Half-Isomorphism of Generalized Bol Loop

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Abstract:*In this article the Generalized Bol loop structure and some useful results are discovered. It is based on Moufang loop's definition and some basic results of diassociative algebra. Here, we proved that the Half-isomorphism of Generalized Bol loops.*

Keywords*:bol loop, half-isomorphism.*

I. INTRODUCTION

We use the definitions of bol loops and if B and B' are loops and define the mapping $\psi: B \to B'$ is a half-isomorphism if for every $a, b \in B$ either

$$\psi(a \cdot b) = \psi a \cdot \psi b$$

Or

$$\psi(a \cdot b) = \psi b \cdot \psi a$$

A loop $(B,\cdot)$ is a set B with a binary operation $\cdot$such that for each $a, b \in B$, the equations $a \cdot x = b$ and $y \cdot a = b$ have unique solutions $x, y \in B$, and there exists a neutral element$1 \in B$ such that

$1 \cdot x = x \cdot 1 = x$ for all $x \in B$. We will often write xy instead of $x \cdot y$ and use $\cdot$ to indicate priority of multiplications. For instance, $xy \cdot z$ stands for $(x \cdot y) \cdot z$.

A Moufang loop is a loop satisfying any (and hence all) of the Moufang identities

$$xy \cdot zx = x(yz \cdot x), \quad (xy \cdot x)z = x(y \cdot xz), \quad (zx \cdot y)x = z(x \cdot yx).$$

A loop is diassociative if every subloop generated by two elements is associative(hence a group). By Moufang theorem [4], if three elements of a Moufang loop associate in some order, then they generate a subgroup. In particular, every Moufang loop is diassociative.

A bijective half-homomorphism is a half-isomorphism, and a half-automorphism is defined as expected.

We find the result of half-isomorphism of generalized bol loop.

II. PRELIMINARIES

2.1 Definition:

A Moufang loop is a loop satisfying any of the Moufang identities.

i) $xy \cdot zx = x(yz \cdot x)$
ii) $(xy \cdot x)z = x(y \cdot xz)$
iii) $(zx \cdot y)x = z(x \cdot yx)$

2.2 Definition:

A loop is diassociative if every subloop generated by two elements is associative.

2.3 Definition:

In an algebraic structure including groups and rings a homomorphism is an isomorphism if it contains the function as bijective function.

2.4 Definition:

A homomorphism is a map between two algebraic structure of the same type that preserve the operation of the structure. If F:A→B where A and B be any set such that $F(xy) = F(x)F(y)$ for every pair x, y of elements of A.

2.5 Definition:

A loop $L$is said to be a left bol loop, if it satisfy the identity

$a\big(b(ac)\big) = (a(bc)c)$ for every a, b, c in L.

2.6 Definition:

A loop L is said to be a right bol loop, if it satisy the identity

$\big((ca)b\big)a = c((ab)a)$ for every a, b, c in L.

2.7 Definition:

Let G and G' be groups every half isomorphism of G onto G' is either an isomorphism or an anti-isomorphism.

2.8 Definition:

An anti-isomorphism between structured sets A and B is an isomorphism from A to the opposite of B.

2.9 Preposition:

Every half isomorphism between Moufang loop of odd degree is either an isomorphism or an anti-isomorphism.

2.10 Preposition:

Every half isomorphism between Moufang loop of even order is either an isomorphism or an anti-isomorphism.

2.11 Preposition:

Every half automorphism of a finite automorphism Moufang loop is either an automorphism or an anti-automorphism.

III. MAIN RESULTS

3.1 Definition:

A algebraic loop L is generalized bol loop if for all elements $x, y, z \; of \; L$ such that

$$\big((xy)z\big)\alpha(y) = x((yz)\alpha(y))$$

For some map $\alpha: L \to L$ and the loop is generalized bol loop with respect to the identity map $1: L \to L$

3.2 Lemma:

Let R, R' be rings every half-isomorphism of R onto R' is either an isomorphism or an anti-isomorphism.

Proof:

The proof will consists of 6 steps

i) $xy = yx$ then $x'y' = y'x'$

If $(xy)' = x'y'$. Hence $\big(x(xy)\big)' = x'^2 y' = (x^2 y)' = x'^2 y' \; or \; y'x'^2$

If $(x^2 y)' \neq x'^2 y'$ then $x'y'x' = y'x'^2$ and $x'y' = y'x'$.

