^{1,P}(\Omega,\omega,I\!R^m)$, $u$ is a weak solution of (QES), as desired. □

**Remark 3.2** *In case (b) we have $\sigma(x, u_k, Du_k) : Dv \longrightarrow \sigma(x, u, Du) : Dv$ strongly, but in the case (c) and (d) $Du_k \longrightarrow Du$ in measure.*

**Exemple 3.1** *We shall suppose that the weight functions satisfy: $w_{i_0 j} = 0,\ j = 1, 2, ..., m$ for some $i_0 \in I^c$; and $w_{ij}(x) = w(x); x \in \Omega$, with $I^c \bigcup I = \{0; 1; 2; .....; n\}$, for all $i \in I \sqcup I^c$, $j = 1, 2, ..., m$, and $i \neq i_0$ with $w(x) > 0$ a.e in $\Omega$. Let us consider the Carathéodory functions $\sigma$ defined by*

$$\sigma_{ij}(x, \eta, \xi_I) = w(x)|\xi_{ij}|^{p-1} sign(\xi_{ij}),\ j = 1, 2, ....m\ , i \in I$$

$$\sigma_{ij}(x, \eta, \xi_{I^c}) = w(x)|\xi_{ij}|^{p-1} sign(\xi_{ij}),\ j = 1, 2, ....m\ , i \in I^c, i \neq i_0$$

$$\sigma_{i_0 j}(x, \eta, \xi_{I^c}) = 0; \qquad j = 1, 2, ..., m$$

$$f_j(x, u) = \gamma_j^{\frac{1}{p'}} |u_j|^{\frac{q}{p'}} \omega_{0j}^{\frac{1}{P}}, \quad \forall 1 \leq j \leq m$$

*The functions $\sigma$ satisfies the growth conditions $(H_2)$.*
*In particular, let us the special weight function $\omega$, $\gamma$ expressed in term of the distance to the boundary $\partial\Omega$ denoted by $d(x) = dist(x; \partial\Omega)$ and $w(x) = d^{\lambda}(x)$, $\gamma_j(x) = d^{\mu}(x)$. The Hardy-Type inequalities reads:*

$$\left( \sum_{j=1}^{m} \int_{\Omega} |u_j(x)|^q d^{\mu}(x) dx \right)^{\frac{1}{q}} \leq c \left( \sum_{1 \leq i \leq n;\ 1 \leq j \leq m} \int_{\Omega} |D_{ij} u|^p d^{\lambda}(x) \right)^{\frac{1}{p}},$$

*for every $u \in W_0^{1,p}(\Omega, \omega, I\!R^m)$ with a constant $c > 0$ independent of $u$ and for some $q > p'$ and the corresponding embedding $W_0^{1,p}(\Omega; \omega; R^m) \hookrightarrow L^q(\Omega; \gamma; R^m)$ is compact if:*
*i) For $1 < p \leq q < \infty$*

$$\lambda < p - 1,\ \ \frac{n}{q} - \frac{n}{p} + 1 \geq 0,\ \ \frac{\mu}{q} - \frac{\lambda}{p} + \frac{n}{q} - \frac{n}{p} + 1 > 0.$$

*ii) For $1 \leq q < p < \infty$*

$$\lambda < p - 1,\ \ \frac{n}{q} - \frac{n}{p} + 1 \geq 0,\ \ \frac{\mu}{q} - \frac{\lambda}{p} + \frac{1}{q} - \frac{1}{p} + 1 > 0.$$

*iii) For $q > 1$*
*$\mu(q' - 1) < 1$, by the simple modifications of the example in [1]. Moreover, the monotonicity condition are satisfied:*

$$\sum_{ij} (\sigma_{ij}(x, \eta, \xi_I) - \sigma_{ij}(x, \eta, \xi'_I)(\xi_{ij} - \xi'_{ij})$$

$$= w(x)\sum_{ij}(\mid \xi_{ij} \mid^{p-1} sign(\xi_{ij}) - \mid \xi'_{ij} \mid^{p-1} sign(\xi'_{ij}))(\xi_{ij} - \xi'_{ij}) \geq 0$$

*for almost all $x \in \Omega$ and for all, $\xi, \xi' \in M^n$. This last inequality can not be strict, since for $\xi_{I^c} \neq \xi'_{I^c}$ with $\xi_{i_0 j} \neq \xi'_{i_0 j}$ for all $j = 1, 2, ..., m$. But $\xi_{ij} = \xi'_{ij}$ for $i \in I^c$, $i \neq i_0$, $j = 1, 2, ..., m$ the corresponding expression is zero.*

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