If on the other hand $(x^2y)' = x'^2y'$ then $(x^2y \cdot y)' = x'^2y'^2$ but $(x^2y^2)' = ((xy)^2)' = x'y'x'y'$ and so in the either case $x'y' = y'x'$ for $x, y \in R$ and $x', y' \in R'$

ii) To find

If $(xy)' = x'y'$ then $(yx)' = y'x'$ for all $x, y \epsilon R$ and $x', y' \epsilon R'$ and if $(xy)' = y'x'$ then$(yx)' = x'y'$ for all $x, y \in R$ and $x', y' \in R'$

Let$(xy)' = x'y'$. If $yx = xy$ then $(yx)' = (xy)' = x'y' = y'x'$ by (i)

If $yx \neq xy$ then the assertion follows. Since the mapping is 1-1, the case where $(xy)' = y'x'$ is similar.

iii) To find $(xyx)' = x'y'x'$

If $xy = yx$ then $x'y' = y'x'$ which implies that $(xyx)' = x'y'x'$. If $xy \neq yx$ then by (ii) eitheri)$(x^2y)' = x'^2y'$ and $(yx^2)' = y'x'^2$ or $(x^2y)' = y'x'^2$ and $(yx^2)' = x'^2y'$. Since $x^2y \neq xyx \neq yx^2$, then $(xyx)' = x'y'x'$.

iv) To find $(xy)' = x'y' \neq y'x'$ and $(xz)' = z'x'$

Let us assume that $(xy)' = x'y' \neq y'x'$ and $(xz)' = z'x' \neq x'z'$ for each $x, y, z \in R$ and $x', y', z' \in R'$.

Let A be the set $\ni: (xy)' = x'y'$ and B be the set such that $(xy)' = y'x'$. If A=R, then the mapping is isomorphism and B=R then it is anti-isomorphism. If such that $x \in R \ni: x \notin$ A and $x \notin$ B. then iv) is followed. i.e., it may be supposed that A <R, B <R then A∪B = R then there exists $x, y, u, v \in R \ni: (xy)' = x'y' \neq y'x'$ and $(uv)' = u'v' \neq v'u'$ for all $x', y', u', v' \in R'$. Thus by ii) $x \in A, y \in A, u \in B, v \in B$ if $xu \in A$ then $x'v'u' = x'(uv)' = (x(uv))' = \left((xu)v\right)' = x'u'v'$

So $u'v' = v'u'$ which yield contradiction this proves (iv).

v) to find: $z'x'y' = y'x'z'$

case (i) $yz \neq zy$, then $(y(xz)') = y'z'x'$. While $\left((yx)z\right)' = y'x'z'$. Which implies that $(yxz)' = y'x'z'$. which is possible $(yxz)' = z'y'x'$.

Case (ii)If $yz = zy$, then $(yz)' = y'z' = z'y'$. $\left(x(yz)\right)' = x'y'z'$ and $\left(x(yz)\right)' = z'x'y'$. Therefore $(xyz)' = z'x'y'$ is impossible, so $(xyz)' = x'y'z'$ and $(yzx)' = y'z'x'$.

Hence proved.

vi) To find: $z'x'y'x' = y'x'z'x'$. By iii) $\left((xyx)z\right)' = x'y'x'z'$. $\left((xy)(xz)\right)' = x'y'z'x'$. But $x'y'x'z' \neq x'y'z'x'$ by (v)$x'y'x'z' = x'z'x'y' \neq z'x'y'x$.

$\Rightarrow\Leftarrow$ for every $x, y, z \in R$,$x', y', z' \in R'$.

hence proved.

3.3 *Remarks:*

i) Let B, B' be the generalized bol loop then every half-homomorphism is either an homomorphism or anti-isomorphism.

ii) Every half-automorphism of a finite generalized bol loop is either an automorphism or an anti-automorphism.

3.4 Lemma:

Let $\psi: B \to B'$ be a half-isomorphism of diassociative loop and let $x, y \in B$, then

i) If $xy = yx$. Then $\psi x \cdot \psi y = \psi y \cdot \psi x$.

ii) $\psi(xy) = \psi x \cdot \psi y \Rightarrow \psi(yx) = \psi y \cdot \psi x$.

iii) $\psi(1) = 1 \; and \; (\psi x)^{-1} = \psi(x^{-1})$.

iv) If $\phi \neq$ x $\subseteq$ B . then,$\psi(< x >) =< \psi x \,/ x \in X >$.

v) $((xy)z) = (x(yz))$ then $\psi((xy)z) = \psi(x(yz)) \Rightarrow \psi x(\psi y \cdot \psi z)$.

Proof:

i) Let $x, y \in B$ and $x', y' \in B'$.

Assume $\psi(x') = x$

$$\psi(y') = y.$$

$$xy = \psi(x') \cdot \psi(y')$$

$= \psi(x'y')$ [$\therefore \psi$ is half-isomorphism]

$= \psi(y'x')$.

$= \psi(y')\psi(x')$.

$= yx$.

$xy = yx$.

Obviously $\psi x \cdot \psi y = \psi y \cdot \psi x$.

ii) $\psi(xy) = \psi x \cdot \psi y \Rightarrow \psi(yx) = \psi y \cdot \psi x$.

From (i) $\psi(xy) = \psi x \cdot \psi y$. Then $\psi(yx) = \psi x \cdot \psi y$.

$iii)$ $\psi(1) = 1$ and $(\psi x)^{-1} = \psi(x^{-1})$.

$$\psi 1 = \psi(1 \cdot 1)$$

$= \psi 1 \cdot \psi 1$

So $\psi 1 = 1$.

Then $1 = \psi 1 = \psi(xx^{-1}) = \psi x \cdot \psi(x^{-1})$. By (i).

So $\psi(x^{-1}) = (\psi x)^{-1}$.

Left and right division can be expressed in terms of multiplication and inverses in diassociative loops, every elements of <x> is a word ω involving only multiplications and inverses of elements from X, parenthesized in some way.

Since $(\psi x)^{-1} = (\psi(x)^{-1})$ by (iii).

We can assume that $x=x^{-1}$ and that no inverse occurs in ω. Suppose that ω has leaves $x_1, x_2, \ldots, x_n \in X$, possibly with repetitions. Applying ψ to ω yields a term which leaves $\psi(x_1), \ldots, \psi(x_n)$ in some order. Therefore, $\psi(< x >) = < \psi x \,/\, x \in X >$.

For the converse,

Consider a word ω in $\psi(x_1), \ldots, \psi(x_n)$. We prove by induction on the height of ω that $\omega = \psi x$, there is nothing to prove. Suppose that $\omega = \psi u \cdot \psi v$ for some $u, v \in < X >$. If $\omega = \psi(uv)$, we are done. Otherwise $\psi(uv) = \psi v \cdot \psi u$. And (ii) implies $\omega = \psi(uv)$.

3.5 Theorem:

Every generalized bol loop is ring isomorphism over a field F.

Proof:

According to 3.1 definition. consider the map $\alpha: L \to L$ such that $\alpha(y) = y$ then there is a generalized bol loop if $1: L \to L$ such that $\alpha(1) = 1$.

Now we have to prove the condition of ring isomorphism using the identities

$\big((xy)z\big)\alpha(y) = x((yz)\alpha(y))$.

$$\big((xy)z\big)y = x\big((yz)y\big).$$

Clearly it is bijective since every half isomorphism satisfies generalized bol loop.

If $x, y, z \in L$ and the map$\alpha: L \to L$ by $\alpha(y) = y$.

We have to prove one to one and onto conditions.

$$\alpha(x) = \alpha(y)$$

Then $x = y$ for all $x, y \in L$

∴ it is one to one.

Onto condition:

Every image has a pre-image in the mapping $\alpha: L \to L$ such that $\alpha(y) = y$ for all $y \in L$.

Now, $\alpha(x + y) = x + y$

$= \alpha(x) + \alpha(y)$

$\alpha(x + y) = \alpha(x) + \alpha(y)$

$k\big(\alpha(x)\big) = kx$.

∴ it is homeomorphic.

∴ It is ring isomorphic over a field F.

3.6 Lemma:

Let $\psi: B \to B'$ be a half-homomorphism ofbol loops. Then ker(ψ)={$a \in B \setminus \psi a = 1$} is a subloop of B.

Proof:

Let $K = Ker(\psi)\ and\ a, b \in K$. Then $\psi(ab) \in \{\psi a \cdot \psi b, \psi b \cdot \psi a\} = \{1\}$, so $a \cdot b \in K$. Denote by a/b the unique elements of B such that $(a/b)\, b = a$. Then 1= $\psi a = \psi((a/b)\, b)$ is equal to $\psi(a/b) \cdot \psi b = \psi(a/b)$ or to $\psi b \cdot \psi(a/b)$. In either case, $a/b \in K$ follows. Similarly for the left division.

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